HOMEWORK PROBLEMS

Due Tuesday Feb. 5th -
1.0.7 (a) Let $R$ be a normal domain with field of fractions $K$ and let $S \subset R$ be a multiplicative subset. Prove that the localization $R_S$ is normal.
(b) Let $R_\alpha$, $\alpha \in A$ be normal domains with the same field of fractions $K$. Prove that the intersection $\cap_{\alpha \in A} R_\alpha$ is normal.

1.1.11 Let $T_N$ be a torus with character lattice $M$. Then every point $t \in T_N$ gives an evaluation map $\phi_t : M \to \mathbb{C}^*$ defined by $\phi_t(m) = \chi^m(t)$. Prove that $\phi_t$ is a group homomorphism and that the map $t \mapsto \phi_t$ induces a group isomorphism $T_N \cong \text{Hom}_\mathbb{Z}(M, \mathbb{C}^*)$.

Due Thursday Feb. 7th -
1.1.6 Different sets of lattice points can parametrize the same affine toric variety, even though these parametrizations behave slightly differently. In this exercise you will consider the parametrizations

$$\Phi_1(s, t) = (s^2, st, st^3) \quad \text{and} \quad \Phi_2(s, t) = (s^3, st, t^3).$$

(a) Prove that $\Phi_1$ and $\Phi_2$ both give the affine toric variety $Y = \mathbb{V}(xz - y^3) \subset \mathbb{C}^3$.
(b) We can regard $\Phi_1$ and $\Phi_2$ as maps $\Phi_1 : \mathbb{C}^2 \to Y$ and $\Phi_2 : \mathbb{C}^2 \to Y$.
Prove that $\Phi_2$ is surjective and that $\Phi_1$ is not. The images are called toric sets - the text gives references to papers studying when a toric set equals the corresponding affine toric variety.

1.1.12 Consider tori $T_1$ and $T_2$ with character lattices $M_1$ and $M_2$. As we saw in class (Example 1.1.13 in the book), the coordinate rings of $T_1$ and $T_2$ are $\mathbb{C}[M_1]$ and $\mathbb{C}[M_2]$. Let $\Phi : T_1 \to T_2$ be a morphism that is a group homomorphism. Then $\Phi$ induces maps $\hat{\Phi} : M_2 \to M_1$ and $\Phi^* : \mathbb{C}[M_2] \to \mathbb{C}[M_1]$ by composition. Prove that $\Phi^*$ is the map of semigroup algebras induced by the map $\hat{\Phi}$ of affine semigroups.

Here is another problem to think about. You do not need to write it up:
1.1.10 Prove that $I = \langle x^2 - 1, xy - 1, yz - 1 \rangle$ is the lattice ideal for the lattice

$$L = \{(a, b, c) \in \mathbb{Z}^3 | a + b + c \equiv 0 \mod 2 \} \subset \mathbb{Z}^3.$$ 

Also compute the primary decomposition of $I$ to show that $I$ is not prime.

Due Tuesday Feb. 12th -
1.2.1 Let $\tau$ be a face of a polyhedral cone $\sigma$. If $v, w \in \sigma$ and $v + w \in \tau$, show that $v, w \in \tau$. Hint: Write $\tau = H_m \cap \sigma$ for $m \in \sigma^\vee$. 

1.2.13 Consider the cone $\sigma = \text{Cone}(3e_1 - 2e_2, e_2) \subset \mathbb{R}^2$.
(a) Describe $\sigma^\vee$ and find generators of $\sigma^\vee \cap \mathbb{Z}^2$. Draw both cones together with facet normals for each edge.
(b) Compute the toric ideal of the affine toric variety $U_\sigma$. Note the connection to problem 1.1.6 from the last problem set.

Here is another problem to think about. You do not need to write it up:
1.2.2 Let $\sigma \subset N_\mathbb{R}$ be a cone.
(a) Show that if $u \in \sigma$, then $u \in \text{Relint}(\sigma)$ if and only if $\langle m, n \rangle > 0$ for all $m \in \sigma^\vee - \sigma^\perp$ if and only if $\sigma^\vee \cap u^\perp = \sigma^\perp$.
(b) Let $\tau \preceq \sigma$ and fix $m \in \sigma^\vee$. Prove that
\[ m \in \tau^* \iff \tau \subset H_m \cap \sigma, \]
\[ m \in \text{Relint}(\tau^*) \iff \tau = H_m \cap \sigma. \]

Due Thur. Feb. 14th -
1.3.7 Let $p$ be a point of an irreducible affine variety $V$. Then $p$ gives the maximal ideal $m = \{ f \in \mathbb{C}[V] | f(p) = 0 \}$ as well as the maximal ideal $m_{V,p} \subset O_{V,p}$. Prove that the natural map $m/m^2 \rightarrow m_{V,p}/m_{V,p}^2$ is an isomorphism of $\mathbb{C}$-vector spaces.

Due Thur. Feb. 19th -
1.3.11 Let $d > 1$ be an integer and $\mu_d = \{ \zeta \in \mathbb{C}^* | \zeta^d = 1 \}$. Consider the lattices:
\[ N' = \mathbb{Z}^2 \subset N = \{(a/d, b/d) | a, b \in \mathbb{Z}, a/d - b/d \in \mathbb{Z}\}, \]
and the cone $\sigma = \text{Cone}(e_1, e_2) \subset N'_\mathbb{R} = N_\mathbb{R}$.
Show that $U_{\sigma,N} = \mathcal{C}_d = \mathbb{C}^2/\mu_d = U_{\sigma,N'}/\mu_d$ where $\mu_d$ acts on $\mathbb{C}^2$ by $\zeta \cdot (x, y) = (\zeta x, \zeta y)$. 