Algebra – Isomorphisms & Homomorphisms

Advanced Mathematics Program, Summer 2019

1. Prove that a group is isomorphic to itself.

2. Prove that $\mathbf{Z}_n \cong \mathbf{Z}_m$ if and only if $m = n$.

3. Recall the Klein 4-group K given by the table

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

   Prove that K is **not** isomorphic to Z_4.

4. Prove that $(\mathbf{R}, +) \cong (\mathbf{R}_{>0}, \cdot)$.

5. Recall that the kernel of a homomorphism $f : G \to H$ is defined as

$$\ker(f) = \{g \in G : f(g) = e_H\}$$

   Prove that $\ker(f)$ is a subgroup of G.

6. Prove that a homomorphism is injective if and only if its kernel is trivial.

7. Consider the map $f : \mathbf{Z} \to \mathbf{Z}_5$ where $f(m)$ is defined to be the remainder when m is divided by 5. More precisely, we can write any integer as $m = 5q + r$ where $q \in \mathbf{Z}$ and $r \in \mathbf{Z}_5$; we define $f(m) = r$.

   (a) Prove that this map is a surjective homomorphism.

   (b) Prove that $\ker(f)$ is the set of integers divisible by 5.

8. Let $f : G \to H$ be a homomorphism and $h \in \ker(f)$. Prove that $ghg^{-1} \in \ker(f)$ for any $g \in G$.