ODEs – Week 4 – Monday (1)

Now we're going to talk about general <u>linear</u> <u>differential</u> <u>equations</u> (not just first-order ones). ~~Mostly we'll focus on~~ second-order ~~equations, like~~ ~~Second-order~~

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$ Second order example

and often times we'll just look at the case when the functions p, q and r are constants. And we'll look at an even easier case of <u>homogeneous</u> differential equations where $g(t) = 0$. (If $g(t) \neq 0$ its nonhomogeneous). This is completely different than the previous "homogeneous." So

$$ay'' + by' + cy = 0$$ Don't restrict to 2^{nd} order here

OR $$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$$

Before solving these things, we've gotta introduce the idea of <u>linear independence</u> of functions. A collection of functions $\{f_1, f_2 \ldots f_n\}$ are linearly independent if ~~when~~ the only time a linear combination of them equals ~~zero~~ zero

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

is when each of the c_i is zero.

write literal examples
$\pi(x+1) + 3(x^2)$
$f_1(x) = x+1$
$f_2(x) = x^2$

ODEs - Week 4 - Monday (2)

Said another way, those functions are linearly independent if you cannot write any one of them as a linear combination of the others. If some set of functions are <u>not</u> linearly independent, the ~~we~~ we'll say they're <u>linearly</u> <u>dependent</u> (one depends on the others):

- Example: The functions $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are linearly independent. If you have a linear combination of them equal 0 like

$$c_1 e^t + c_2 e^{-t} = 0$$

Then

$$c_1 e^{2t} + c_2 = 0$$

$$\rightarrow \quad c_2 = -c_1 e^{2t}$$

But $-c_1 e^{2t}$ is never ~~zero~~ constant unless $c_1=0$, so the only solution here is $c_1 = c_2 = 0$.

- Example: The three functions $f_1(x) = x+1$ $f_2(x) = x+x^2$ $f_3(x) = x^2-1$ are <u>not</u> linearly independent. Write it down

$$c_1(x+1) + c_2(x+x^2) + c_3(x^2-1) = 0$$

$$\Rightarrow \quad c_1(\cancel{x+1}) + c_2(x)(\cancel{1+x}) + c_3(x-1)(\cancel{x+1}) = 0$$

$$\Rightarrow \quad c_1 + c_2 x + c_3(x-1) = 0$$

So we need $\begin{cases} c_1 - c_3 = 0 \\ c_2 + c_3 = 0 \end{cases}$ So $\begin{matrix} c_1 = c_3 = 1 \\ c_2 = -1 \end{matrix}$ ✓

ODEs - Week 4 - Monday (3)

So now that we've talked about that, let's solve a nice example DE to give you an idea of why we talked about that.

(*) "By inspection, find some solutions to the DE $y'' - 9y = 0$."
$y(0) = 2 \quad y'(0) = -1$

It may help to recall that homework problem where the solution to $y' = ky$ was $y = e^{kt}$. .....

Some solutions are $y = e^{3t}$ and $y = e^{-3t}$. Verify these!
Also, any linear combination of these solutions will be a solution! So $y = c_1 e^{+3t} + c_2 e^{-3t}$ is a solution. Verify this!

This suggests the following major theorem:

Theorem:

- (Principal of Superposition)
  If y_1 and y_2 are solutions to a linear homogeneous differential equation, then so is $y = c_1 y_1 + c_2 y_2$.
- An n^{th}-order linear homogeneous differential equation has exactly n linearly independent solutions. If $y_1, y_2, \dots, y_n$ are such linearly independent solutions, then

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_2$$

  will be the general solution to the DE.

"Solve the IVP (*) with $y(0) = 2 \quad y'(0) = -1$."

"Solve the IVP (*) with $y(0)=2$ & $y'(0)=-1$."

