ODEs - Week 5 - ~~Monday~~ Tuesday (1)

Method of Undetermined Coefficients

Now we're going to look at nonhomogeneous linear differential equations

$$(*) \qquad y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y' + a_0(t)\, y = g(t).$$

For this general linear DE, there is the associated homogeneous linear DE

$$(**) \qquad y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y' + a_0(t)\, y = 0.$$

We've gone over a few techniques to solve these ones, like by looking at the characteristic polynomial or by reduction of order, so you can probably find a solution y_c to $(**)$. This solution y_c, we'll call that the complementary solution to $(*)$. (Some texts call this the homogeneous solution and denote it y_h instead). Note that y_c is not actually a solution to $(*)$, so maybe "complementary solution" is a poor choice of name. See the mathematical Red Herring Principle.

ODEs - Week 5 - Tuesday (2)

Theorem

> The general solution $y(t)$ of $(*)$ is of the form
> $$y(t) = y_c(t) + Y_p(t)$$
> where $Y_p(t)$ is a particular solution to $(*)$.

The point of today and tomorrow (and maybe Thursday) is to ~~how~~ talk about how to find $Y_p(t)$. In particular how to find the RIGHT $Y_p(t)$ so that $y = y_c + Y_p$ is truly the general solution. Now I'm not going to justify the above theorem (It's not too enlightening), but if you're interested, read through the section of Paul's notes titled

"Nonhomogeneous Differential Equations".

So then how do we find Y_p? We make an intelligent guess. If your DE ~~looks like~~ has a constant function $g(t)$, then a good guess would be that Y_p looks something like $g(t)$. This is a good ~~idea~~ idea for some $g(t)$ (we'll talk about which ones later), and for other $g(t)$ we'll need to use another trick.

ODEs - Week 5 - Tuesday (3)

Let's do an example just to get a flavor of the technique of solving a DE by the method of Undetermined Coefficients.

"Find a particular solution to $y''-4y'-12y=3e^{5t}$."

It's a good habit to get into to find the complementary solution first. Since $(\lambda^2-4\lambda-12)=(\lambda-6)(\lambda+2)$,

$$y_c = C_1e^{-2t}+C_2e^{6t}.$$

Now by our previous discussion, a good guess for Y_p might be something like Ae^{5t} for some constant coefficient A that we need to determine. Let's see if this is a good guess.

$$Y_p = Ae^{5t} \qquad Y_p' = 5Ae^{5t} \qquad Y_p'' = 25Ae^{5t}$$

$$(25Ae^{5t})-4(5Ae^{5t})-12(Ae^{5t}) = 3e^{5t}$$

Comparing all the coefficients of e^{5t}, we need A so that

$$25A-20A-12A=3$$

$$-7A=3$$

$$A=-3/7.$$

ODEs - Week 5 - Tuesday (4)

So a particular solution is $y_p(t) = -\frac{3}{7}e^{5t}$, and our general solution ~~[illegible]~~ is

$$y = C_1 e^{-2t} + C_2 e^{6t} - \frac{3}{7}e^{5t}.$$

Note 1: if your solving an IVP, you <u>must</u> get to this general solution before applying your initial conditions (say $y(0) = \frac{18}{7}$ & $y'(0) = -1/7$). Don't just stop at y_c and then solve for C_1 and C_2.

Note 2: A is a different sort of coefficient than C_1 and C_2. While y is a solution forall C_1, C_2, it's only a solution for a particular value of A. You've gotta find A.

Let's walk through another.

"Find a particular solution to $y'' - 4y' - 12y = \sin(2t)$."

We get the same complementary solution as before.

Maybe it's a good idea to guess $A\sin(2t)$ for our particular solution. Let's try it out.

ODEs - Week 5 - Tuesday (5)

$$Y_p(t) = A\sin(2t) \quad Y_p' = 2A\cos(2t) \quad Y_p'' = -4A\sin(2t)$$

$$(-4A\sin(2t)) - 4(2A\cos(2t)) - 12(A\sin(2t)) = \sin(2t)$$

$$(-16A)\sin(2t) + (-8A)\cos(2t) = \sin(2t)$$

I see trouble though. To solve for A, we should compare the coefficients of $\sin(2t)$ on the left with those on the right, so $-16A = 1$, and $A = -\frac{1}{16}$. But we need to do the same for that $\cos(2t)$, where $A = 0$ ‼. Shoot. When something like this happens, it means we didn't guess right, so let's go back and guess something better. Since there's a $\cos(2t)$ hanging out, maybe we should guess $Y_p = A\sin(2t) + B\cos(2t)$. Then

$$Y_p' = 2A\cos(2t) - 2B\sin(2t) \quad Y_p'' = -4A\sin(2t) - 4B\cos(2t)$$

Then we punch 'em in!

