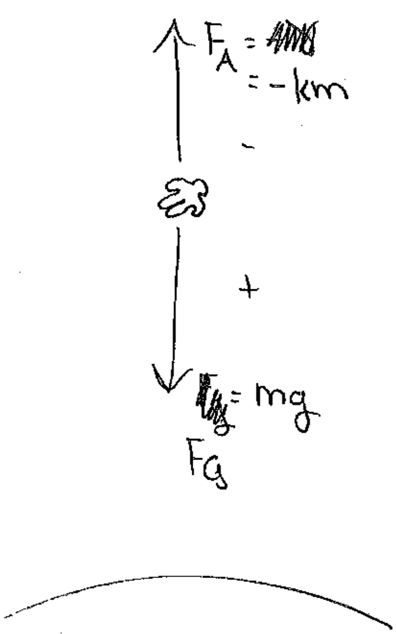
ODEs - Week 3 - ~~Monday~~ Tuesday (1)

The process of describing a physical situation using differential equations is called <u>modeling</u>. This is basically the whole point of having non-mathematicians take this course. If you've ever heard someone say something like "mathematics is the language of the universe," this is kinda the basis for that. If you use ~~a~~ things (explicitly) from this course in your careers / lives, this be it.

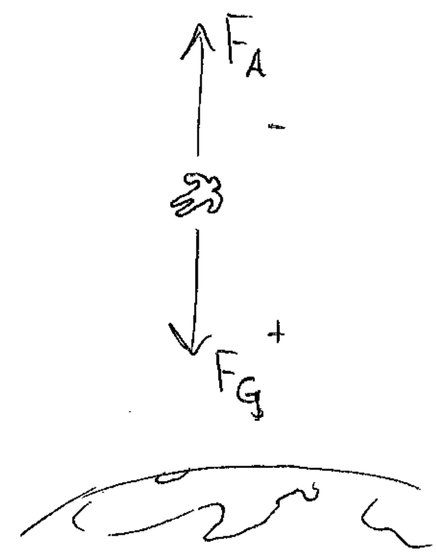
Let's start with a classic example: a falling body. Suppose that an object is falling ~~two~~ towards the earth.



To model this, set some conventions, like we'll say the object has positive velocity whilst hurtling towards the earth. We'll say ~~the downward force~~ forces acting in the downward direction are positive, while forces in the upward direction are negative.

ODEs - Week 3 - Tuesday (2)

So F_G will be the force acting downward on the body due to gravity, and F_A will account for the force due to air resistance. Recall Newton's Second Law of motion $F = m\underline{a}$. This is the differential equation we'll use to model this situation. ~~Say our body has~~ The acceleration in this law can be written as $\frac{\partial v}{\partial t}$ if we are interested in the velocity of the object, making it a first-order DE: $F(t,v) = m\frac{\partial v}{\partial t}$. OR if we are interested in the position of the object, this must be a second-order DE $F(t,p,\dot{p}) = m\frac{\partial^2 p}{\partial t^2}$, where $\frac{\partial^2 p}{\partial t^2} = p'' = \ddot{p}$ is the acceleration in terms of position p.



Let's consider the first-order DE $F(t,v) = m\frac{\partial v}{\partial t}$. So $F(t,v)$ the total force on the body (which we'll say has mass m) is comprised of the two component forces F_G and F_A. F_G is pretty straight-forward: $F_G = mg$ where $g \approx 9.81\ m/s^2$, gravitational acceleration near the surface of the earth.

ODEs - Week 3 - Tuesday (3)

Figuring out what F_A should be is less straightforward, but some thought indicates that it can be modeled as some multiple of the velocity ~~mass~~ of the object, so $F_A = -kv$. So our full model for the velocity of the body becomes

$$F(t,v) = m\frac{dv}{dt} \rightarrow 9.8m - kv = m\frac{dv}{dt}$$

$$\rightarrow \frac{dv}{dt} + \left(\frac{k}{m}\right)v = 9.8,$$

and this is a first-order linear DE! Chill.

"Suppose that a plane is equipped with cannon ~~canon~~ pointed towards the earth. The plane shoots a 2 kg cannonball at the earth at an initial speed of 200 ft/s. Further suppose that the force of air ~~resis~~ resistance on the cannonball ~~will~~ is appoximately 10 times ~~twice~~ the velocity of the cannonball at any given time; Using the model for a falling body just developed, find function ~~an equation~~ that returns the velocity of the cannonball after t seconds of being fired from the plane."

