ORBITS LENGTHS OF MODULAR REDUCTIONS OF PAIRS OF POLYNOMIAL DYNAMICAL SYSTEMS

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Abstract. We obtain various bounds on orbit length of modular reductions of algebraic dynamical systems generated by polynomials with integer coefficients. In particular we extend a recent result of Chang (2015) in two different directions.

1. INTRODUCTION

Let

\[ F = (F_1, \ldots, F_m), \quad F_1, \ldots, F_m \in \mathbb{K}[X], \]

be a system of \( m \) polynomials in \( m \) variables \( X = (X_1, \ldots, X_m) \) over a field \( \mathbb{K} \). The iterations of this system are given by

\[ F^{(0)}_i = X_i \quad \text{and} \quad F^{(k)}_i = F_i \left(F^{(k-1)}_1, \ldots, F^{(k-1)}_m\right) \]

for \( i = 1, \ldots, m \) and \( k \geq 1 \). We refer to [AnaKhr09, Sch95, Sil07] for a background on the dynamical systems associated with these iterations.

Given a point \( w \in \mathbb{K}^m \) we define its orbit with respect to the system \( F \) as the set

\[ \text{Orb}_F(w) = \{ w_n \mid \text{with } w_0 = w \text{ and } w_k = F(w_{k-1}), \ k = 1, 2, \ldots \}. \]

The set \( \text{PrePer}_\mathbb{K}(F) \) of preperiodic points of \( F \) is the set of points \( w \in \mathbb{K}^m \) for which \( \text{Orb}_F(w) \) is a finite set.

Sets \( \text{PrePer}_\mathbb{K}(F) \) are classical objects of study and in particular for polynomial systems over \( \mathbb{C} \). For example, by the celebrated result of Northcott [Nor50], if \( \mathbb{K} \) is an algebraic number field, for any system of nonlinear polynomials the set \( \text{PrePer}_\mathbb{K}(F) \) is finite, see also [Sil07, Theorem 3.12]. The Uniform Boundedness Conjecture of Morton and Silverman [MS94] asserts that the cardinality \( \# \text{PrePer}_\mathbb{K}(F) \) can be bounded only in terms of degrees of the polynomials in \( F \) and the

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Recently, several very deep results have been obtained towards this conjecture, see [BDeM11, BDeM13, GHT13, GHT15, GKN16, GKNY16, GNT15, Ing12] and references there in. In a similar spirit, Dvornicich and Zannier [DvZan07] show that under some very natural necessary conditions a polynomial \( f \) may have only finitely many preperiodic points in the set \( \mathcal{U} \) of roots of unity (or more generally in the cyclotomic closure \( \mathbb{K}[\mathcal{U}] \) of an algebraic number field \( \mathbb{K} \)). On the other hand, if \( \mathbb{K} = \mathbb{F}_q \) is a finite field of \( q \) elements then all orbits \( \text{Orb}_F(w) \) are finite and in fact \( \#\text{Orb}_F(w) \leq q^m \).

We also note the result of Ingram [Ing12] which shows that the set of \( t \in \mathbb{Q} \) for which the critical points of a parametric polynomial \( f_t(X) \in \mathbb{C}[X] \) are preperiodic (such polynomials are called \textit{post-critically finite}) is a set of bounded height.

Recently, there has been active interest in the study of orbits of reductions \( F_p \) modulo distinct primes \( p \) of a polynomial system \( F \) defined over \( \mathbb{Q} \), see [AkbGhi09, BGH+13, Cha15, DOSS15, Sil08]. We use \( \text{Orb}_{F_p}(w) \) to denote the orbit of the reduction of \( w \in \mathbb{Z}^m \) modulo \( F_p \) in the dynamical system over \( \mathbb{F}_p \) generated by the reduction of polynomial system \( F \in \mathbb{Z}[X] \) modulo \( p \). Alternatively, \( \text{Orb}_{F_p}(w) \) is the reduction modulo \( p \) of the elements of the orbit (1.2).

Silverman [Sil08] has shown that under some natural conditions on a fixed \( w \in \mathbb{Z}^m \), for almost all primes \( p \) (in the sense of asymptotic relative density) we have \( \#\text{Orb}_{F_p}(w) \geq (\log p)^{1+o(1)} \). This result has been improved slightly by Akbary and Ghioca [AkbGhi09].

