ODEs - Week 1 - Monday (1)

Logistics:

- Mike Pierce - mpierce@math.ucr.edu
- Office Hours: 8am-Noon @ Orbach Library 330
- Homework: due each week on Tuesday in discussion (or whenever Eddie says).
- Final - 15th of September (Saturday) 1-3pm in here. Gonna be kinda tough and long, but don't worry about it. So long as I look at your final and am confident you'll be fine in the classes for which this is a prereq, you'll pass. And if you can't make the final for any (good!) reason let me know before the final time!!
- $G = E + \frac{1}{2}H(100-E)$ Explain this with examples. 40% on final (subject to change)

  Good question: ask me how you're doing in the course.
- Don't be dishonest & talk to me about accessbility.
- Textbook(s)
  - Paul's Online Lecture notes for Differential Equations Google
  - Schaum's Outlines - tinyurl.com/schaumode
  - WeBWorK - Link in syllabus; good for practice.
  - Wolfram Alpha - good to check! answers
  - Arnol'd - ODEs - tinyurl.com/arnoldode

Survey: tak(en/ing) 10A? Majors? who read the syllabus?!

ODEs - Week 1 - Monday (2)

Before Solutions

- What's an ODE vs. a PDE?

$$ay'' + by' + c = g(t)$$

$$\cos(xy)\,y'' - \sin(y) = 2$$

$$\frac{\partial y}{\partial t}x + e^{y}\frac{\partial^2 y}{(\partial t)^2} = y$$

$$\frac{\partial^2 p}{\partial t\,\partial \mu} = \frac{\partial p}{\partial t} - p\left(\frac{\partial p}{\partial \mu}\right)^2$$

- The order of a DE is the highest derivative of the function (y in this class) that appears in the DE

"What are the orders of the above DEs?"

- A linear DE is our favorite kind!

$$a_n(t)\,y^{(n)} + a_{n-1}(t)\,y^{(n-1)} + \cdots + a_1(t)\,y' + a_0(t)\,y = g(t)$$

- After talking about solutions, talk about vector fields. See next page.

ODEs - Week 1 - ~~Monday~~ Tuesday (1)

$F = \frac{\partial^2 p}{\partial t^2} m$

ODE vs PDE

$y^{(n)} = y''''' \cdots ''$

"Model the World!"

$y' = \dot{y} = \frac{\partial y}{\partial x}$ or t

EXAMPLES

- What's a differential equation?

Write: it's an equation that relates a function to its derivatives

- What's it mean to find a solution to a differential Eq?

It's a Continuous! function y that "satisfies" the DE (on some interval!)

TALK ABOUT ODEs vs PDEs order linear before solutions

A differential equation might have many solutions.

Exercise: Show that

$y = \ln(x)$ is a solution to $xy'' + y' = 0$ on the domain $(0, \infty)$, but not on $(-\infty, \infty)$.

So is $y = 5\ln(x)$. REAL SOLUTIONS ONLY!

- The general solution of a differential equation is the function y ~~that~~ with some constants floating around $(c_1\ c_2, \ldots)$ that captures ~~the most~~ every(!) solution of a differential equation.

Get wild with constants

The general solution of $xy'' + y' = 0$ is $y = c_1 \ln(x) + c_2$ on $(0, \infty)$

- A particular solution to a differential equation is a particular function y that satisfies the DE and also satisfies some initial conditions that fix what those constants must be.

"What is the (particular) solution to $xy'' + y' = 0$ if we know that $y(1) = 3$ and $y(e) = 5$?"