Notice that the previous theorem, along with that exercise telling us e^{-kt} and e^{kt} are linearly independent, tells us $y = c_1 e^{3t} + c_2 e^{-3t}$ is <u>the</u> <u>general</u> <u>solution</u> to our DE. To find the particular solution (solve for c_1 and c_2) we're just got to take a single derivitive of our general solution and use those initial conditions

$$\begin{cases} y = c_1 e^{3t} + c_2 e^{-3t} \\ y' = 3c_1 e^{3t} - 3c_2 e^{-3t} \end{cases} \Longrightarrow \begin{cases} 2 = c_1 e^{3(0)} + c_2 e^{-3(0)} \\ -1 = 3c_1 e^{3(0)} - 3c_2 e^{-3(0)} \end{cases}$$

$$\Rightarrow \begin{cases} 2 = c_1 + c_2 \\ -1 = 3c_1 - 3c_2 \end{cases} \Rightarrow \quad \begin{matrix} c_1 = 5/6 \\ c_2 = 7/6 \end{matrix}$$

So we get $y = \frac{5}{6} e^{3t} + \frac{7}{6} e^{-3t}$.

If there's still time,

talk about determinants...

ODEs - Week 4 - ~~Monday~~ Tuesday ~~(4)~~ (1)

Now suppose that you have an <u>n^{th}-order</u> <u>linear</u> <u>homogenous</u> DE, and I hand you n functions $f_1, \ldots, f_n$ and tell you that these are each a solution to your DE. You can check yourselfs that each one is a solution, but how do you know that's <u>all</u> the solutions? How do you know that $c_1 f_1 + \cdots + c_n f_n$ is the general solution? How do you know if the $f_1, \ldots, f_n$ are linearly independent? (Note that ~~these~~ if you can answer one of these, you've answered them all).

For this, we'll need to develop some linear algebra.

Do y'all know how to calculate the determinant of a matrix?

- $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

I think it would be fine to spend <u>more</u> time on determinants. In fact, do this first in lecture.

- $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \times \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Or you could expand on any other row too, just gotta remember the sign.

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ and the pattern continues for larger matrices.

$\det\begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 6 \\ 0 & 1 & 2 \end{pmatrix}$?

ODEs - Week 4 - ~~Monday (5)~~ Tuesday (2)

You can take determinants of matrices with functions as their entries too.

"Calculate the determinant of the matrix

$$\begin{pmatrix} e^t & \cos^2(t) \\ e^t & \sin^2(t) \end{pmatrix}.$$ "

Given a set of functions $\{f_1, \dots, f_n\}$ that each have $n-1$ derivatives, the <u>Wronskian</u> of that set is the determinant

$$\det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}$$

"Calculate the Wronskian of the set of functions of x given by the expressions $\{x, x^2, x^3\}$."

Theorum

If $f_1, \dots, f_n$ are solutions to an n^{th}-order <u>linear</u> <u>homogeneous</u> differential equation, then these solutions are linearly independent if and only if their Wronskian is not identically zero.

ODEs - Week 4 - Tuesday (3)

"Example: verify that $y = c_1 \cos(2x) + c_2 \sin(2x)$ is the general solution to the differential equation $y'' + 4y = 0$."

- Check that it <u>is</u> a solution
- Check there are enough
- Check for linear independence w/ Wronskian.

You may ask why $c_1 e^{2ix} + c_2 e^{-2ix}$ isn't the general solution, but that's just a rewritten form of the other solution using the identities

needs i because 4 is positive...

$$i\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \qquad \left(e^{i\pi} = -1 \right)$$

Complex stuff will start to appear in the course now. Be aware though, that when asking for a solution to a DE, the implication is ~~that~~ ~~the~~ ~~most~~ for a <u>real</u> solution.

ODEs – Week 4 – Tuesday (4)

Just some justification for the Wronskian, without too much linear algebra. Suppose you have some nth-order linear homogeneous DE with ~~general~~ solutions $y_1, \ldots, y_n$, but you don't know if they're linearly independent. Suppose too that this is some initial value problem with initial conditions

$$y(x_0) = y_0 \qquad y'(x_0) = y_0' \qquad \ldots \qquad y^{(n-1)}(x_0)\ y_0^{(n-1)}$$

(this is weird notation since each $y_0^{(i)}$ is a number and not a derivative but bear with me.) Applying these initial conditions to our alleged general solution $c_1y_1 + \cdots + c_ny_n$ we get a system of equations ~~(~~ n equations with n unknowns $c_1 \ldots c_n$

$$n \text{ equations} \begin{cases} y_0 = c_1y_1(x_0) + c_2y_2(x_0) + \cdots + c_ny_n(x_0) \\ y_0' = c_1y_1'(x_0) + c_2y_2'(x_0) + \cdots + c_ny_n'(x_0) \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ y_0^{(n-1)} = c_1y_1^{(n-1)}(x_0) + c_2y_2^{(n-1)}(x_0) + \cdots\ c_ny_n^{(n-1)}(x_0) \end{cases}$$

This can be solved! The general method is called Cramer's rule, (you don't need to know that nor the details for that) But the idea is that when you solve for $c_1, \ldots, c_n$ each is a fraction, and the denominator is the Wronskian.