Chang [Cha15] has given a result of a new type involving two distinct orbits. The method of [Cha15] is based on a result of Ghioca, Krieger and Nguyen [GKN16] on the finiteness of the set of \( t \in \mathbb{C} \) for which \( 0 \in \text{PrePer}_C(f_t) \cap \text{PrePer}_C(g_t) \) for the polynomials \( f_t(X) = X^d + t \) and \( g_t(X) = X^d + a(t) \) with \( a \in \mathbb{Z}[T] \) and a fixed integer \( d \geq 2 \). This result has been extended by Ghioca, Krieger, Nguyen and Ye [GKNY16] to much wider families of polynomials.

Let \( \overline{\mathbb{F}}_p \) denote the algebraic closure of \( \mathbb{F}_p \). Then, by [Cha15, Theorem 1], there are constants \( c_1, c_2 \) depending on \( d \) and \( a(T) \) such that for almost all primes \( p \), there is a set \( \mathcal{T} \subseteq \overline{\mathbb{F}}_p \) with \( \#\mathcal{T} \leq c_1 \) such that for every \( t \in \overline{\mathbb{F}}_p \setminus \mathcal{T} \) we have

\[
\max \{ \#\text{Orb}_{f_t,p}(0), \#\text{Orb}_{g_t,p}(0) \} \geq c_2 \log p. \tag{1.3}
\]

Here we consider a more general case of \( r \geq 1 \) distinct \( n \)-parametric \( m \)-dimensional polynomial systems

\[
F_{t,\nu}(X) = (F_{1,\nu}(X, t), \ldots, F_{m,\nu}(X, t)), \quad \nu = 1, \ldots, r, \tag{1.4}
\]
with polynomials
\[(1.5) \quad F_{i,\nu}(\mathbf{X}, \mathbf{T}) \in \mathbb{Z}[\mathbf{X}]; \mathbf{T}], \quad i = 1, \ldots, m, \ \nu = 1, \ldots, r,\]
where \(\mathbf{T} = (T_1, \ldots, T_n)\), specialised at the values of the parameter \(t \in \mathbb{C}^n\).

It is also convenient to denote
\[0_m = (0, \ldots, 0).\]

Here we extend [Cha15, Theorem 1] in several different directions:

- We use some results of [DOSS15] to obtain an analogue of the result of Chang [Cha15, Theorem 1] for \(r\) distinct \(n\)-parametric \(m\)-dimensional polynomial systems \(F_{t,\nu}, \nu = 1, \ldots, r\), for which
\[0_m \in \bigcap_{\nu=1}^{r} \text{PrePer}_\mathbb{C}(F_{t,\nu})\]
for only finitely many values of the parameter \(t \in \mathbb{C}^n\);

- We obtain a somewhat dual result of similar flavour, which applies to one polynomial system and several initial points.

- We use a result on divisibility of resultants which is due to Gómez-Pérez, Gutierrez, Ibeas and Sevilla [GGIS09] in the settings of [Cha15] with two parametric families of univariate polynomials to get a trade-off between the size of the exceptional set \(T \subseteq \mathbb{F}_p\) and \(\max\{\#\text{Orb}_{t,\nu}(0), \#\text{Orb}_{g_{t,\nu}}(0)\}\) in [Cha15, Theorem 1].

Note that our results can be derived for any fixed initial point \(w_0 \in \mathbb{Z}^m\), not necessary for \(w_0 = 0_m\). In fact no special adjustment is needed, one simply considers the polynomial systems \(F_{t,\nu}(\mathbf{X} - w_0) + w_0, \nu = 1, \ldots, r\), with shifted arguments and polynomials.

Throughout the paper, given functions
\[\Phi, \Psi : \mathbb{N} \to \mathbb{N},\]
the symbols \(\Phi = O(\Psi)\) and \(\Phi \ll \Psi\) both mean that there is a constant \(c \geq 0\) such that \(\Phi(k) \leq c\Psi(k)\) for all \(k \in \mathbb{N}\). To emphasise the dependence of the implied constant \(c\) on a list of parameters \(\rho\), we write \(\Phi = O_\rho(\Psi)\) or \(\Phi \ll_\rho \Psi\).