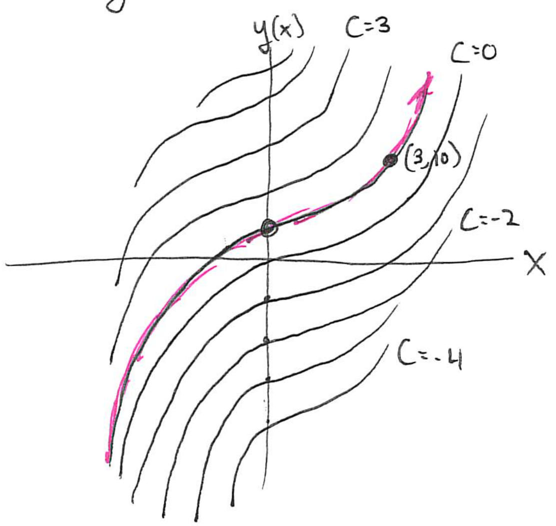
This is an initial value problem OR boundary value problem.

$y' = x^2$

$\int y' \, dx = \int x^2 \, dx$

$y = \frac{1}{3}x^3 + C$ on $\mathbb{R}$

general solution



family of solution curves of $y' = x^2$.

"Integral Curves"

"Suppose $y(\overset{0}{\cancel{3}}) = \overset{1}{1\cancel{0}}$. What is y?"

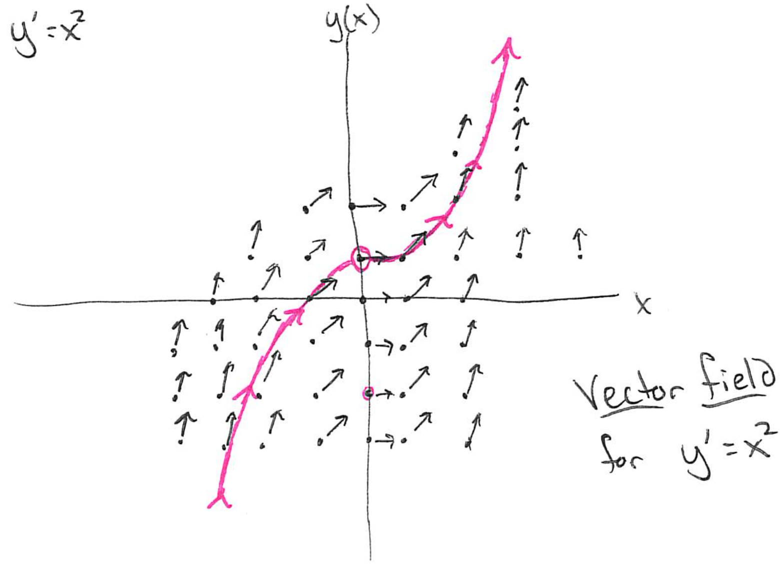
$C = 1$

$y = \frac{1}{3}x^3 + 1$

particular solution

Another way to think about it that generalizes to this class (and to situations where you can't find a solution)

$y' = x^2$



Vector field for $y' = x^2$

Remember y' is "slope"/rate-of-change

Vector fields like wind on earth or ocean currents.

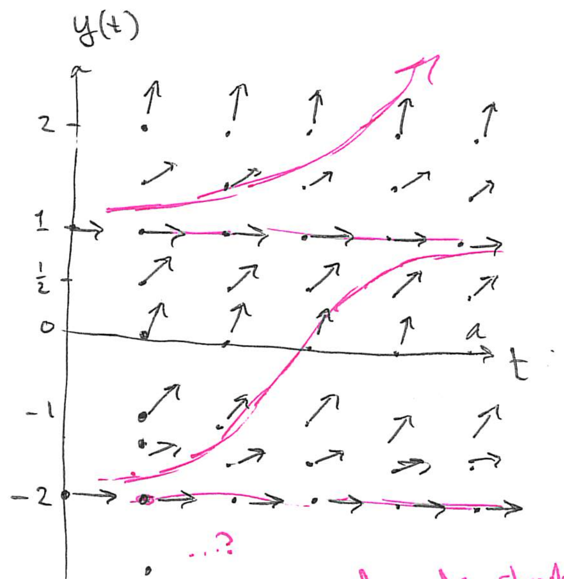
Drop a leaf at $(0,1)$ ~~(3,10)~~

Vector Fields

Tuesday

Solving a DE, IVP, you can think of as finding a vector field for the DE and then dropping a leaf at your initial condition.