ODEs - week 3 - Tuesday (4)

So $m = 2\,kg$, $F_G = 9.81 \cdot 2\,kg \approx 20\ N$

$F_A = -\overset{10}{\cancel{2}}v$.

$F = ma \Rightarrow 20 - \overset{10}{\cancel{2}}v = 2\dot{v}$ $\qquad \mu = e^{\int 5\,dt} = e^{5t}$

$\Rightarrow \dot{v} + 5v = 10$

$\Rightarrow e^{5t}\dot{v} + e^{5t}v = e^{5t} \cdot 10$

$\Rightarrow \int \frac{\partial}{\partial t}(v e^{5t})\,dt = \int 10 e^{5t}\,dt$

$\Rightarrow v e^{5t} = +\frac{10}{5} e^{5t} + C$

$\Rightarrow v(t) = \dfrac{C + \overset{2}{\cancel{10}} e^{5t}}{e^{5t}} = Ce^{-5t} + \overset{2}{\cancel{10}}$

$V(0) = 200$

So $200 = Ce^{-5(0)} - \overset{2}{\cancel{10}}$, and $C = 198$, so our function is $v(t) = 198e^{-5t} + \overset{2}{\cancel{10}}$.

$198e^{-5t} + \overset{2}{\cancel{10}}$

$t \geq 0$

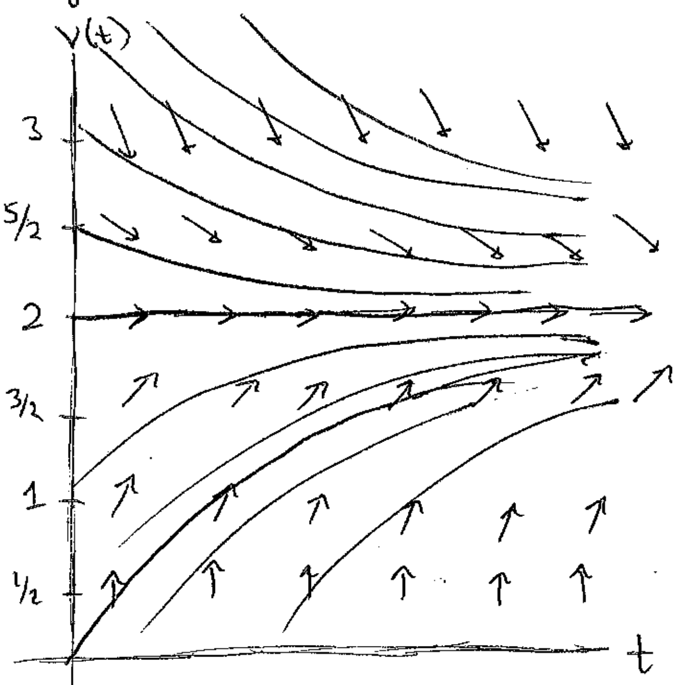
Domain?

ODEs – Week 3 – Tuesday (5)

Let's learn something else real quick, through some exploration. We have our differential equation:

$\dot{v} + 5v = 10$ OR $\dot{v} = 10 - 5v$ & General Solution $\overset{2}{\cancel{10}} + Ce^{-5t}$ $C = 198$

Let's sketch the vector field associated to our DE that models our falling object. Note $t \geq 0$ since ... time and $v \geq 0$ since the cannonball is always falling, so we only need to consider the first-quadrant.



Note that as $t \to \infty$, each particular solution approaches $v(t) = 2$ ft/s. How do we interpret this in terms of the physical situation we ~~are~~ modeling?

For our particular solution ~~before~~ the cannonball ~~slows~~ ~~down~~ descellantes after being fired. For a different initial condition say $v(0) = 0$ so the cannonball just gets droped from the plane, the cannonball has positive acceleration.

ODEs - Week 3 - Tuesday (6)

Regardless of the initial conditions though, for any ~~initial conditions~~ particular solution $v(t)$ we get $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 2 + Ce^{-5t} = 2$. In this particular physical situation of a falling object, this limit is called the *terminal velocity* of the object. It's the velocity any object falling towards will appraoch as it falls.