2. Main results

2.1. Multivariate systems. We start with a generalisation of the result of Chang [Cha15, Theorem 1] and obtain a version of the lower bound (1.3) for several parametric multivariate polynomial systems as in (1.4) and (1.5).
Theorem 2.1. Let $F_{t,\nu}$, $\nu = 1, \ldots, r$, be $r \geq 1$ parametric systems of polynomials as in (1.4) and (1.5) with
\[
\max_{i=1, \ldots, m} \deg F_{i,\nu} \leq d \quad \text{and} \quad \max_{i=1, \ldots, m} h(F_{i,\nu}) \leq h.
\]
Assume that there exists $K \in \mathbb{N}$ such that
\[
\# \left\{ t \in \mathbb{C}^n : 0_m \in \bigcap_{\nu=1}^{r} \text{PrePer}_C(F_{t,\nu}) \right\} \leq K.
\]
Then, for any integer $L$, there exists an integer $\mathfrak{A} \geq 1$ with
\[
\log \mathfrak{A} \ll_{d, h, n, m, r} (LdL)^{3n+2}
\]
such that for a prime $p$ with $p \nmid \mathfrak{A}$, for all but at most $K$ values of $t \in \mathbb{F}_p^n$, we have
\[
\max \left\{ \#\text{Orb}_{F_{t,\nu},p}(0_m) : \nu = 1, \ldots, r \right\} > L.
\]
Corollary 2.2. Under the conditions of Theorem 2.1, for any prime $p$ we have
\[
\max \left\{ \#\text{Orb}_{F_{t,\nu},p}(0_m) : \nu = 1, \ldots, r \right\} \gg_{d, h, n, m, r} \log \log p
\]
for all but at most $K$ values of $t \in \mathbb{F}_p^n$.

For almost all primes, we have a stronger result.

Corollary 2.3. Under the conditions of Theorem 2.1, for any fixed $\varepsilon > 0$ and sufficiently large integer $Q \geq 2$, for all but $Q^{\varepsilon}$ primes $p \leq Q$ we have
\[
\max \left\{ \#\text{Orb}_{F_{t,\nu},p}(0_m) : \nu = 1, \ldots, r \right\} \gg_{d, h, m, n, r} \log p
\]
for all but at most $K$ values of $t \in \mathbb{F}_p^n$.

It is interesting to compare the bound of Corollary 2.3 with the result of Silverman [Sil08] and its improvement due to Akbary and Ghioca [AkbGhi09].

We now obtain a dual result for a polynomial system but with several initial points.

Theorem 2.4. Let $\{F_t\}_{t \in \mathbb{C}^n} = \{(F_1(X, t), \ldots, F_m(X, t))\}_{t \in \mathbb{C}^n}$ be a parametric system with polynomials as in (1.4) and (1.5) and let $a_\nu \in \mathbb{Z}^m$, $\nu = 1, \ldots, r$, be $r$ integer vectors with
\[
\max_{i=1, \ldots, m} \deg F_i \leq d \quad \text{and} \quad \max_{i=1, \ldots, m} \max_{\nu=1, \ldots, r} h(F_i, h(a_\nu)) \leq h.
\]
Assume that there exists $K \in \mathbb{N}$ such that
\[
\# \left\{ t \in \mathbb{C}^n : \{a_1, \ldots, a_r\} \subseteq \text{PrePer}_C(F_t) \right\} \leq K.
\]
Then, for any integer $L$, there exists an integer $A \geq 1$ with

$$\log A \ll_{d,h,n,m,r} (Ld^L)^{3n+2}$$

such that for a prime $p$ with $p \not| A$, for all but at most $K$ values of $t \in \mathbb{F}_p^n$, we have

$$\max \{ \#\text{Orb}_{F_t,p}(a_\nu) : \nu = 1, \ldots, r \} > L.$$
Corollary 2.6. Under the conditions of Theorem 2.5, for any integers $E, L, Q \geq 1$ the number $R$ of primes $p \in [Q, 2Q]$ such that
\[
\max \{\#\text{Orb}_{f_t,p}(0), \#\text{Orb}_{g_t,p}(0)\} \leq L
\]
for at least $E$ values of $t \in \mathbb{F}_p$, satisfies
\[
ER \ll_{d,h} L^2 d^{2L} / \log Q + K.
\]