$y' = (y+2)(1-y)^2$ on $(0,\infty)$



Solution curve depends strongly on IV...

Similar to example 1 in Paul's Notes

Ex 2)) $y' = y - x$

ODEs - Week 1 - Wednesday (1)

Separable Differential Equations

A Differential equation is separable if it can be written as

$$N(y)\,y' = M(x)$$

I.e. you can "separate the xs from the ys."

We're talking about these first because you'll look at a DE and be like "none of the other techniques work, so let's see if it's separable."

goofy *

$$y' = \frac{dy}{dx} \quad \text{so} \quad \int N(y)\,dy = \int M(x)\,dx$$

Note that the left-hand-side may have a bunch of ys in it, so you may not be able to write down an explicitic solution to the DE $y = \ldots$, but instead only an implicit solution $n(y) = m(x) + C$. Just do what you can to make your solution explicit if possible.

Warning! Be mindful of your domain!

ODEs - Week 1 - Wednesday (2)

"Solve the following differential equation: $y' = 6y^2x$."

$$\frac{1}{y^2}\frac{\partial y}{\partial x} = 6x$$

We divided by y, so $y \neq 0$...

$$\int \frac{1}{y^2}\,\partial y = \int 6x\,\partial x$$

$y=0$ is a separate solution to the ODE that we might have to consider...

$$-\frac{1}{y} + \cancel{c} = 3x^2 + C$$

$$y = \frac{1}{-3x^2 \pm C}$$

$C \neq +3x^2$

"Solve the initial value problem: solve $y' = 6y^2x$ where $y(1) = 1/9$."

$$\frac{1}{9} = \frac{1}{-3(1)^2 + C}$$

$$\frac{1}{9} = \frac{1}{-3 + C}$$ so $C = 12$

$$y = \frac{1}{12 - 3x^2} = \frac{1}{3(4-x^2)} = \frac{1}{3(2-x)(2+x)}$$

But this solution has two holes in its domain @ ± 2, so its domain consists of three ~~conti~~ intervals

$(-\infty, -2)$ $(-2, 2)$ $(2, \infty)$

ODEs – Week 1 – Wednesday (3)

Recall a solution to a differential has to be defined ~~on its~~ on a single connected domain, so we've got to pick one of these intervals. Our initial condition happens at $x = 1$, so we should pick the interval that includes 1. So our solution is $y = \frac{1}{3x^2 - 12}$ on $(-2, 2)$. $\square$

Another?

"Solve $y' = \frac{e^x}{y}$ where $y(0) = 1$."

$$\int y \, dy = \int e^x \, dx$$

$$\frac{1}{2} y^2 = e^x + C$$

$$y \overset{?}{=} \pm \sqrt{2e^x + C}$$

Since $y(0) = 1 > 0$ we need the positive square-root.

$$1 = +\sqrt{2e^0 + C} \Rightarrow C = -1$$

So our solution is $y = \sqrt{2e^x - 1}$. But we need to define the domain of this solution. The domain must contain zero, and $2e^x - 1 > 0$ b/c we need real solutions, so the domain is $(\ln(\frac{1}{2}), \infty)$.

ODEs - Week 1 - Wednesday (4)

Another?

"Solve $y' = xe^{x^2 - \ln(y^2)}$."

Have them try to separate this on their own.

$$y' = xe^{x^2 - \ln(y^2)}$$

$$= xe^{x^2} e^{-\ln(y^2)}$$

$$\Rightarrow \quad e^{\ln(y^2)}\, dy = xe^{x^2}$$

$$\Rightarrow \quad \int y^2\, dy = \int xe^{x^2}\, dx$$

$$\frac{1}{3}y^3 = \frac{1}{2}e^{x^2} + C$$

$$y = \sqrt[3]{\frac{3}{2}e^{x^2} + C} \quad \checkmark$$

How does everybody feel?