In general though, ~~this~~ a solution that ~~they~~ other particular solutions are asymptotic to are called *equilibrium solutions*. These are important to consider when studying the long-term behavior of a physical system. These are the constant solutions to the DE.

~~Need one more?~~

ODEs - Week ~~4~~ 3 - Wednesday (1)

Today let's talk about tanks of saltwater.

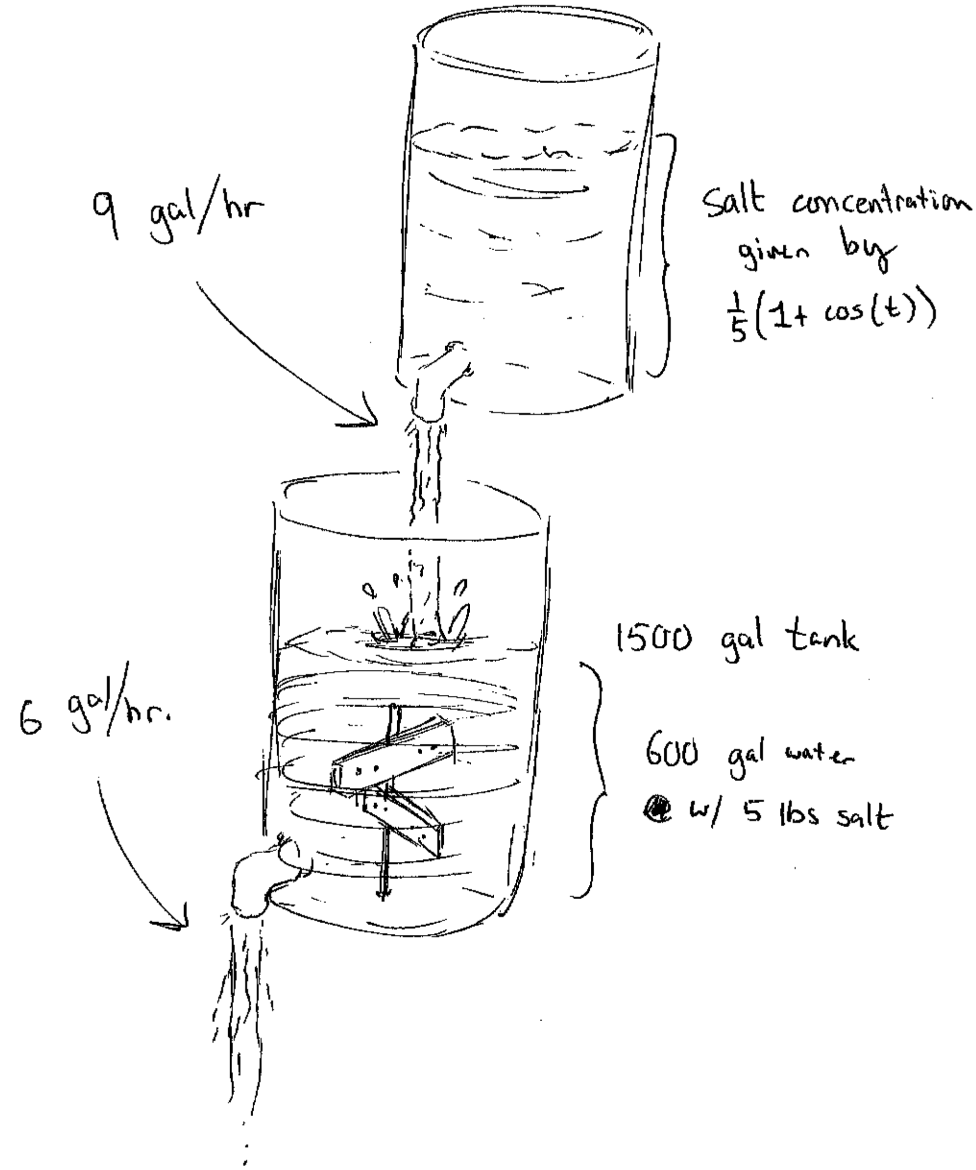
"A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and water entering the tank has a salt concentration of $\frac{1}{5}(1+\cos(t))$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?"