For example, we see that for any function $\psi$ with $\psi(z) \to \infty$ as $z \to \infty$ for all but $o(Q/\log Q)$ primes $p \in [Q, 2Q]$ we have
\[
\max \{\#\text{Orb}_{f_t,p}(0), \#\text{Orb}_{g_t,p}(0)\} \leq \frac{\log Q - 2 \log \log Q}{2 \log d} - \psi(Q)
\]
for at most $K + O_{d,h}(1)$ values of $t \in \mathbb{F}_p$, which is a more explicit form of the bound (1.3).

Theorem 2.7. Let $\{f_t\}_{t \in \mathbb{C}}$ be a parametric family of univariate polynomials defined by (1.4) and (1.5) with a polynomial $f(X,T) \in \mathbb{Z}[X,T]$ and let $a,b \in \mathbb{Z}^m$ be two integers with
\[
\deg f \leq d \quad \text{and} \quad \max\{h(f), \log |a|, \log |b|\} \leq h.
\]
Assume that there exists $K \in \mathbb{N}$ such that
\[
\# \{t \in \mathbb{C}^m : \{a,b\} \subset \text{PrePer}_C(f_t)\} \leq K.
\]
Then, for any integer $L$, there exists an integer $\mathfrak{B} \geq 1$ with
\[
\log \mathfrak{B} \ll_{d,h,m} L^2 d^{2L}
\]
such that for a prime $p$ and a positive integer $N$ with $p^N \nmid \mathfrak{B}$, for all but at most $N + K - 1$ values of $t \in \mathbb{F}_p$ we have
\[
\max \{\#\text{Orb}_{f_t,p}(a), \#\text{Orb}_{f_t,p}(b)\} > L.
\]

As we have mentioned, the result of Baker and DeMarco [BDeM11, Theorem 1.1] shows that the class of polynomials to which Theorem 2.7 applies is not void.

Finally, as before, we also have:

Corollary 2.8. Under the conditions of Theorem 2.7, for any integers $E, L, Q \geq 1$ the number $R$ of primes $p \in [Q, 2Q]$ such that
\[
\max \{\#\text{Orb}_{f_t,p}(a), \#\text{Orb}_{f_t,p}(b)\} \leq L
\]
for at least $E$ values of $t \in \mathbb{F}_p$, satisfies
\[
ER \ll_{d,h} L^2 d^{2L} / \log Q + K.
\]
3. Auxilary results

3.1. Heights of polynomials and their iterates. For an integer vector \( a = (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell \) we define its height \( h(a) \) as

\[
    h(a) = \max_{j=1,\ldots,\ell} \log \max \{1, |a_j|\}.
\]

For a polynomial \( \Psi \in \mathbb{Z}[X] \), we define its height, denoted by \( h(\Psi) \), as the height of the vector formed by its coefficients.

The following bound on the height of a product of polynomials is important for our results. It follows from [KPS01, Lemma 1.2].

**Lemma 3.1.** Let \( \Psi_1, \ldots, \Psi_s \in \mathbb{Z}[Z] \) be polynomials in \( n \) variables \( Z = (Z_1, \ldots, Z_n) \). Then

\[
    -2 \sum_{i=1}^s \deg \Psi_i \log(n + 1) \leq h \left( \prod_{i=1}^s \Psi_i \right) - \sum_{i=1}^s h(\Psi_i)
    \leq \sum_{i=1}^s \deg \Psi_i \log(n + 1).
\]

We also frequently use the trivial bound on the height of a sum of polynomials

\[
    h \left( \sum_{i=1}^s \Psi_i \right) \leq \max_{i=1,\ldots,s} h(\Psi_i) + \log s. \tag{3.1}
\]

Moreover, we need a bound of [DOSS15] on the degree and height of iterations of polynomial systems.

