Physics, Topology, Logic, and Computation: A Rosetta Stone

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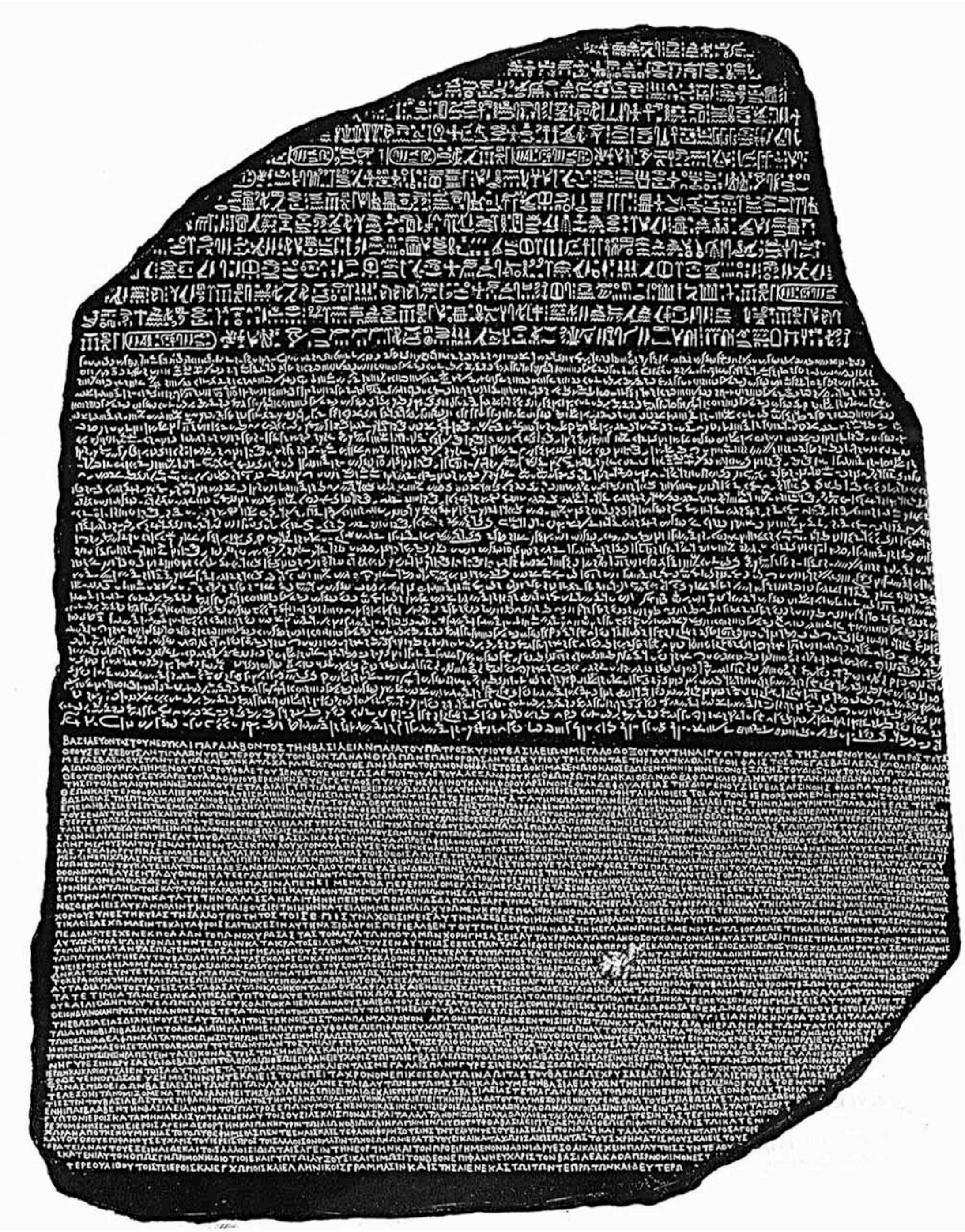
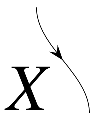


The Rosetta Stone (pocket version)

Category Theory	Physics	Topology	Logic	Computation
object	system	manifold	proposition	data type
morphism	process	cobordism	proof	program