Let $Q(t)$ denote the amount of salt dissolved in the tank at a given time t. The main equation we can use to model this situation is given by

$$\begin{pmatrix}\text{total rate}\\ \text{of change}\\ \text{of } Q(t)\end{pmatrix} = \begin{pmatrix}\text{rate at}\\ \text{which } Q(t)\\ \text{enters tank}\end{pmatrix} - \begin{pmatrix}\text{rate at}\\ \text{which } Q(t)\\ \text{leaves tank}\end{pmatrix}.$$

Notice too that this is a simple model of the situation. Really salt water would not instantly mix when poured into a new tank, (i.e. the concentration would vary throughout the tank) but ~~that~~ considering that would make the problem way hard, so we won't, and assume "... a well mixed solution leaves the tank...".

Figure A



9 gal/hr
Salt concentration given by $\frac{1}{5}(1+\cos(t))$
1500 gal tank
6 gal/hr.
600 gal water w/ 5 lbs salt

ODEs - Week 3 - Wednesday (2)

Now the total rate of change of $Q(t)$ is just $\dot{Q}(t)$ which we'll write as $\dot{Q}$. Next

$$\begin{pmatrix} \text{the rate at which} \\ Q(t) \text{ exits/enters} \\ \text{a tank} \end{pmatrix} = \begin{pmatrix} \text{flow rate of} \\ \text{the fluid} \end{pmatrix} \times \begin{pmatrix} \text{concentration} \\ \text{of the} \\ \text{substance} \end{pmatrix}$$

So writing down the rate at which $Q(t)$ enters the tank is alright; it's just $9 \times \frac{1}{5}(1 + \cos(t))$. Now writing down the rate is a bit tougher: the flow-rate is 6 gal/hr, but since saltwater is simultaneously entering and leaving the tank, how can we write down the "concentration of the substance in the fluid leaving the tank"? Well, think about the word "concentration":

$$\text{Concentration} = \frac{\text{Amount of salt in tank at time } t}{\text{Amount of water in the tank at time } t}$$

The numerator of this is just $Q(t)$ by definition. For the denominator, we start at 600 gallons, and every hour 9 gal enter and 6 gallons leave, so we're looking at $600 + 3t$ gallons in the tank at time t.

ODEs – Week 3 – Wednesday (3)

So our ~~differential equation~~ Initial Value Problem that models the amount of salt in the tank at time t is given by

$$\dot{Q} = \frac{9}{5}\left(1+\cos(t)\right) - \frac{6Q}{600+3t} \; ; \quad Q(0)=5$$

$$\dot{Q} + 2\frac{Q}{200+t} = \frac{9}{5}\left(1+\cos(t)\right) \; ; \quad Q(0)=5$$

~~It's separable too, but also linear!~~

It's linear! So we can solve for Q.

$$\mu = e^{\int \frac{2}{200+t}\,dt} = e^{2\ln(200+t)} = (200+t)^2$$

$$(200+t)^2\dot{Q} + 2(200+t)Q = \frac{9}{5}(200+t)^2\left(1+\cos(t)\right)$$

$$\Rightarrow \int \frac{d}{dt}\left((200+t)^2 Q\right)dt = \frac{9}{5}\int (200+t)^2\left(1+\cos(t)\right)dt$$

$$\Rightarrow Q = \frac{1}{(200+t)^2}\,\frac{9}{5}\left(\int (200+t)^2\,dt + \underbrace{\int (200+t)^2\cos(t)\,dt}_{\text{By parts twice.}}\right)$$

$$\Rightarrow Q = \frac{9}{5}\left(\frac{1}{200+t}\right)^2\left(\frac{1}{3}(200+t)^3 + (200+t)^2\sin(t) + 2(200+t)\cos(t) - 2\sin(t) + C\right)$$

Then after some more arduous calculations, we find that $C = -4600720$ using our initial condition $Q(0)=5$.

$\int u\dot{v} = C + uv - \dot{u}\bar{v} + \ddot{u}\bar{\bar{v}} - \dddot{u}\bar{\bar{\bar{v}}} + \dots$

$v = \frac{d}{dt}(\bar{v})$

$\dot{u} = \frac{d}{dt}(u)$

ODEs - Week 3 - Wednesday (4)

So our particular solution is

$$Q(t) = \frac{9}{5}\left(\frac{200+t}{3} + \sin(t) + \frac{2\cos(t)}{200+t} - \frac{2\sin(t)}{(200+t)^2} \right) - \frac{460072}{(200+t)^2}.$$

where the domain is only $t>0$.