**Lemma 3.2.** Let \( \Psi_1, \ldots, \Psi_s \in \mathbb{Z}[Z] \) be polynomials in \( s \) variables \( Z = (Z_1, \ldots, Z_s) \) of degree at most \( D \geq 2 \) and of height at most \( H \). Then, for any positive integer \( k \), the polynomials \( \Psi_1^{(k)}, \ldots, \Psi_s^{(k)} \) defined as in (1.1), are of degree at most

\[
    \max_{j=1,\ldots,s} \deg \Psi_j^{(k)} \leq D^k
\]

and of height at most

\[
    \max_{j=1,\ldots,s} h(\Psi_j^{(k)}) \leq H \frac{D^k - 1}{D - 1} + D(D + 1) \frac{D^{k-1} - 1}{D - 1} \log(s + 1).
\]

3.2. Modular reduction of systems of polynomial equations. We recall the following result of [DOSS15] concerning the reduction modulo prime numbers of systems of multivariate polynomials over the integers.
Lemma 3.3. Let $\Psi_1, \ldots, \Psi_s \in \mathbb{Z}[T]$ in $n$ variables $T = (T_1, \ldots, T_n)$ of degree at most $D \geq 2$ and of height at most $H$, whose zero set in $\mathbb{C}^n$ has a finite number $K$ of distinct points. Then there exists $A \in \mathbb{N}$ satisfying

$$\log A \leq C_1(n)D^{3n+1}H + C_2(n, s)D^{3n+2},$$

with

$$C_1(n) = 11n + 4 \quad \text{and} \quad C_2(n, s) = (55n + 99)\log((2n + 5)s)$$

and such that, if $p$ is a prime number not dividing $A$, then the zero set in $\mathbb{F}_p^s$ of the system of polynomials $\Psi_i \pmod{p}$, $i = 1, \ldots, s$, consists of exactly $K$ distinct points.

3.3. Common zeros and resultants of polynomials. One of our main results relies on a generalisation of the well known fact that if two univariate polynomials $f(T), g(T) \in \mathbb{Z}[T]$ have a common zero modulo $p$ then their resultant $\text{Res}(f, g)$ is divisible by $p$. We need the following extension of this property, due to Gómez-Pérez, Gutierrez, Ibeas and Sevilla [GGIS09], to polynomials with several common roots modulo a prime.

Lemma 3.4. Let $p$ be a prime and let $f, g \in \mathbb{Z}[T]$ be two univariate polynomials such that their reduction modulo $p$ do not vanish identically and have at least $N$ common roots in $\mathbb{F}_p$ counted with multiplicities. Then $p^N \mid \text{Res}(f, g)$.

We remark that for applications, the result of [KS99, Lemma 5.3] (which counts only simple roots) is sufficient.

4. Proofs of main results

4.1. Proof of Theorem 2.1. Consider the systems

$$R_\nu = (F_{1, \nu}(X, T), \ldots, F_{m, \nu}(X, T), T_1, \ldots, T_n), \quad \nu = 1, \ldots, r,$$

of $m+n$ polynomials in $m+n$ variables, each.

Let $T$ be set of those $t \in \mathbb{C}^n$ for which $0_m$ is a preperiodic point of every system $F_{t, \nu}$, $\nu = 1, \ldots, r$. By our assumptions, we have that $\#T \leq K$.

For every choice of nonnegative integers $k_1, \ldots, k_r < L$, we consider the system of $(m+n)r$ equations formed by the iterations

$$(4.1) \quad R^{(L)}_\nu(0_m, T) = R^{(k_\nu)}_\nu(0_m, T), \quad \nu = 1, \ldots, r.$$}

Observe that in each group of $m + n$ equations corresponding to the same value of $\nu$, the bottom $n$ equations in (4.1) are automatically
satisfied. So we have $mr$ equations in $n$ variables:

$$
(4.2) \quad F_{i,\nu}^{(L)}(0_m, T) = F_{i,\nu}^{(k_\nu)}(0_m, T) \quad i = 1, \ldots, m, \nu = 1, \ldots, r.
$$

Furthermore, we consider now the system of $mr$ equations

$$
(4.3) \quad \prod_{k_\nu < L} \left( F_{i,\nu}^{(L)}(0_m, T) - F_{i,\nu}^{(k_\nu)}(0_m, T) \right) = 0,
$$

$$
i = 1, \ldots, m, \nu = 1, \ldots, r,
$$

which by the above, has at most $K$ solutions $t \in T$.