ODEs - Week 1 - Thursday (1)

A first-order differential equation $y' = f(x,y)$ is homogenous if $f(x,y) = f(bx, by)$ for every real number b.

Slightly more generally, a differential equation is homogenous of degree n if for every real b

$$b^n f(x,y) = f(bx, by).$$

Then a DE $M(x,y)\,dx + N(x,y)\,dy = 0$ homogenous if M and N are homogenous of degree n for the same n.

Informally (quick-glance) a DE might be homogenous if in each term of the DE the symbols x and y occur with the same "total degree".

$$x^2y' + xy - x^2 = y^2 \qquad y' = \frac{xy+y^2}{x^2} \qquad x + yy' = \frac{x^2}{y} + \frac{3y^{100}}{x^{99}}$$

$$\cos\left(\frac{x}{y}\right)y' + \frac{x}{y} = 2 \qquad \frac{x}{y}y' = 3 \qquad \cancel{\sin(x^2) + y^2y' = xy}$$

Give students a minute to copy these.

ODE - Week 1 - Thursday (2)

If you've got a first-order homogenous DE you can make a substitution ~~and~~ to get a separable DE:

$$v = \frac{y}{x} \quad \text{OR} \quad v = \frac{x}{y}$$

This is just how I remember them, but we're gonna rearrange these.

whichever feels best. Sometimes one is easier than the other. Gotta get rid of all the y's though!

Solve $y' = \dfrac{2y^4 + x^4}{xy^3}$.

Notice that $y' = \dfrac{2y}{x} + \dfrac{x^3}{y^3}$ is totally homogenous.

Let's do the substitution $v = \frac{y}{x}$.

$$y' = 2v + \left(\frac{1}{v}\right)^3$$

Notice (1) we could've done the other sub and gotten $2\frac{1}{v} + v^3$ instead which looks easier, maybe not really but more pertinently notice (2) there's still a y'! We've gotta get rid of that too!

If $v = \frac{y}{x}$ then $y = xv$ and so $y' = xv' + v$.

$$xv' + v = 2v + \frac{1}{v^3}.$$

ODEs - Week 1 - Thursday (3)

$$xv' + v = 2v + \frac{1}{v^3}$$

$$x \frac{dv}{dx} = v + \frac{1}{v^3}$$

$$'' = \frac{v^4+1}{v^3}$$

$u = v^4+1$
$du = 4v^3\,dv$

$$\frac{1}{4}\int \frac{4v^3}{v^4+1}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{4}\ln(u) = \ln(x) + C$$

$$\Rightarrow \frac{1}{4}\ln(v^4+1) = \ln(x) + C$$

$$\Rightarrow \frac{1}{4}\ln\left(\frac{y^4}{x^4}+1\right) = \ln(x) + C$$

$$\Rightarrow \ln\left(\frac{y^4}{x^4}+1\right) = \ln(x^4) + C$$

explain this :)

$$\rightarrow '' = \ln(Cx^4)$$

$$\Rightarrow \frac{y^4}{x^4} + 1 = Cx^4$$

$$\Rightarrow y = \sqrt[4]{Cx^8 + x^4}$$

$$\Rightarrow = x\sqrt[4]{Cx^4+1}$$:) ✓

How did that make everyone feel?

ODEs – Week 1 – Thursday (4)

Let's do another!

"Solve $xyy' + 4x^2 + y^2 = 0$ where $y(\tfrac{1}{2}) = 1$, $x > 0$."