Now we must answer the question though. "How much salt is in the tank when it overflows?" Well, what time t does it overflow?

$$600 + 3t = 1500 \implies t = 300.$$

So the answer to our question is $Q(300) \approx 279.797$ lbs.

What an event that was! :)

You wanna try to work through an easier one?

"A tank initially holds 10 gallons of fresh water. A salt-water solution containing $\frac{1}{2}$ lb salt per gallon begins to pour into the tank at a rate of 2 gal/minute, while the well-stirred mixture leaves the tank at the same rate. Write down a function that returns the amount of salt in the tank after t minutes since the salt-water solution began to pour. Do another for concentration."

ODEs – Week 3 – Wednesday (5)

$$\dot{Q} = \overset{2\left(\frac{1}{2}\right)}{\cancel{\tfrac{4}{2}\left(\tfrac{1}{8}\right)}} - 2\,\frac{Q}{10} \quad ; \quad Q(0)=0$$

Letting $Q(t)$ denote the "amount" of salt at any given time t, we have

(The rate at which $Q(t)$ is changing in the tank) = (The rate at which fluid enters the tank)(amount of salt per gallon in that tank) − (The rate at which fluid leaves the tank)(...

concentration!

(... the concentration, the amount of salt per gallon leaving the tank. I.e. Salt at time t / Volume at time t)

So we arrive at $\dot{Q} = 2\left(\frac{1}{2}\right) - 2\frac{Q}{10}$ ~~and~~ $Q(0)=0$.

$$\dot{Q} + \frac{1}{5}Q = 1 \qquad \mu = e^{\int \frac{1}{5}dt} = e^{\frac{1}{5}t}$$

Separable if you want

$$\frac{d}{dt}\left(Q e^{\frac{1}{5}t}\right) = e^{1/5 t}$$

$$Q = \frac{1}{e^{1/5 t}}\left(5e^{1/5 t} + C\right)$$

$$Q = 5 + Ce^{-1/5 t}$$

And since $Q(0)=0$, $C=-5$

So our "amount" is

$$5 - 5e^{-1/5 t}$$

and since the ~~concentration~~ volume is constant at 10, the concentration is

$$\frac{1}{2}\left(1 - e^{-1/5 t}\right)$$

ODEs - Week 3 - Wednesday (6)

"Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 liters of a dye solution with a concentration of 1 g/liter. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 liters/min. and the well-stirred solution flowing out at the same rate. How much time must pass before the concentration of dye in the tank reaches 1% its original value."

$$\dot{Q} = 2(0) - 2\frac{Q}{200} \qquad Q(0) = 200$$

$$\dot{Q} = -\frac{Q}{100}$$

$$\ln(Q) = -\frac{t}{100} + c$$

$$Q = Ce^{-t/100}$$

$$Q = 200e^{-t/100}$$

$$Q(t) = 1\% \text{ of } 200$$
$$= 2\,?$$

$$\frac{1}{100} = e^{-t/100}$$

$$+\frac{\ln(100)}{100} = t$$

$$\ln(100 - e^{100}) = t$$

ODEs - Week 3 - Thursday (1)

First, ~~basic~~ a basic model of (unchecked) population growth. The primary idea behind it is that, " the population increases at a rate proportional to the population." I.e. if some population has size $P(t)$ at time t, then, unchecked $\dot{P} = kP$ for some k.

" The population of a certain country is known to increase at a rate proporptional to the population. If after two years the population has doubled, and after three years the population is 20,000, how many people were initially living in the country? "

Extract information! Let $P(t)$ denote the population of the country t years after inception. Using our basic model the situation suggests, $\dot{P} = kP$. This is a seperable differential equation. Lets solve this!

ODEs - Week 3 - Thursday (2)

$$\dot{P} = kP$$

$$\frac{1}{P}\dot{P} = k \Rightarrow \int \frac{1}{P}\,\partial P = \int k\,\partial t$$

$$\Rightarrow \ln(P) = kt + C$$

$$\Rightarrow P = Ce^{kt} \quad \text{(familiar?)}$$

Notice that C is the initial population!

