Quiz 3 for Section 3

History of Mathematics

UCR Math-153-03, Spring 2019

1. (a) Let $n > 1$ be an integer. Explain why $k^2 \equiv (n-k)^2 \pmod{n}$ for all $k \in \mathbf{Z}$.

   (b) Find all integers $a \in \{0, 1, \ldots, 16\}$ such that $a \equiv b^2 \pmod{17}$ for some integer b.

   We notice that $(n-k)^2 = n^2 - 2nk + k^2$. The first two summands of this are divisible by n, so $n^2 - 2nk + k^2 \equiv k^2 \pmod{n}$.

   To find all integers $a \in \{0, 1, \ldots, 16\}$ such that a is a square modulo 17, we must simply calculate the squares of $\{0, 1, \cdots, 8\}$ modulo 17 We only need to check up to 8 by part (a). These squares are

x	$x^2 \pmod{17}$
0	0
1	1
2	4
3	9
4	16
5	8
6	2
7	15
8	13

   so the numbers a such that $a \equiv b^2 \pmod{17}$ for some integer b are $a \in \{0, 1, 2, 4, 8, 9, 13, 15, 16\}$.

There is another question on the back.

2. The Cubic Formula shows that one root of the polynomial $x^3 - 3x + 1 = 0$ has the form

$$\sqrt[3]{\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)}$$

where $i^2 = -1$. Express this as a real number; your answer should have the form $k\cos\theta$ for some explicit value of θ and $k \in \mathbf{R}$. (HINT: $e^{i\alpha} = \cos\alpha + i\sin\alpha$)

Utilizing the hint, and the fact that sine is an odd function while cosine is an even function,

$$\begin{aligned}
& \sqrt[3]{\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)} \\
= \quad & \sqrt[3]{\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)} \\
= \quad & \sqrt[3]{e^{i\frac{2\pi}{3}}} + \sqrt[3]{e^{-i\frac{2\pi}{3}}} \\
= \quad & e^{i\frac{2\pi}{9}} + e^{-i\frac{2\pi}{9}} \\
= \quad & \cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right) + \cos\left(-\frac{2\pi}{9}\right) + i\sin\left(-\frac{2\pi}{9}\right) \\
= \quad & \cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right) + \cos\left(\frac{2\pi}{9}\right) - i\sin\left(\frac{2\pi}{9}\right) \\
= \quad & 2\cos\left(\frac{2\pi}{9}\right).
\end{aligned}$$

Quiz 3 for Section 2

History of Mathematics

UCR Math-153-02, Spring 2019

1. (a) Let $n > 1$ be an integer. Explain why $k^2 \equiv (n-k)^2 \pmod{n}$ for all $k \in \mathbf{Z}$.

   (b) Find all integers $a \in \{0, 1, \ldots, 18\}$ such that $a \equiv b^2 \pmod{19}$ for some integer b.

   We notice that $(n-k)^2 = n^2 - 2nk + k^2$. The first two summands of this are divisible by n, so $n^2 - 2nk + k^2 \equiv k^2 \pmod{n}$.

   To find all integers $a \in \{0, 1, \ldots, 18\}$ such that a is a square modulo 19, we must simply calculate the squares of $\{0, 1, \cdots, 9\}$ modulo 19 We only need to check up to 9 by part (a). These squares are

x	$x^2 \pmod{19}$
0	0
1	1
2	4
3	9
4	16
5	6
6	17
7	11
8	7
9	14

   so the numbers a such that $a \equiv b^2 \pmod{19}$ for some integer b are $a \in \{0, 1, 4, 6, 7, 9, 11, 14, 16, 17\}$.

There is another question on the back.

2. The Cubic Formula shows that one root of the polynomial $x^3 - 3x + 1 = 0$ has the form

$$\sqrt[3]{\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(\frac{2\pi}{3}\right) - \mathrm{i}\sin\left(\frac{2\pi}{3}\right)}$$

where $\mathrm{i}^2 = -1$. Express this as a real number; your answer should have the form $k\cos\theta$ for some explicit value of θ and $k \in \mathbf{R}$. (HINT: $\mathrm{e}^{\mathrm{i}\alpha} = \cos\alpha + \mathrm{i}\sin\alpha$)

Utilizing the hint, and the fact that sine is an odd function while cosine is an even function,

$$\begin{aligned}
& \sqrt[3]{\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(\frac{2\pi}{3}\right) - \mathrm{i}\sin\left(\frac{2\pi}{3}\right)} \\
= \quad & \sqrt[3]{\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right)} \\
= \quad & \sqrt[3]{\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}} + \sqrt[3]{\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}} \\
= \quad & \mathrm{e}^{\mathrm{i}\frac{2\pi}{9}} + \mathrm{e}^{-\mathrm{i}\frac{2\pi}{9}} \\
= \quad & \cos\left(\frac{2\pi}{9}\right) + \mathrm{i}\sin\left(\frac{2\pi}{9}\right) + \cos\left(-\frac{2\pi}{9}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{9}\right) \\
= \quad & \cos\left(\frac{2\pi}{9}\right) + \mathrm{i}\sin\left(\frac{2\pi}{9}\right) + \cos\left(\frac{2\pi}{9}\right) - \mathrm{i}\sin\left(\frac{2\pi}{9}\right) \\
= \quad & 2\cos\left(\frac{2\pi}{9}\right).
\end{aligned}$$