ODEs - Week 4 - Wednesday (1)

So now that we've developed tools to know when we have a solution to an ~~[illegible]~~ n^{th}-order linear homogenous DE. So your DE looks something like

$$\overset{a_n(x)}{a_n} y^{(n)} + \overset{a_{n-1}(x)}{a_{n-1}} y^{(n-1)} + \cdots + \overset{a_1(x)}{a_1} y' + \overset{a_0(x)}{a_0} y = 0$$

Those a_i are generally functions of x, but we'll look at the easier case where they're constants first. Also, again remember to divide through by a_n to make life easier

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Now we've solved examples of such DEs before by "inspection" or by "guessing smart." Recall $y'' - 9y = 0$ had solutions like $y = e^{3t}$ and $y = e^{-3t}$? That's exactly how we're gonna solve these in general to (not each time though): Let's just make a really wild guess that $y = e^{\lambda x}$ is a solution to

$$y^{(n)} + \cdots + a_1 y' + a_0 = 0$$

for some number λ. Well, let's check if it's a solution.

ODEs - Week 4 - Wednesday (2)

So if $y = e^{\lambda x}$, then $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$, and so on. In general $y^{(n)} = \lambda^n e^{\lambda x}$.

$$\lambda^n e^{\lambda x} + a_{n-1}\lambda^{n-1} e^{\lambda x} + \cdots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0$$

$$\Rightarrow \quad e^{\lambda x}\left(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0\right) = 0$$

Since $e^{\lambda x} \neq 0$ ever, we must have that the polynomial we get on the right must be zero! That is to say λ must be a root of the polynomial

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0$$

(~~written with coefficients in t~~). More importantly, if λ is a root of this polynomial, then $e^{\lambda t}$ is a solution to the original DE! This polynomial, having degree n will have exactly n (potentially complex) roots each "corresponding" to a solution. Better yet, these will correspond to all of the solutions! (Recall an n^{th}-order linear homogeneous DE has n solutions). And we can easily find this polynomial, called the <u>characteristic polynomial</u> from the original DE.

ODEs - Week 4 - Wednesday (3)

Let's see ~~some~~ an example~~s~~ before we talk about some of the subtlies here.

"Solve the differential equation $2y''-2y'-24y=0$ where $y(0)=3$ and $y'(0)=-2$."

First, what's the characteristic polynomial associated to this DE?

$$2y''-2y'-24y=0$$

$$y''-y'-12y=0$$

the characteristic polynomial $\Longrightarrow$ $$\lambda^2-\lambda-12=0$$

$$(\lambda-4)(\lambda+3)=0$$

The roots of the char poly are 4 and -3, giving us a general solution of $y=c_1e^{4t}+c_2e^{-3t}$. To find the values of c_1 c_2 corresponding to the initial conditions we need to take a single derivative of our general solution $y'=4c_1e^{4t}-3c_2e^{-3t}$ and get the system of equations

$$\begin{cases} y(0)=3=c_1e^{4(0)}+c_2e^{-3(0)} \\ y'(0)=-2=4c_1e^{4(0)}-3c_2e^{-3(0)} \end{cases}$$

$$\Longrightarrow \begin{cases} 3=c_1+c_2 \\ -2=4c_1-3c_2 \end{cases} \Longrightarrow \begin{cases} c_1=1 \\ c_2=2 \end{cases}$$

$$y(t)=e^{4t}+2e^{-3t}$$

ODEs – Week 4 – Wednesday (4)

How does everyone feel?

Okay, so lets talk about the subtleties ~~subtlties~~ I mentioned earlier.