Now note that if

$$
\max \{ \#\text{Orb}_{F_{i,\nu}(0_m, T)} : \nu = 1, \ldots, r \} \leq L
$$

for some parameter $t \in \mathbb{F}_p^n$, then there are some nonnegative integers $k_1, \ldots, k_r < L$ for which we have (4.1), and thus (4.3) (considered over $\mathbb{F}_p^n$ with reductions modulo $p$ of the corresponding polynomials).

Applying Lemma 3.2 to the systems $R_{i,\nu}$ in $n + m$ variables, we obtain that for $i = 1, \ldots, m$, $\nu = 1, \ldots, r$ and an integer $k \geq 0$ we have

$$
(4.4) \quad \deg F_{i,\nu}^{(k)}(0_m, T) \leq d^k
$$

and

$$
(4.5) \quad h \left( F_{i,\nu}^{(k)}(0_m, T) \right) \leq h \left( \frac{d^k - 1}{d - 1} + d(d + 1) \frac{d^{k-1} - 1}{d - 1} \log(n + m + 1) \right).
$$

From (4.4), we immediately conclude

$$
(4.6) \quad \deg \left( \prod_{k < L} \left( F_{i,\nu}^{(L)}(0_m, T) - F_{i,\nu}^{(k)}(0_m, T) \right) \right) \ll_{d,h,n,m} Ld^L,
$$

and furthermore by (3.1) and (4.5), we have

$$
\begin{align*}
\quad & h \left( F_{i,\nu}^{(L)}(0_m, T) - F_{i,\nu}^{(k)}(0_m, T) \right) \\
\quad & \leq h \left( \frac{d^L - 1}{d - 1} + d(d + 1) \frac{d^{L-1} - 1}{d - 1} \log(n + m + 1) + \log 2 \right) \\
\quad & \ll_{d,h,n,m} q^L,
\end{align*}
$$

for $i = 1, \ldots, m$ and $\nu = 1, \ldots, r$.

Hence, by Lemma 3.1, we immediately obtain

$$
(4.7) \quad h \left( \prod_{k < L} \left( F_{i,\nu}^{(L)}(0_m, T) - F_{i,\nu}^{(k)}(0_m, T) \right) \right) \ll_{d,h,n,m,r} Ld^L,
$$

for $i = 1, \ldots, m$ and $\nu = 1, \ldots, r$. 


Now we apply Lemma 3.3 with $s = mr$. Hence, if $p \nmid \mathfrak{a}$, where $\mathfrak{a}$ is as in Lemma 3.3, and thus

$$\log \mathfrak{a} \ll_{d,h,n,m,r} (Ld^L)^{3n+2},$$

then the system (4.3) (considered over $\mathbb{F}_p^n$ again) has at most $K$ zeros in $\mathbb{F}_p^n$. The bound (4.8) gives the desired inequality.

4.2. Proof of Corollary 2.2. We can assume that $p$ is sufficiently large. Theorem 2.1 applied with

$$L = \left\lfloor \frac{\log \log p}{3(n + 1) \log d} \right\rfloor$$

implies $\log \mathfrak{a} \ll_{d,h,m,n,r} (\log p)^{1-1/(3n+3)} \epsilon (\log Q)^{3n+2}$. Since $p$ is large enough we have $p \nmid \mathfrak{a}$ and the result now follows.

4.3. Proof of Corollary 2.3. Theorem 2.1 applied with

$$L = \left\lfloor \frac{\log Q}{3(n + 1) \log d} \right\rfloor$$

implies $\log \mathfrak{a} \ll_{d,h,m,n,r} Q^{(1-1/(3n+3)) \epsilon (\log Q)^{3n+2}}$. The divisibility $p \mid \mathfrak{a}$ is possible for at most $2 \log \mathfrak{a} \ll_{d,h,m,n,r} Q^{(1-1/(3n+3)) \epsilon (\log Q)^{3n+2}}$ primes $p$ and since $Q$ is large enough the result now follows.