ODEs - Week 5 - Tuesday (6)

$$(-4A\sin(2t) - 4B\cos(2t)) - 4(2A\cos(2t) - 2B\sin(2t)) - 12(A\sin(2t) + B\cos(2t)) = \sin(2t)$$

$$(-4A + 8B - 12A)\sin(2t) + (-4B - 8A - 12B)\cos(2t) = \sin(2t)$$

Comparing coefficients we get a system of equations

$$\begin{cases} -16A + 8B = 1 \\ -8A - 16B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{20} \\ B = \frac{1}{40} \end{cases} \checkmark$$

So our particular solution is $Y_p = -\frac{1}{20}\sin(2t) + \frac{1}{40}\cos(2t)$.

One more:

" ~~Solve~~ Find a particular solution to $y'' - 4y' - 12y = 2t^3 - t + 3$."

Our $g(t)$ is a degree-3 polynomial, so what should our guess be for Y_p? A general degree 3 polynomial

$$Y_p = At^3 + \underbrace{Bt^2}_{\text{Still need this}} + Ct + D$$

... have them work it out.

ODEs - Week 5 - Wednesday (1)

Protips
- turn paper sideways
- Use \\ + ... + \\\\ for newlines

Today we're going to keep going with the method of undetermined coefficients. Last time we saw examples of solving DEs like

$$y'' + ay' + by = g(t)$$

where $g(t)$ was an exponential, trig, or polynomial function. It turns out these are the only three sorts of functions that work consistently using the method of undetermined coefficients. Let's lay out some general rules for what you should guess Y_p to be now.

If $g(t)$ looks like...	You should guess Y_p is
$ae^{\beta t}$	$Ae^{\beta t}$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n^{th}-degree polynomial $p(t)$ $a_nt^n + \cdots + a_1t + a_0$	$A_nt^n + \cdots + A_1t + A_0$ (n^{th} degree polynomial)

Some important notes

1. The coefficients a and $b, \ldots$ don't matter in you guess, but β super does.

ODEs - Week 5 - Wednesday (2)

5 • If your (suggested) guess looks like your complementary solution, you've got to add some factors of (powers of) t to your guess until it doesn't. (We'll do an example of this).

2 • If $g(t)$ looks like $a\sin(\beta t) + b\cos(\gamma t)$, you've gotta do Both suggested guesses

$$Y_p = A\cos(\beta t) + B\sin(\beta t) + C\cos(\gamma t) + D\sin(\gamma t).$$

3 • More generally, if $g(t)$ is a sum of ~~exponenti~~ terms of the listed forms, your guess for Y_p should be a ~~guesses~~ sum of the suggested guesses for Y_p. (Then combine coefficients if you can)

4 • If $g(t)$ is a product of things in the list, then your guess needs to be product of the suggested guesses. And if one of the factors is ~~a trig function~~, sine or cosine with a polynomial factor, you need a different polynomial on each of sine and cosine in your guess. So if $g(t) = (t^2+2)\cos(3t)$, your guess needs to be

$$Y_p(t) = (At^2 + Bt + C)\cos(3t) + (Dt^2 + Et + F)\sin(3t).$$

ODEs - Week 5 - Wednesday (3)

Questions before we look at some examples?

" Find a particular solution to $y'' - 4y' - 12y = te^{4t}$. "

Recall $y_c = C_1 e^{-2t} + C_2 e^{6t}$, so since the exponents aren't $4t$, we don't need to worry about adding factors of t. ~~For the e^{4t} bit we should guess~~ For the polynomial t bit we should guess $At + B$, and for the exponential e^{4t} we should guess Ce^{4t}, so our total guess is

$$Y_p = Ce^{4t}(At + B)$$
$$= e^{4t}(At + B) \quad \text{absorb...}$$

Then we rock 'n' roll from there.

$$Y_p' = 4e^{4t}(At+B) + e^{4t}(A) \qquad Y_p'' = 16e^{4t}(At+B) + 4e^{4t}A + 4e^{4t}A$$
$$= e^{4t}(4At + 4B + A) \qquad = e^{4t}(16At + 16B + 8A)$$

$$e^{4t}(16At + 16B + 8A) - 4e^{4t}(4At + 4B + A) - 12(e^{4t}(At + B)) = te^{4t}$$

Notice all the e^{4t}'s go away, and we just have a bunch of coefficients of t and constants to compare.