"The population doubled after two years": $P(2) = 2P(0)$

~~three years later~~ Using this, we get that

$$\cancel{C}e^{(2)k} = 2\cancel{C}e^{(0)k}$$

$$e^{2k} = 2 \Rightarrow 2k = \ln(2)$$

$$k = \frac{1}{2}\ln(2) \text{ OR } k = \ln(\sqrt{2}).$$

$$P(t) = Ce^{\ln(\sqrt{2})t} = C\sqrt{2}^{\,t} \quad \ddot{\smile}$$

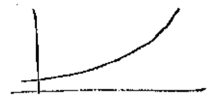
Then since "three years later, the population was 20000." $P(3) = 20000$.

$$P(3) = C\sqrt{2}^{(3)} = 20000$$

$$\Rightarrow C = \frac{20000}{\sqrt{8}} = \frac{10000}{\sqrt{2}}$$

So since $P(0) = C\sqrt{2}^{(0)} = C$, $\frac{10000}{\sqrt{2}}$ is our initial population.

ODEs – Week 3 – Thursday (3)

Neat! But in the $\dot{P} = kP$ model, populations grow exponentially. $P = Ce^{kt}$  . That's not real. Sometimes ~~the~~ factors inhibit a population's growth.

$$\dot{P} = \begin{pmatrix} \text{rate at which} \\ P(t) \text{ enters} \\ \text{the region} \end{pmatrix} - \begin{pmatrix} \text{rate at which} \\ P(t) \text{ leaves} \\ \text{the region} \end{pmatrix}$$

looks like mixing

birth
migration in

death / natural disasters
migration out

This is a slightly better model that represents <u>checked population growth</u>. But modeling either one of the two terms just depends on the ~~population~~ particular situation.

"A population of insects in a region grows at a rate proportional to its current population. In the absense of any outside factors, the population would triple in two weeks time. On any given day there is a net migration into the area of 15 insects, 16 insects are eaten by local birds, and 7 die of natural causes. If there are initially 100 insects in the area, will the population survive? If not, when will it die out?"

ODEs - Week 3 - Thursday (4)

The initial condition is that $P(0) = 100$. Then the general differential equation to model the situation is

$$\dot{P} = \begin{pmatrix} \text{rate } P(t) \\ \text{enters} \end{pmatrix} - \begin{pmatrix} \text{rate } P(t) \\ \text{leaves} \end{pmatrix}$$

$$\uparrow$$

$$= (kP + 15) - (\cancel{}\, 16 + 7)$$

$$= kP - 8$$

"absense of outside factors" look @ $\dot{P} = kP$

"the population would triple in two weeks" : We solve this DE just like the last one to see that $P = Ce^{kt}$, and so

$$3P(0) = P(14)$$

$$3Ce^{k(0)} = Ce^{k(14)}$$

$$3 = e^{14k} \Rightarrow k = \frac{\ln(3)}{14} \quad \text{gross} ☹$$

So we've got $\dot{P} = \frac{\ln(3)}{14} P - 8$. Solving this DE, well its separable, so lets do it!

$$\dot{P} = \frac{\ln(3)}{14}\left(P - \frac{112}{\ln(3)}\right)$$

$$\int \frac{1}{P - \frac{112}{\ln(3)}}\, \partial P = \int \frac{\ln(3)}{14}\, \partial t$$

$$\ln\left(P - \frac{112}{\ln(3)}\right) = \frac{\ln(3)}{14} t + C \quad \ldots$$

$$\Rightarrow \quad P = Ce^{\frac{\ln(3)}{14}t} + \frac{112}{\ln(3)}$$

Now our initial condition was that $P(0) = 100$, so

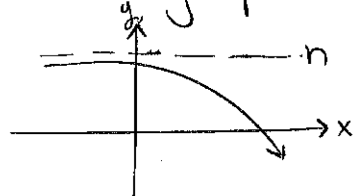
$$100 = Ce^{\frac{\ln(3)}{14}(0)} + \frac{112}{\ln(3)}$$

