Some consequences of the Polynomial Freiman-Ruzsa Conjecture

Mei-Chu Chang

notations

- n-fold sum set

$$nA = A + \cdots + A = \{a_1 + \cdots + a_n : a_1, \cdots, a_n \in A\}$$

- n-fold product set

$$A^n = A \cdots A = \{a_1 \cdots a_n : a_i \in A\}$$

- inverse set

$$A^{-1} = \{a^{-1} : a \in A\}$$

- $$A^{[n]} = (\{1\} \cup A \cup A^{-1})^n$$

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- $V = \mathbb{Z}$-module, $A \subset V$ finite, $|A + A| < K|A|$

Freiman-Ruzsa Theorem

$\Longrightarrow \exists B \subset \mathbb{R}^d$ a box

$\exists \phi : \mathbb{Z}^d \to V$ a group homo s.t.

$$A \subset \phi(B)$$

$$d \leq K$$

$$|B| < e^{c(K)}|A|.$$

$$d \leq c \log K$$

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Freiman-Ruzsa Conjecture

$\Longrightarrow \exists B \subset \mathbb{R}^d$ convex

$\exists \phi : \mathbb{Z}^d \to V$ a group homo

$\exists A_1 \subset A$, $|A_1| > K^{-c}|A|$ s.t. $A_1 \subset \phi(B \cap \mathbb{Z}^d)$

$$d \leq c \log K$$

$$|B \cap \mathbb{Z}^d| < K^c|A|.$$

Freiman-Ruzsa

- $V = \mathbb{Z}$-module, $A \subset V$ finite, $|A + A| < K|A|$

Freiman-Ruzsa Lemma
$|A| > cK^2/\varepsilon$
$\Longrightarrow A \subset \phi(\mathbb{Z}^d)$ with $d \leq [K - 1 + \varepsilon]$.

Weak Freiman-Ruzsa Conjecture (WFRC)
$\Longrightarrow \exists A_1 \subset A$ with $|A_1| > K^{-c}|A|$
$\exists\ \xi_1, \dots, \xi_d \in V$ with $d < c \log K$, s. t.

$$A_1 \subset \mathbb{Z}\xi_1 + \cdots + \mathbb{Z}\xi_d.$$

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"RESULTS"

- assume WFRC, i.e. assume
  "$|AA| < K|A|$
  $\implies \exists G < \mathbb{R}^*$, rk $G < c \log K$, $|A \cap G| > K^{-c}|A|$". Then the following hold
- $|AA| < |A|^{1+\delta} \implies |nA| > |A|^{n(1-\varepsilon)}$
- $|AA| < |A|^{1+\delta} \implies \forall B \subset A$ with $|B| > |A|^{\varepsilon}$, we have $|nB| > |B|^{n(1-\varepsilon)}$.

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- (without assuming WFRC)
$|AA| < |A|^{1+\delta} \implies |nA| > |A|^m$

- (without assuming WFRC)
$A \subset \{$ algebraic numbers of degree $\leq d\}$,
$|AA| < |A|^{1+\delta} \implies \forall B \subset A, |nB| > |A|^{-\varepsilon}|B|^n$

- $A \subset SL_3(\mathbb{C})$, $|AA| < K|A|$
$\implies \exists A_1 \subset A$ with $|A_1| > K^{-c}|A|$
$A_1 \subset \xi N$, where $N < SL_3(\mathbb{C})$ is nilpotent

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additive energy

- additive energy for $S_1, \dots, S_n \subset \mathbb{C}$

$$E(S_1, \dots, S_n) = |\{(x_1, y_1, \dots, x_n, y_n) \in S_1^2 \times \cdots \times S_n^2 : x_1 + \cdots + x_n = y_1 + \cdots + y_n\}|$$

- $|S_1 + \cdots + S_n| \geq \frac{|S_1|^2 \cdots |S_n|^2}{E(S_1, \cdots, S_n)}$

Lemma

$G < \mathbb{C}^*$, rk $G < c \log K$

$A_1 \subset G$ finite.