4.4. Proof of Theorem 2.4. The proof follows the same way as for Theorem 2.1. Consider the system

$$R = (F_1(X, T), \ldots, F_m(X, T), T_1, \ldots, T_n)$$

of $m + n$ polynomials in $m + n$ variables, each.

Let $\mathcal{T}$ be set of those $t \in \mathbb{C}^n$ for which $a_1, \ldots, a_r$ are preperiodic points of $F_t$. By our assumptions, we have that $\# \mathcal{T} \leq K$.

For every choice of nonnegative integers $k_1, \ldots, k_r < L$, we consider the system of $(m + n)r$ equations formed by the iterations

$$R^{(L)}(a_\nu, T) = R^{(k_\nu)}(a_\nu, T), \quad \nu = 1, \ldots, r.$$  

Observe that in each group of equations the bottom $n$ equations are automatically satisfied. So we have $mr$ equation (formed by the first $m$ components of $R^{(k_\nu)}$) in $n$ variables:

$$F_i^{(L)}(a_\nu, T) = F_i^{(k_\nu)}(a_\nu, T), \quad i = 1, \ldots, m, \ \nu = 1, \ldots, r.$$  

We consider now the system of $mr$ equations

$$\prod_{k_\nu \leq L} \left( F_i^{(L)}(a_\nu, T) - F_i^{(k_\nu)}(a_\nu, T) \right) = 0,$$  

$$i = 1, \ldots, m, \ \nu = 1, \ldots, r,$$  

where $i = 1, \ldots, m$. The result now follows.
which by the above, has at most $K$ solutions $t \in \mathcal{T}$.

Now note that if

$$\max \{ \#\text{Orb}_{F_t,p}(a_{\nu}) : \nu = 1, \ldots, r \} \leq L$$

for some parameter $t \in \overline{F}_p^m$, then there are some nonnegative integers $k_1, \ldots, k_r < L$ for which we have (4.9), and thus (4.11) (considered over $\overline{F}_p^m$ with reductions modulo $p$ of the corresponding polynomials).

As before, applying Lemma 3.2 to the system $R$ in $n + m$ variables, we see that for any integer $k \geq 1$ we have a full analogue of (4.6) and (4.7), that is,

$$\deg \left( \prod_{k < L} \left( F^{(L)}_i(a_{\nu}, T) - F^{(k)}_i(a_{\nu}, T) \right) \right) \ll_{d,h,n,m,r} Ld^L$$

and

$$h \left( \prod_{k < L} \left( F^{(L)}_i(a_{\nu}, T) - F^{(k)}_i(a_{\nu}, T) \right) \right) \ll_{d,h,n,m,r} Ld^L,$$

for $i = 1, \ldots, m$ and $\nu = 1, \ldots, r$.

Now we apply Lemma 3.3 with $s = mr$. Hence, if $p \nmid \mathfrak{A}$, where $\mathfrak{A}$ is as in Lemma 3.3, and thus

(4.12) \hspace{1cm} \log \mathfrak{A} \ll_{d,h,n,m,r} (Ld^L)^{3n+2},

then the system (4.11) (considered over $\overline{F}_p^m$ again) has at most $K$ zeros in $\overline{F}_p^n$. The bound (4.12) gives the desired inequality.