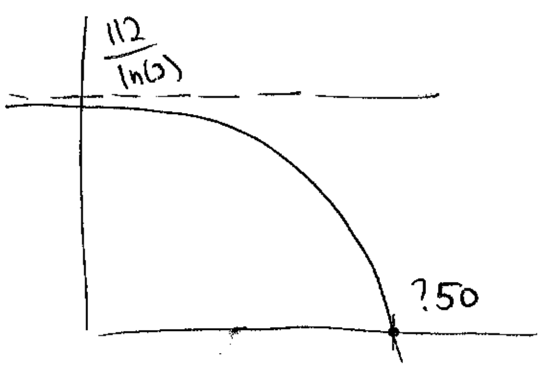
$$\Rightarrow \quad C = \left(100 - \frac{112}{\ln(3)}\right) \approx -1.9\overset{5}{\cancel{47}}$$

$$\Rightarrow \quad P = -1.95e^{\frac{\ln(3)}{14}t} + \frac{112}{\ln(3)}$$

"Does the population die out" Graph it!

Generally $y = -e^x + n$





$$0 = -1.95e^{\frac{\ln(3)}{14}t} + \frac{112}{\ln(3)}$$

$$e^{\frac{\ln(3)}{14}t} = \frac{112}{\ln(3)\cdot 1.95}$$

$$t = \frac{14}{\ln(3)}\ln\left(\frac{112}{1.95\ln(3)}\right) \approx 50 \text{ days}$$

How do y'all feel?

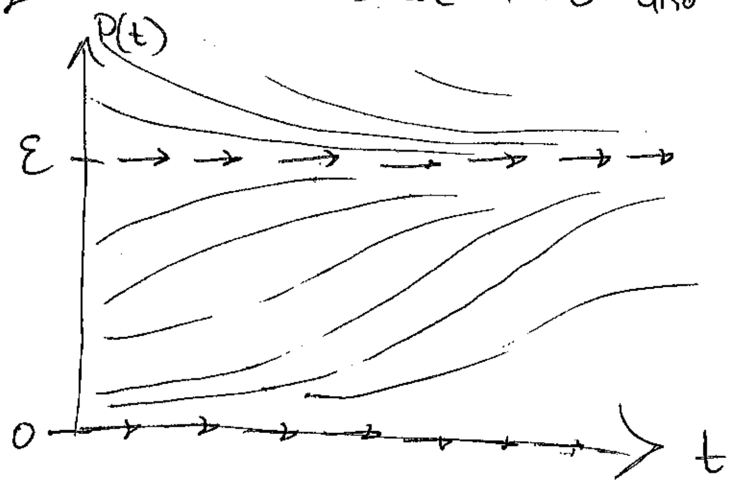
ODEs – Week 3 – Thursday (6)

The model still isn't that good though. It tells us that the population P is given by an exponential, which isn't really true. Populations either "gradually" die off  or have a carrying capacity .

A more reasonable model is the logistic growth equation.

$$\dot{P} = k\left(1 - \frac{P}{\mathcal{E}}\right)P$$

where k is still a constant and $\mathcal{E}$ is the carrying capacity ($\mathcal{E}$ for equilibrium) of the population. Sketching a vector field of the population, we see why this is better. We get equilibrium solutions at $P=0$ and $P=\mathcal{E}$



Happy?

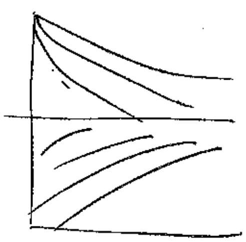
ODEs - Week 3 - Thursday (7)

Now I'm going to go through some general facts and terminology at you (if we have time).

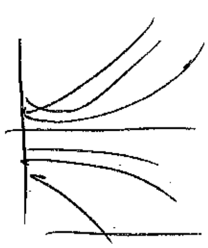
In order to have these situations that have equilibrium solutions, we need DEs thats don't depend explicitly on t; only on the function in question. Such DEs are called autonomous.

$$\dot{y} = f(y) \quad (\text{with no } t).$$

We can start to ~~does~~ classify the sorts of equilibrium ~~points~~ solutions such autonomous DEs have. If all the curves near an equilibrium solution get nearer to it (like $P = \mathcal{E}$ in the last example), it's called an ~~eq~~ asymptotically stable equilibrium solution.



Then there are unstable equilibrium solutions, where all the curves near the solution move away from it ($P = 0$ in the last example).



Or there's semi-stable equilibrium solutions where curves move closer on one side and move further away on the other.