Then

$$E(\underbrace{A_1, \dots, A_1}_{n}) \leq K^{C(n)} |A_1|^{n-1} + \frac{2n\,!}{n!} |A_1|^n$$

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- bounding the number of solutions in of

$$x_1 + \cdots + x_n - x_{n+1} - \cdots - x_{2n} = 0, \quad x_i \in A_1$$

Theorem Evertse-Schlickewei-Schmidt
$\Gamma < (\mathbb{C}^*)^n$, rank $\Gamma = r$. Then the number of nondegenerate solutions of

$$a_1 x_1 + \cdots + a_n x_n = 1, \ (x_1, \cdots, x_n) \in \Gamma$$

is bounded by

$$e^{(6n)^{3n}(r+1)}$$

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    - $$G < \mathbb{C}^*, \quad \text{rk} G = d < c \log K$$

    let $\Gamma = G \times \cdots \times G$. then $\text{rk}\Gamma \leq nd$
    - the number of solutions is bounded by

$$e^{(6n)^{3n}(nd+1)} < e^{cn(6n)^{3n} \log K} = K^{C(n)}$$

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Proof of Lemma.

• decompose in minimal vanishing subsums

$x_1 + \cdots + x_n - x_{n+1} - \cdots - x_{2n} = 0, \quad x_i \in A_1$

decomposition $\longleftrightarrow \{1, \dots, 2n\} = \bigcup_{\alpha=1}^{\beta} E_\alpha$

$|E_\alpha| \geq 2, \forall \alpha$

$\beta \leq n$

• for $|E_\alpha| \geq 3$, rewrite

$$\sum_{i \in E_\alpha} \pm x_i = 0$$

as (w.m.a. $1 \in E_\alpha$)

$$\sum_{i \in E_\alpha \setminus \{1\}} \pm \frac{x_i}{x_1} = 1$$

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• If $\exists \alpha$ s.t. $|E_\alpha| \geq 3$, $\sharp\{\text{sol 's}\}$ is bounded by

$$|A_1|^{\beta} \prod_{\alpha=1}^{\beta} K^{C(|E_\alpha|)} \leq |A_1|^{n-1} K^{C(n)}.$$

• If $\forall \alpha, |E_\alpha| = 2$, then $\sharp\{\text{sol's}\}$ is bounded by

$$\frac{2n\,!}{n!} \, |A_1|^n.$$

Hence

$$E(\underbrace{A_1, \ldots, A_1}_{n}) \leq K^{C(n)} |A_1|^{n-1} + \frac{2n\,!}{n!} \, |A_1|^n.$$

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WFRC for multiplicative sets

• assume $A \subset \mathbb{R}_+$ finite, $|AA| < K|A|$
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Example 1.

Assume WFRC.

$\forall n \in \mathbb{Z}_+$ and $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $A \subset \mathbb{C}^*$ finite, large and

$$|AA| < |A|^{1+\delta},$$

then

$$|nA| > |A|^{n(1-\varepsilon)}.$$

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Example 1.

Assume WFRC.
$\forall n \in \mathbb{Z}_+$ and $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $A \subset \mathbb{C}^*$ finite, large and

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- additive energy for $S_1, \dots, S_n \subset \mathbb{C}$
  $E(S_1, \dots, S_n)$
$$= |\{(x_1, y_1, \dots, x_n, y_n) \in S_1^2 \times \cdots \times S_n^2 : x_1 + \cdots + x_n = y_1 + \cdots + y_n\}|$$

- $|S_1 + \cdots + S_n| \geq \frac{|S_1|^2 \cdots |S_n|^2}{E(S_1, \cdots, S_n)}$

Lemma

$G < \mathbb{C}^*$, rk $G < c \log K$
$A_1 \subset G$ finite.