- If the roots ~~are~~ of the characteristic equation are all real and all distinct, it's straightforward.
- But what if there are complex roots?
- What if there are repeated roots?

Not every polynomial factors completely over the real numbers $\mathbb{R}$ but they do all factor over the complex numbers $\mathbb{C}$ ($\mathbb{C}$ is special this way. ~~we say that~~ this property that $\mathbb{C}$ has is known as being algebraically closed.)

Anyways you might get complex roots !!. Recall that these complex roots come in pairs though: If $\lambda_1 = a+bi$ is a root of the characteristic equation, so is $\lambda_2 = a-bi$.

So what's our solution in this case (remember that we're demanding real solutions)? Well I'll tell you: it's

$$c_1 e^{at}\cos(bx) + c_2 e^{at}\sin(bx).$$

Lol but why?

ODEs – Week 4 – Wednesday (5)

It's because of those goofy identities I told you once that relate trig functions to complex numbers

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad i\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$$

$$e^{(a+bi)\theta} = e^{a\theta+bi\theta} = e^{a\theta}e^{bi\theta} = e^{a\theta}\left(\cos(b\theta)+i\sin(b\theta)\right)$$

Supposing your general solution looks like $c_1e^{(a+bi)t}+c_2e^{(a-bi)t}$, you can rewrite this as:

$$c_1e^{at}e^{bit}+c_2e^{at}e^{-bit}$$

$$= e^{at}\left(c_1\left(\cos(bt)+i\sin(bt)\right)+c_2\left(\underbrace{\cos(-bt)}_{\cos(bt)}+\underbrace{i\sin(-bt)}_{-i\sin(bt)}\right)\right)$$

$$= e^{at}\left((c_1+c_2)\cos(bt)+i(c_1-c_2)\sin(bt)\right)$$

Notice that $\overbrace{c_1+c_2}^{c_3}$ and $\overbrace{i(c_1-c_2)}^{c_4}$ are just constants, so we can replace them with new constants

$$c_3e^{at}\cos(bt)+c_4e^{at}\sin(bt) \quad \checkmark$$

ODEs - Week 4 - ~~Th~~ Wednesday (6)

Perfect! Now what about duplicate roots? Say we're looking at the DE $y'' + 6y' + 9y = 0$. The characteristic polynomial factors as $(\lambda+3)(\lambda+3) = 0$ giving us only one solution $c_1 e^{-3t}$. But the DE is degree 2, so there's another solution. Surprise! The other solution just has a mysterious factor of t hanging out on it: $c_2 t e^{-3t}$.

"Verify $y = c_1 e^{-3t} + c_2 t e^{-3t}$ is a solution to $y'' + 6y' + 9y = 0$."

We might see why this is later. :)

Problems if there's time left over

Solve

$y''' + 4y' = 0$ complex w/ $\lambda_1 = 0$

duplicate roots

$x^4 + 4x^2 + 4$

$y'' - 4y' + 4y = 0$

ODEs - Week 4 - Thursday (1)

Reduction of Order

Okay now we're gonna discuss a general techique to solve n^{th}-order linear homogeneous DEs that don't necessarily have constant coefficients. This method will also explain why we "add a factor of t" in that last section if the characteristic equation had repeated roots. We'll look at 2^{nd}-order DEs, but this technique can work in general. Suppose your DE looks like

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = 0$$

and further suppose you know a solution $y_1(x)$ already (because someone told you it's a solution, or because you stared at the DE for a long time and figured it out).

The whole idea is that you guess that there's a second solution that is only a function times your first solution. That is, guess that $y_2(x) = \varphi(x)\,y_1(x)$ is a solution too. Then you can, using the fact that y_1 is a solution, try to figure out $\varphi(x)$.

ODEs – Week 4 – Thursday (2)

"Find the general solution to

$$2t^2y'' + ty' - 3y = 0$$

given that $y_1(t) = \frac{1}{t}$ is a solution."