4.5. Proof of Theorem 2.5. As in Theorem 2.1, consider the two dimensional dynamical systems

$$R = (f(X,T), T), \quad \text{and} \quad Q = (g(X,T), T).$$

By the finiteness assumption, the polynomials

$$\Phi_L(T) = \prod_{k=0}^{L-1} \left( f^{(L)}(0, T) - f^{(k)}(0, T) \right),$$

$$\Psi_L(T) = \prod_{k=0}^{L-1} \left( g^{(L)}(0, T) - g^{(k)}(0, T) \right),$$

have at most $K$ common zeros $t \in \mathbb{C}$. This implies that at least one among $\Phi_L(T)$ and $\Psi_L(T)$ is not zero. If one of them is identically zero, then the degree of the other is bounded by $K$ and the claim follows straightforwardly by taking $\mathfrak{A} = 1$. 
Suppose then without loss of generality that $\Psi_L(T) \neq 0 \neq \Phi_L(T)$, and write
\[
\Phi_L(T) = \tilde{\Phi}_L(T)H_L(T) \quad \text{and} \quad \Psi_L(T) = \tilde{\Psi}_L(T)H_L(T),
\]
for nonzero polynomials $\tilde{\Phi}_L(T), \tilde{\Psi}_L(T), H_L(T) \in \mathbb{Z}[T]$ such that the polynomials $\tilde{\Phi}_L(T)$ and $\tilde{\Psi}_L(T)$ have no common root in $\mathbb{C}$ and $H_L(T)$ has at most $K$ distinct zeros.

Let $M$ the number of their common zeros in $\mathbb{F}_p$. At most $K$ of them come from the polynomial $H_L(T)$. Hence, the polynomials, $\tilde{\Phi}_L(T)$ and $\tilde{\Psi}_L(T)$ have at least $M - K$ common zeros.

In particular, by Lemma 3.4, we deduce that $p^{M-K} \mid \mathfrak{B}$, where
\[
\mathfrak{B} = \left| \text{Res} \left( \tilde{\Phi}_L(T), \tilde{\Psi}_L(T) \right) \right| > 0.
\]

Hence, for a bound $N$ such that $p^N \nmid \mathfrak{B}$, we must have $M \leq N + K - 1$. One checks that this is also true if one of the polynomials $\tilde{\Phi}_L(T)$ and $\tilde{\Psi}_L(T)$ vanishes identically modulo $p$.

To finish the proof we need to bound the size of $\mathfrak{B}$. As in the proof of Theorem 2.1, applying Lemma 3.2 to the system $R$ and $Q$ in two variables, we get
\[
\deg \Phi_L, \deg \Psi_L \leq Ld^L
\]
and
\[
(4.13) \quad h(\Phi_L(T)), h(\Psi_L(T)) \ll_{d,h} Ld^L.
\]

We apply now Lemma 3.1 and using (4.13), we conclude that
\[
(4.14) \quad h(\tilde{\Phi}_L), h(\tilde{\Psi}_L) \ll_{d,h} Ld^L.
\]

We now use the trivial bound
\[
|\det B| \leq s! H^s \leq s^s H^s
\]
on the determinant of an $s \times s$ matrix $B$ with complex entries of absolute value at most $H$ (note that the Hadamard inequality does not lead to any advantage here). We apply it to the Sylvester determinant formula for the resultant $\mathfrak{B}$ (with $\log H \ll_{d,h,m} Ld^L$ and $s \leq Ld^L$). Hence we derive
\[
\log \mathfrak{B} \ll_{d,h} L^2 d^{2L},
\]
which concludes the proof.

4.6. Proof of Corollary 2.6. Theorem 2.5 implies
\[
(E - K + 1)R \log Q \leq \log \mathfrak{A} \ll_{d,h} L^2 d^{2L}
\]
and the result now follows.
4.7. **Proof of Theorem 2.7.** By consider the polynomials
\[
\Phi_L(T) = \prod_{k=0}^{L-1} \left( f^{(L)}(a, T) - f^{(k)}(a, T) \right),
\]
\[
\Psi_L(T) = \prod_{k=0}^{L-1} \left( f^{(L)}(b, T) - g^{(k)}(b, T) \right),
\]
which have at most \(K\) common zeros \(t \in \mathbb{C}\), and then follow the same argument as in the proof of Theorem 2.5. In particular, we have full analogues of the bounds (4.13) and (4.14).

4.8. **Proof of Corollary 2.8.** Similarly to the proof of Corollary 2.6 we note that Theorem 2.7 implies
\[
(E - K + 1)R \log Q \leq \log \mathfrak{A} \ll_{d,h} L^2 d^{2L}
\]
and the result now follows.

5. **Comments**

We remark that considering the systems of equations (4.2) and (4.10) separately for each choice of the parameters \(k_1, \ldots, k_r\), and \(k\), respectively, instead of the systems of equations (4.3) and (4.11), one can slightly improve polynomial factors in the dependence on \(L\) in the bounds of Theorems 2.1 and 2.4.

**References**