Then
$$E(\underbrace{A_1, \dots, A_1}_{n}) \leq K^{C(n)} |A_1|^{n-1} + (2n)! \, |A_1|^n$$

- $$|nA_1| \geq \frac{|A_1|^{2n}}{E(A_1, \dots, A_1)}$$

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Proof. Take $K = |A|^\delta$.

$$\begin{aligned} |nA| &\geq |nA_1| \\ &\geq \frac{|A_1|^{2n}}{K^{C(n)}|A_1|^{n-1} + \frac{2n\,!}{n!}\,|A_1|^n} \\ &> \min\left(\frac{n!}{2n\,!}\,|A_1|^n,\ K^{-C(n)}|A_1|^{n+1}\right) \\ &> \min\left(\frac{n!}{2n\,!}\,K^{-c_1 n}|A|^n,\ K^{-C(n)}|A|^{n+1}\right) \end{aligned}$$

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$\forall n \in \mathbb{Z}_+$ and $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.
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Proof of Example 2

$z_1, \dots, z_s$ be maximal in A s.t.
$z_i A_1 \cap z_j A_1 = \emptyset, \forall i \neq j$

$$s \leq \frac{|AA_1|}{|A_1|} \leq K^c \frac{|AA|}{|A|} < K^{c+1}$$

$$A \subset \bigcup_{i=1}^{s} z_i A_1 A_1^{-1}$$

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$$\left|B_1 := B \cap z_i A_1 A_1^{-1}\right| \ge \frac{|B|}{s} \ge \frac{|B|}{K^{c+1}}$$

$A_1 A_1^{-1} \subset G \Longrightarrow z_i^{-1} B_1 \subset A_1 A_1^{-1} \subset G$
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Theorem

Assume WFRC.
$\forall \varepsilon > 0$, $\exists \delta > 0$ *s.t. if*

$$A \subset SL_3(\mathbb{C}), \quad |A| < \infty$$

then either

(i) $|A \cap \xi N| > |A|^{1-\varepsilon}$, *where* $N < SL_3(\mathbb{C})$ *is nilpotent*

or

(ii) $|A^2| > |A|^{1+\delta}$.

Proof.

- $A \subset GL_3(\mathbb{C}), |A| < \infty$

$$\Rightarrow \text{(i)}, \text{or} \ \ \exists \tilde{g} \in A^{[3]}, \ \ \text{s.t.} \ \ |Tr\,(\tilde{g}A)| > |A|^{\delta}$$

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outline

- By Balog-Szemeredi-Gowers (since $|AA| < K|A|$)

$\exists A_1 \subset A$ with $|A_1| > K^{-c}|A|$, and

$\exists X \subset SL_3(\mathbb{C})$ s.t.

$$|X| < K^c, \quad A_1 A_1 \subset XA_1 \cap A_1 X$$

— replace A by A_1, we have

- $$\left|A^{[s]}\right| < K^{cs}|A|, \ \forall s \in \mathbb{Z}^+$$

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(★) for fixed $h \in A$, $|D(hD)^{\ell-1}| \gtrsim |D|^\ell$
(★) $\Longrightarrow$ Theorem:

$$
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K^{3\ell c}|A| \geq |A|^{3\ell} \\
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\end{aligned}
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$$\ell \gg 0 \quad \Longrightarrow \quad \rightarrow\leftarrow$$

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set of diagonal matrices with large product

($\bigstar$) for fixed $h \in A$, $|D(hD)^{\ell-1}| \gtrsim |D|^{\ell}$:

- $g = g^{(1)} h g^{(2)} h \cdots h g^{(\ell)}, \qquad g^{(1)}, \cdots, g^{(\ell)} \in D$
- $g^{(s)}_{\lambda_1}, g^{(s)}_{\lambda_2}, g^{(s)}_{\lambda_3}$:diagonal elements of $g^{(s)} \in D$
- $\sum_{i,j} g_{ij}$

$$= \sum_{i_1, \ldots, i_\ell} h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_{\ell-1} i_\ell} g^{(1)}_{\lambda_{i_1}} g^{(2)}_{\lambda_{i_2}} \cdots g^{(\ell)}_{\lambda_\ell}$$

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Theorem

Evertse-Schlickewei-Schmidt
$\Gamma < (\mathbb{C}^*)^n$, *rank* $\Gamma = r$. *Then the number of nondegenerate solutions of*

$$a_1 x_1 + \cdots + a_n x_n = 1, \;\; (x_1, \cdots, x_n) \in \Gamma$$

is bounded by

$$e^{(r+1)(6n)^{3n}}$$

Goal 1.