First, sanity check, just make sure $\frac{1}{t}$ really is a solution

$$y_1 = \frac{1}{t} \qquad y_1' = -\frac{1}{t^2} \qquad y_1'' = \frac{2}{t^3}$$

and $2t^2\left(\frac{2}{t^3}\right) + t\left(-\frac{1}{t^2}\right) - 3\left(\frac{1}{t}\right)$

$$= \frac{4}{t} - \frac{1}{t} - \frac{3}{t} = 0 \quad \checkmark$$

Now let's <u>guess</u> that there's another solution of the form $y_2 = \varphi\frac{1}{t}$ for some function $\varphi(t)$.

$$y_2 = \varphi\frac{1}{t} \qquad y_2' = -\frac{\varphi}{t^2} + \frac{\varphi'}{t} \qquad y_2'' = -\left(-\frac{2\varphi}{t^3} + \frac{\varphi'}{t^2}\right) + \left(\frac{\varphi''}{t} - \frac{\varphi'}{t^2}\right)$$

$$= \frac{\varphi''}{t} - 2\frac{\varphi'}{t^2} + 2\frac{\varphi}{t^3}$$

Then let's punch this into our DE with the dream of solving for $\varphi(t)$.

$$2t^2y'' + ty' - 3y = 0$$

$$\longrightarrow \quad 2t^2\left(\frac{\varphi''}{t} - 2\frac{\varphi'}{t^2} + 2\frac{\varphi}{t^3}\right) + t\left(\frac{\varphi'}{t} - \frac{\varphi}{t^2}\right) - 3\left(\frac{\varphi}{t}\right) = 0$$

$$\Longrightarrow \quad 2t\varphi'' - 4\varphi' + 4\underline{\underline{\frac{\varphi}{t}}} + \varphi' - \underline{\underline{\frac{\varphi}{t}}} - 3\underline{\underline{\frac{\varphi}{t}}} = 0$$

Notice how the stuff in red goes away. This is because $\frac{1}{t}$ was already a solution to the DE.

$$\Longrightarrow \quad 2t\varphi'' - 3\varphi' = 0$$

Then we can solve this DE with a quick substitution to reduce the order. Let $\beta = \varphi'$, so $\beta' = \varphi''$, and

$$2t\beta' - 3\beta = 0$$

which is generally linear (in this case it's separable!) and we can proceed.

$$2t\beta' - 3\beta = 0$$

$$\Rightarrow \int \frac{2}{\beta}\, d\beta = \int \frac{3}{t}\, dt$$

$$\Rightarrow \ln(\beta^2) = \ln(Ct^3)$$

$$\Rightarrow \beta = Ct^{3/2}$$

$$\Rightarrow \varphi' = Ct^{3/2}$$

$$\Rightarrow \varphi = C_1 t^{5/2} + C_2$$

So $y_2 = \varphi \frac{1}{t} = C_1 t^{3/2} + C_2 t^{-1}$ is another solution to the DE. And this constant C_2 allowed $\frac{1}{t}$ to be part of this solution too. $t^{3/2}$ and t^{-1} are linearly independent, so $y = C_1 t^{3/2} + C_2 t^{-1}$ is our general solution!

How do y'all feel? :)

Let's do one now

ODEs - Week 4 - Thursday (5)

"Find the general solution to $t^2y''+2ty'-2y=0$ given that $y_1(t)=t$ is a solution."

So say we'll guess another solution $y_2 = \omega t$

$y_2=\omega t \quad y_2' = \omega' t + \omega \quad y_2'' = \omega'' t + 2\omega'$

And punch it in

$$t^2(\omega'' t + 2\omega') + 2t(\omega' t + \omega) - 2(\omega t) = 0$$

to first

$$\omega'' t^3 + 2\omega' t^2 + 2\omega' t^2 + \underline{2\omega t} - \underline{2\omega t} = 0$$

$$t\omega'' + 4\omega' = 0 \qquad t \neq 0$$

Let $\gamma = \omega'$, so $\gamma' = \omega''$.

$$t\gamma' = -4\gamma$$

$$\Rightarrow \int \frac{1}{\gamma}\, d\gamma = \int \frac{-4}{t}\, dt$$

$$\Rightarrow \gamma = \frac{c}{t^4} \Rightarrow \omega = \int \frac{c}{t^4}\, dt = \frac{c_1}{t^3} + c_2$$

So our general solution is $y(t) = c_1 \frac{1}{t^2} + c_2 t$.