ON PROBLEMS OF ERDÖS AND RUDIN

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1. INTRODUCTION

Let

$$\mathrm{S} = \{n^2 \,|\, n \in \mathbb{N}\} \tag{1.1}$$

be the set of squares. A well-known (and still open) conjecture of W. Rudin is that S is a $\wedge_p-$ set for all $p < 4$. This means that for all $p < 4$, there is a constant c_p such that for all finite scalar sequences $(a_n)_{n\in\mathbb{N}}$

$$\left(\int\limits_{\Pi} \Big|\sum a_n\, e^{in^2x}\Big|^p dx\right)^{\frac{1}{p}} \quad \leq \quad c_p\left(\sum |a_n|^2\right)^{\frac{1}{2}}. \tag{1.2}$$

Here Π denotes the usual circle group. In 'Harmonic Analysis language', the problem is thus whether $L^p_S(\Pi) = L^2_S(\Pi)$ if $p < 4$ and S as above. Presently, there is no exponent $p > 2$ known for which $L^p_S(\Pi) = L^2_S(\Pi)$ holds. See [Ru].

Rudin's problem implies an affirmative answer to the following question:

For all $\varepsilon > 0$, does there exist a constant c_ε such that

$$\left(\int\limits_{\Pi} \Big|\sum_{j=1}^{k} e^{in_j^2x}\Big|^4 dx\right)^{\frac{1}{4}} \quad < \quad c_\varepsilon k^{\frac{1}{2}+\varepsilon} \tag{1.3}$$

for any k distinct integers $n_1, \ldots, n_k$?

Our purpose here is to give a combinatorial interpretation of (1.3) in the spirit of "sum and product sets along graphs" as considered in [E-S] by Erdös and Szemeredi.

We recall the setup.

Let $A = \{a_i \in \mathbb{Z} \,|\, a_i < a_j, \text{ if } i < j\}$ be a set of n distinct integers and $G \subset \{(i,j) \,|\, i,j \in \mathbb{Z}, 1 \leq i,j \leq n\}$ a graph.

Denote

$$\mathrm{Sum}_G\, A = \{a_i + a_j |\, (i,j) \in G\}, \tag{1.4}$$

$$\mathrm{Diff}_G\, A = \{a_i - a_j |\, (i,j) \in G\}, \tag{1.5}$$

$$\mathrm{Prod}_G\, A = \{a_i a_j |\, (i,j) \in G\}. \tag{1.6}$$