Objects

- **String diagrams have 'strings' or 'wires':**

X 

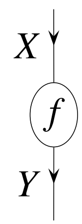
- **Quantum mechanics has Hilbert spaces:** $X \cong \mathbb{C}^n$
- **Topology has manifolds:** X 
- **Linear logic has propositions:**

X = **"I have an item of type** X**."**

- **Computation has datatypes:** `interface X;`
- **SET has sets:** X

Morphisms

- **String diagrams have vertices:**

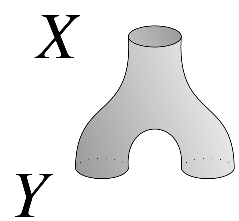


- **Quantum mechanics has linear transformations:**

$$f : X \to Y \quad \cong \quad f : \mathbb{C}^n \to \mathbb{C}^m$$

(An $m \times n$ matrix with complex entries)

Morphisms

- **Topology has cobordisms:** 

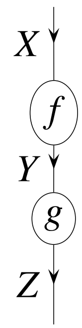
- **Linear logic has constructive proofs:**

$$X \vdash Y$$

- **Computation has programs:** `Y f(X);`
- **SET has functions:** $f : X \to Y$

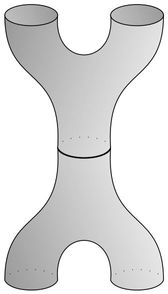
Morphisms compose associatively

- **String diagrams:**



- **Quantum mechanics: matrix multiplication**

- **Topology:**



Morphisms compose associatively

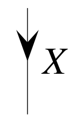
- **Linear logic:** $$\frac{Y \vdash Z \quad X \vdash Y}{X \vdash Z}\ (\circ)$$

- **Computation:**

```
Y f(X x);
Z g(Y y);
...
z = g(f(x));
```

- **SET:** $(g \circ f) : X \to Z$

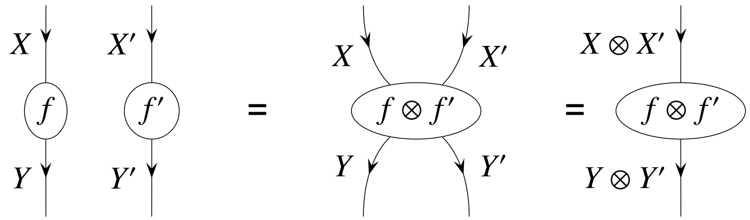
Identity morphisms

- **String diagrams:** 

- **Quantum mechanics: identity matrix** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- **Topology:** 
- **Linear logic:** $\dfrac{}{X \vdash X}$ (i)
- **Computation:** `X id(X x) { return x; }`
- **SET:** $1_X : X \to X$

Monoidal categories

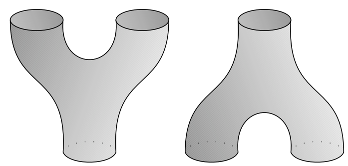
- **String diagrams:**



- **Quantum mechanics: tensor product**

$$\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \otimes \begin{pmatrix} e & f & g \\ h & j & k \end{pmatrix} = \left(\begin{array}{ccc|ccc} ae & af & ag & be & bf & bg \\ ah & aj & ak & bh & bj & bk \\ \hline ce & cf & cg & de & df & dg \\ ch & cj & ck & dh & dj & dk \end{array}\right)$$

Monoidal categories

- **Topology:** 

- **Linear logic: AND** $$\frac{X \vdash Y \quad X' \vdash Y'}{X \otimes X' \vdash Y \otimes Y'} \ (\otimes)$$

- **Computation: parallel programming**

```
Pair<X, X'> pair;
```

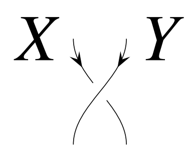
- **SET:** $f \times f' : X \times X' \to Y \times Y'$

Monoidal unit

- **String diagram:**
- **Quantum mechanics:** $I = \mathbb{C}$**, the phase of a photon**
- **Topology:**
- **Linear logic:** I**, trivial proposition**
- **Computation:** `I = void` **or** `I = unit` **type**
- **SET: one-element set** I

