Examples of Obstructions to Apex Graphs, Edge-Apex Graphs, and Contraction-Apex Graphs

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Some Vocabulary

- The **order** of a graph is the count of its vertices ($|V(G)|$).
- The **size** of a graph is the count of its edges ($|E(G)|$).
- If two vertices are connected by an edge, we say they are **adjacent** (or that they are **neighbors**).
- An edge that connects two vertices we will say is **incident** to each of those vertices.
- The **degree** of a vertex is the count of edges that are incident to it. This is also called the **valency** of the vertex.
- Two graphs $G_1$ and $G_2$ are **isomorphic** if there exists a bijection $f : V(G_1) \leftrightarrow V(G_2)$ that preserves vertex adjacency.
The notion of a graph minor is a generalization of the idea we have of a subgraph.

A graph $H$ is a minor of a graph $G$ if performing some sequence of vertex deletions, edge deletions, or edge contractions on $G$ results in a graph isomorphic to $H$.

A simple minor of a graph is the result of any one of these actions.
Graph Minors
Robertson and Seymour’s Graph Minor Theorem

Graph Minor Theorem
The set of finite, undirected graphs form a well-quasi-ordering under the graph minor relationship. That is, given a countable sequence of graphs $G_i$ ($i = 1, 2, \ldots$) there must exists some $j > i \geq 1$ such that $G_i$ is isomorphic to a minor of $G_j$.

Corollary: There are Finitely Many Minor-Minimal Non-$P$ Graphs
Take some graph property $P$ such that any graph either does or does not have $P$, and such that $P$ is minor-closed. Consider the set of all graphs $\{G_i\}$ such that each $G_i$ does not have $P$, but such that every (proper) minor of each $G_i$ has $P$. This set of graphs $\{G_i\}$ must be finite.
Minor-Minimal Non-Planar

We can imagine that there are some graphs that are not planar, but that every minor of these graphs are planar. According to Robertson and Seymour’s Theorem, there must be a finite number of these graphs.

Figure: The Two Minor-Minimal Non-Planar Graphs
A graph is **intrinsically linked** if, independent of its embedding in $\mathbb{R}^3$, it must have a pair of linked cycles.

There are exactly seven minor-minimal intrinsically linked graphs.

This set of graphs is called the **Petersen Family**.

*Figure: Source: Wikimedia Commons*
Minor-Minimal Intrinsically Knotted

- A graph is **intrinsically knotted** if, independent of its embedding in $\mathbb{R}^3$, it must contain a knotted cycle.
- Classifying the entire set of minor-minimal intrinsically knotted graphs is still an open problem.
- We only know that there are at least 263.
A graph is **apex** if there is a vertex in the graph that we may remove to make the graph planar.

We can think of these graphs as “one vertex away” from being planar.
Minor-Minimal Non-Apex Graphs

This means that a **minor-minimal non-apex** (MMNA) graph is a graph that is *not* apex, but such that every minor of the graph *is* apex.

**Figure**: Three Examples of Minor-Minimal Non-Apex Graphs
Examples of Minor-Minor Non-Apex Graphs

All intrinsically linked graphs are non-apex. It also turns out that all minor-minimal intrinsically linked graphs are minor-minimal non-apex.
Examples of Minor-Minor Non-Apex Graphs

There are exactly three disconnected MMNA graphs. They are each a disjoint union of a pair of graphs from \( \{K_5, K_{3,3}\} \).
Examples of Minor-Minor Non-Apex Graphs

There are no MMNA graphs of connectivity 1.
Examples of MMNA Graphs with Connectivity 2

Since apexness is based on graph planarity, many MMNA graphs appear to have $K_5$ and $K_{3,3}$ as “building blocks”.

![Diagram of MMNA graphs with various connectivity levels.](image-url)
Examples of MMNA Graphs with Connectivity 2

Since apexness is based on graph planarity, many MMNA graphs appear to have $K_5$ and $K_{3,3}$ as “building blocks”.

![MMNA Graph Examples](image-url)
Examples of MMNA Graphs with Higher Connectivity
Minor-Minimal Non-Apex Graphs

Number of MMNA Graphs by Size and Order (61 so far)
A Few Initial Restrictions on MMNA Graphs

**Lemma**
The minimum vertex degree in any MMNA graph is 3.

**Theorem**
Given a graph $G$ with order $v$ and minimum vertex degree $\delta$, the size of $G$ is at least $\lceil \frac{\delta v}{2} \rceil$.