Find a new D s.t. $|DD| < K^c|D|$:

- $D \subset A^{-1}A \subset A^{[2]}$ be the diagonal set with $|D| > |A|^\rho$

$$D_s := \mathcal{D} \cap A^{[s]}$$

$$D_s \supset D_2 \supset D$$

- take $B \subset A^{[2]}$ min s.t. $A^{[2]} \subset B\mathcal{D}$.
    - $g\mathcal{D} \cap g'\mathcal{D} = \emptyset, \qquad \forall g \neq g' \in B$
    - $A^{[2]} \subset B\, D_4$
    - $|A| \leq |A^{[2]}| \leq |B|\,|D_4|$

$$|D_8|\,|B| = |D_8 B| \leq |A^{[10]}| < K^{10c}|A|$$

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Lemma

$A \subset A_1 \times R$

$\pi : A \twoheadrightarrow A_1$

Assume $|2A| = |A + A| < K|A|$

$\implies \exists C \subset A$ with $|C| > \frac{1}{2 \log K} |A|$, s.t.

$\forall x \in C,\ |\pi^{-1}(\pi(x))| \sim h$

Proof

$m := \max\{|\pi^{-1}(x)| : x \in A_1\}$

$$|2A| \geq |A_1|\ m$$

- $|A| > |A_1|\ \frac{m}{K}$
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Find $F \subset D$ large s.t.

$$\text{rank}\langle f_{i,j} : (f_{i,j}) \in F \rangle \text{ small}$$

- $$\begin{array}{ccccc} \pi : D & \longrightarrow & D_1, & \qquad & (g_{i,j}) \mapsto g_{1,1} \\ \cup & & \cup & & \\ E & \longrightarrow & E_1 & & \end{array}$$

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WFRC $\Longrightarrow \exists F' \subset E_1$ s.t. $|F'| > K^{-c}|E_1|$ and $F' \subset \Gamma_1 < \mathbb{C}^*$, $\text{rank}(\Gamma_1) < c \log K$

- $$\begin{array}{ccccc} \pi_1 : & D & \to & D_1, & \qquad (g_{i,j}) \mapsto g_{1,1} \\ & \cup & & \cup & \\ & E & \to & E_1 & \\ & \cup & & \cup & \\ & F & \to & F' & \end{array}$$

$$|F| > \frac{1}{K^C}|E| > \frac{1}{K^C \log K}|D|$$

$$\pi_1(F) \subset \Gamma_1, \quad \mathrm{rank}(\Gamma_1) < c \log K$$

- $$\pi_2 : F \to F_2, \qquad (g_{i,j}) \mapsto g_{2,2}$$

$$\vdots$$

- ► $\pi_i(F) \subset \Gamma_i, \qquad \mathrm{rank}(\Gamma_i) < c \log K, \quad i = 1, 2, 3$

$$\langle f_{i,j} : (f_{i,j}) \in F \rangle < \langle \Gamma_i : i = 1, 2, 3 \rangle$$

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Konyagin

- $V = \mathbb{Z}$-module, $A \subset V$ finite, $|A+A| < K|A|$

$\Longrightarrow \exists A' \subset A$, s.t.

1. $|A'| > e^{-cK^{\frac{1}{4}}(\log K)^{\frac{3}{4}}} |A|$
2. $\dim\langle A'\rangle < K^{-\frac{1}{4}}(\log K)^{\frac{3}{4}}$

$$K^{-\frac{1}{4}}(\log K)^{\frac{3}{4}} < c \log |A| \Longleftrightarrow K < c(\log |A|)^{4-\epsilon} \qquad (\bigstar)$$

$$(\bigstar) \Longrightarrow \exists A' \subset A, \quad |A'| > |A|^{\frac{1}{2}}, \quad \dim\langle A'\rangle < c \log |A|)^{1-\epsilon}$$

- Ch- $\quad K < c \log |A|$
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$$K < |A|^{\delta} \ ?$$

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