This is homogeneous, so we'll do our favorite substitution

$$\frac{y}{x} \mapsto v \qquad y \mapsto xv \qquad y' \mapsto v + xv'$$

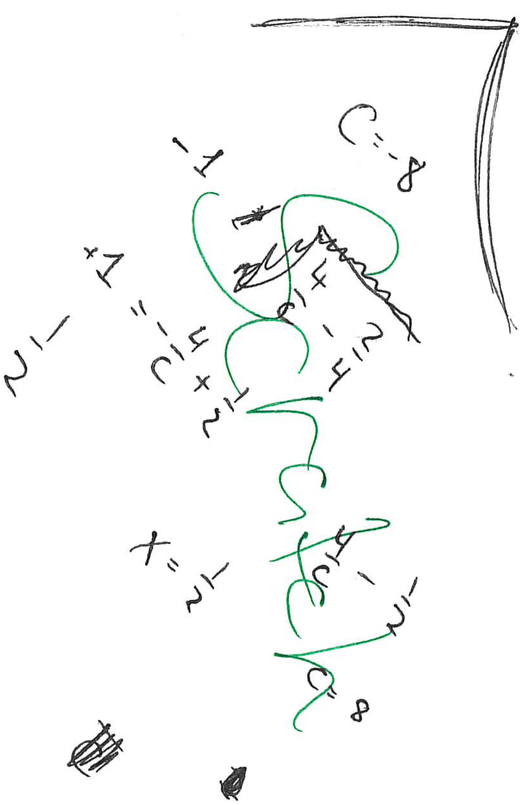
$$xyy' + 4x^2 + y^2 = 0 \implies y' + 4\frac{x}{y} + \frac{y}{x} = 0$$

$y \neq 0$?
$x \neq 0$?
$y = 0$ is not a solution anyways

$$\implies v + xv' + 4\frac{1}{v} + v = 0$$

$$\implies xv' = \frac{-4}{v} + -2v$$

$$\implies xv' = -\frac{4 + 2v^2}{v}$$



$$\implies -\int \frac{v}{4+2v^2}\, dv = \int \frac{1}{x}\, dx$$

$u = 4 + 2v^2$
$du = 4v\, dv$

$$\implies -\frac{1}{4}\int \frac{du}{u} = \int \frac{1}{x}\, dx$$

$$\implies -\frac{1}{4}\ln(u) = \ln(x) + C$$

ODEs - Week 1 - Thursday (5)

$$-\frac{1}{4}\ln(4+2w^2) = \ln(cx)$$

$$\Rightarrow \ln\left(4+2\left(\frac{y}{x}\right)^2\right) = -4\ln(Cx)$$

$$\Rightarrow 4+2\frac{y^2}{x^2} = \frac{1}{Cx^4}$$

$$\Rightarrow \cancel{4}^{2}x^2 + \cancel{2}y^2 = \frac{1}{Cx^2}$$

$$\Rightarrow y = \pm\sqrt{\frac{1}{Cx^2} - 2x^2}$$

Looking at our initial conditions, $y>0$, so we need to choose the + sqrt. Also since $y(\frac{1}{2}) = 1$,

$$1 = \sqrt{\frac{4}{c} - \frac{1}{2}}$$

$$1 = \frac{8-c}{2c} \Rightarrow 2c = 8-c$$

$$c = \frac{8}{3}$$

$$y = \sqrt{\frac{3}{8x^2} - 2x^2} \quad x>0$$

But we have to restrict the domain further b/c we need $\frac{3}{8x^2} - 2x^2 > 0$

$$\frac{3}{8} - 2x^4 > 0 \Rightarrow 2x^4 < \frac{3}{8} \quad x < \sqrt[4]{\frac{3}{16}}$$

So our domain is $\left(0, \sqrt[4]{\frac{3}{16}}\right)$

ODEs - Week 1 - Thursday (6)

How does everyone feel?

In case there's still time left...

Solve $(x^2-y^2)\,dy + (-2xy)\,dx = 0$.

It's written in a goofy form, but $\frac{dy}{dx} = y'$ still.

Didn't even make it here.