Algebra – Cosets & Normal Sugroups

Advanced Mathematics Program, Summer 2019

1. Let's get some hands-on practice now.

   (a) For the group $\mathbf{Z}_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$, look at the subgroup $\{0,4,8\} = H < \mathbf{Z}_{12}$. What are the left cosets of H in $\mathbf{Z}_{12}$?

   (b) Remember the quaternion group $Q = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$? It has a subgroup $H = \{1, -1\}$. What are the left cosets of this subgroup?

   (c) Consider the set of integers that are divisible by three: $3\mathbf{Z} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$. This is a subgroup of $\mathbf{Z}$. Can you describe the left cosets of $3\mathbf{Z}$?

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2. For the group $\mathbf{Z}_6 = \{0,1,2,3,4,5\}$, look at the subgroup $\{0,3\} = H < \mathbf{Z}_6$. What are the *right* cosets of H in $\mathbf{Z}_6$? How do they compare to the left cosets?

3. For the group $S_3 \cong \{e, a, b, ab, ba, aba\}$, look at the subgroup $\{e, a\} = H < S_3$. What are the *right* cosets of H in S_3? How do they compare to the left cosets?

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4. Recall from last time that for a homomorphism $\varphi\colon G \to H$, the kernel of φ, defined as $\ker(\varphi) = \{x \in G : \varphi(x) = e_H\}$ was a subgroup of G. Prove that, even better, the kernel is a *normal* subgroup of G.

5. (Lagrange's Theorem +) Let G be a *finite* group, and let H be a subgroup of G.

   (a) Why must the order of H divide the order of G?

   (b) Prove that every group of prime order is cyclic.

   (c) How many groups are there of order 37, up to isomorphism?

6. Realize that if G is an abelian group, then *every* subgroups $H < G$ will be normal.

7. Suppose $H < G$ and $[G : H] = 2$. Prove that H is a normal subgroup of G.

8. Suppose that G is a finite group, and that you have a chain of nested subgroups $H < G$ and $K < H$. Prove that $[G : K] = [G : H][H : K]$.

9. Show that if a group has at least two elements and has no proper subgroups must be finite and of prime order.