**Theorem**
Given a graph $G$ with order $v$, the maximum size of $G$ is $\frac{v(v-1)}{2}$. 
Minor-Minimal Non-Apex Graphs

Number of MMNA Graphs by Size and Order (61 so far)
Minor-Minimal Non-Apex Graphs

Number of MMNA Graphs by Size and Order (61 so far)
Brute-Force Search on 8, 9, and 10 Vertices

For doing this project, it is convenient to establish some functions to determine if a given graph $G$ is MMNA.

```mathematica
MMNAGraphQ[G_Graph] := Module[{},
    Return[
        (!ApexGraphQ[G]) && !(MemberQ[APEXGRAPHQ /@ SIMPLEMINORS[G], False])
    ];
]

APEXGRAPHQ[G_Graph] := Module[{},
    Return[
        MemberQ[
            PLANARGRAPHQ /@ UNION[{G}, Table[VERTEXDELETE[G, i], {i, VERTEXLIST[G]}]], True]
    ];
]
```
Brute-Force Search on 8, 9, and 10 Vertices

Number of MMNA Graphs by Size and Order (61 so far)

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Brute-Force Search on 8, 9, and 10 Vertices

Number of MMNA Graphs by Size and Order (72 so far)
Brute-Force Search on 17–21 Edges

Number of MMNA Graphs by Size and Order (72 so far)
Brute-Force Search on 17–21 Edges

Number of MMNA Graphs by Size and Order (109 so far)
Triangle-Wye and Wye-Triangle Transforms

- Let $T$ be a 3-cycle (triangle) in a graph. We can perform a **triangle-wye** move on $T$ by deleting the edges of $T$, and adding a new vertex to our graph adjacent to the vertices of $T$.

- Let $v$ be a degree 3 vertex in a graph. We can perform a **wye-triangle** move on $v$ by adding edges to connect the vertices adjacent to $v$, and then deleting $v$. 

![Image of triangle-wye and wye-triangle transforms]
Triangle-Wye Often Preserves NA

**Theorem**

Let $G$ be a non-apex graph with triangle $T$ and let $G'$ be the result of performing triangle-wye on $T$ where the vertex added to $G'$ is $v$. Graph $G'$ is non-apex if and only if $G' - v$ is non-planar.

- So unless the vertex that gets added when we perform triangle-wye causes a graph to become apex, then the graph will remain non-apex.
- Since performing triangle-wye on a graph preserves its size, no graph in a triangle-wye family can be a minor of another, increasing the likelihood that some members of the triangle-wye family of an MMNA graph are MMNA.
We can write some functions to automate the process of constructing the triangle-wye family of a graph.

```mathematica
TriangleWyeFamily[G_Graph] := Module[{},
    Return[
        List[#, TriangleWyeFamily[#]] & @ TriangleWyeCousins[G]
    ];
];

TriangleWyeCousins[G_Graph] := Module[{},
    Return[
        DeleteGraphDuplicates[Table[TriangleWye[G, t], {t, TriangleList[G]}]]
    ];
];
```
Building Triangle-Wye Families

Number of MMNA Graphs by Size and Order (109 so far)
Building Triangle-Wye Families

Number of MMNA Graphs by Size and Order (157 so far)

Number of Vertices (Order)

Number of Edges (Size)
The Current MMNA Total

- We continued to construct MMNA graphs by “gluing together” \( K_5 \) and \( K_{3,3} \) subgraphs.

- We did a more liberal computer search on graphs with a minimum vertex degree of 4, 5, and 6.

- We performed a “smart search” for MMNA graphs by looking at the simple minors of extensions of MMNA graphs we already had.

- We (at least partially) created the wye-triangle-wye family for each of the new MMNA graphs.
The Current MMNA Total

Number of MMNA Graphs by Size and Order (157 so far)
The Current MMNA Total

Number of MMNA Graphs by Size and Order (263 so far)
Edge-Apex and Contraction-Apex

- A graph is **edge-apex** if there is some edge in the graph that we may delete to make the graph planar.
- A graph is **contraction-apex** if there is some edge in the graph that we may contract to make the graph planar.

**Figure:** An Edge-Apex Graph
MMNE and MMNC are not as nice as MMNA

- The minimum allowed vertex degree of MMNE graphs and MMNC graphs is 2.
- Neither of the properties edge-apex or contraction-apex are closed under taking minors (but just barely).

Figure: A Contraction-Apex Graph with a Non-Contraction-Apex Minor
Properties of MMNE and MMNC Graphs

- Given a degree 2 vertex in an MMNE graph, the neighbors of that vertex must themselves be neighbors.
- The three disconnected graphs that are the disjoint union of a pair of graphs from \( \{K_5, K_{3,3}\} \) are exactly the disconnected MMNE and MMNC graphs.
- There are three MMNE and MMNC graphs of connectivity 1 that are each the result of gluing together graphs in \( \{K_5, K_{3,3}\} \) on a single vertex.
Examples of MMNE Graphs
The Current MMNE Total

Number of MMNE Graphs by Size and Order (55 so far)
The Current MMNC Total

Number of MMNC Graphs by Size and Order (82 so far)