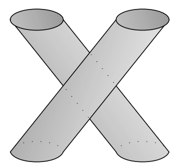
Braided monoidal categories

- **String diagrams:**



- **Quantum mechanics: swap the particles. Bosons commute, fermions anticommute; quantized magnetic flux tubes in thin films, or "anyons", can have arbitrary phase multiplier.**

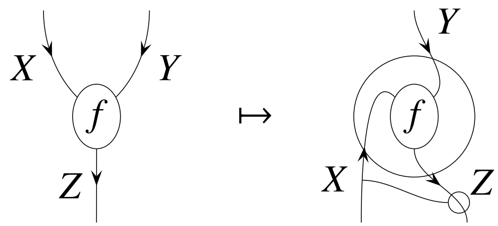
Braided monoidal categories

- **Topology:** 

- **Linear logic:** $\dfrac{W \vdash X \otimes Y}{W \vdash Y \otimes X}$ (b)
- **Computation:** `pair.swap();`
- **SET:** $b(\langle x, y \rangle) = \langle y, x \rangle$

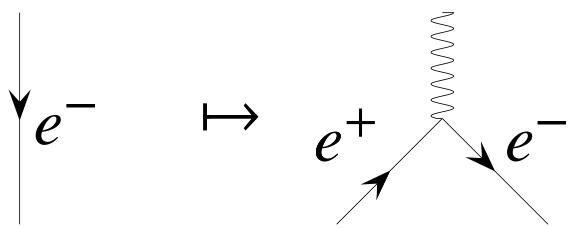
Braided monoidal closed categories

- **String diagrams:**

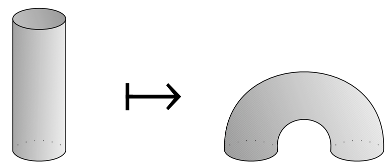


- **Quantum mechanics: antiparticles**

$$1_X : X \to X \quad \cong \quad \mathbf{pair} : I \to X^* \otimes X$$



Braided monoidal closed categories

- **Topology:** 

- **Linear logic: IMPLIES** $\dfrac{X \otimes Y \vdash Z}{Y \vdash X \multimap Z}$ (c)

- **Computation: Currying**

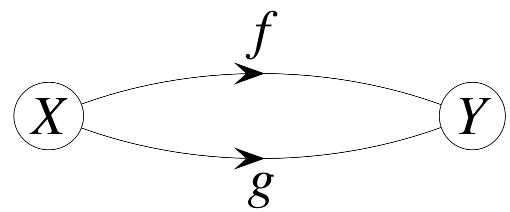
```
z = f(x, y);
```

or

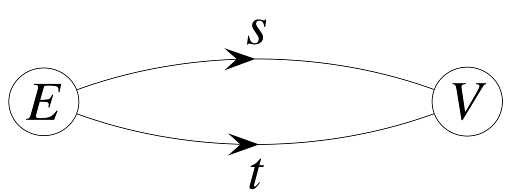
```
z = f(y)(x);
```

- **SET:** $f : X \times Y \to Z \quad \cong \quad f : Y \to Z^X$

Model Theory



Model Theory



Th(Graph)

Model Theory

Th(Graph)	SET
object V	**set of vertices**
object E	**set of edges**
morphism $s : E \to V$	**function that picks out the source of each edge**
morphism $t : E \to V$	**function that picks out the target of each edge**

Model Theory (in Java)

```
interface ThGraph {
   // Internal interfaces
   interface V;
   interface E;

   // Methods
   V s(E);
   V t(E);
}
```

A functor is a structure-preserving map. In Java terms, a functor picks out a class that implements the interface.

Model Theory (Classical)

Syntax [Programming language]	Semantics [CE-SET]
data type	computably enumerable set of values
method	partially recursive function

Model Theory (Quantum)

Syntax [Programming language]	Semantics [QM]
data type	Hilbert space of values
method	linear transformation

Model Theory (Quantum)

Syntax [Topology]	Semantics [QM]
manifold	Hilbert space of states
cobordism	linear transformation

Topological Quantum Field Theory