ODEs – Week 5 – Wednesday (4)

$$(16A - 16A - 12A)t + (16B + 8A - 16B - 4A - 12B) = (1)t + (0)$$

$$\Rightarrow \begin{cases} -12A = 1 \\ 4A - 12B = 0 \end{cases} \Rightarrow \begin{cases} A = -1/12 \\ B = 4(-1/12)/12 = -1/36 \end{cases}$$

and so our particular solution is

$$Y_p(t) = e^{4t}\left(-\frac{1}{12}t - \frac{1}{36}\right) = \frac{1}{36}e^{4t}(3t+1)$$

"Suppose you're looking at $y'' - 4y' - 12y = g(t)$.
What should be your guess for Y_p if $g(t)$ is"

(a) $16e^{7t}\sin(10t)$ $\quad Ae^{7t}\cos(10t) + Be^{7t}\sin(10t)$

(b) $(9t^2 - 103t)\cos(t)$ $\quad (At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t)$

(c) $-e^{-3t}(3-5t)\cos(9t)$ $\quad e^{-3t}(At+B)\cos(9t) + e^{-3t}(Ct+D)\sin(9t)$

(d) $e^{6t}(t+2)$ $\quad te^{6t}(At+B)$ $\quad$ work it out to see why?

(e) $t\cos(2t) + \sin(3t)$ $\quad (At+B)\cos(2t) + (Ct+D)\sin(t) + \dots$

$$\dots + E\cos(3t) + F\sin(3t).$$

ODEs - Week 5 - Wednesday (5)

"Work out why guessing $Y_p = e^{6t}(At+B)$ doesn't work for $y''-4y'-12y = e^{6t}(t+2)$."

Just do it.

This'll take the rest of class time.

Do one more with a sum first!

" $y''-4y'-12y = t^2+6 - \cos(2t) + 3e^t$ "

ODEs - Week 5 - Thursday (1)

- use $\dot{\mu}$ instead of μ'
- tell kids not to take notes $\ddot{\smile}$

Variation of Parameters

Today will be fun. We're going to learn how to solve <u>any</u> linear differential equation (assuming we have solutions to the corresponding homogeneous differential equation.) This method can cover more cases than the method of undetermined coefficients, but is much more burdensome. We'll walk through the procedure for a 2^{nd} degree linear DE.

$$y'' + q(t)y' + r(t)y = g(t)$$

~~Suppose we have the complementary solution~~

$$\cancel{y_c = c_1y_1 + c_2y_2}.$$

First, an important "Big Idea." We're looking for Y_p, and we know that we can get a linear approximation to Y_p as a linear combo of two functions. That is, at some point t_0, we can write

$$Y_p\cancel{(t_0) \approx \mu_1 y_1(t_0) + \mu_2 y_2}(t_0)$$

~~for some constants μ_1 and μ_2. And this is~~ a linear ~~approximation,~~

ODEs - Week 5 - Thursday (2)

$$y_p \approx \mu_1 y_1 + \mu_2 y_2$$

for constants μ_1 and μ_2 that satisfy

$$\begin{cases} y_p(t_0) = \mu_1 y_1(t_0) + \mu_2 y_2(t_0) \\ y_p'(t_0) = \mu_1 y_1'(t_0) + \mu_2 y_2'(t_0) \end{cases}$$

Now this is only a great approximation near t_0. It would be great if we could find functions $\mu_1(t)$ and $\mu_2(t)$ that varied on the domain so we have equality

$$\begin{cases} \underline{y_p(t) = \mu_1(t) y_1(t) + \mu_2(t) y_2(t)} \leftarrow \text{guess} \\ y_p'(t) = \mu_1(t) y_1'(t) + \mu_2(t) y_2'(t) \end{cases}$$

where we just replace constants μ_i with functions $\mu_i(t)$. But the second equation isn't precisely the derivative of the first. It's missing some terms from the product rule. So for our dream to come true, we need those terms to zero out. We need

$$\boxed{\mu_1'(t) y_1(t) + \mu_2'(t) y_2(t) = 0.} \quad (*)$$

Let's remember that ~~this~~ (*) is part of our dream.

ODEs - Week 5 - Thursday (3)

So we're looking at $y'' + q(t)y' + r(t)y = g(t)$.

We guess that $Y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ for some functions u_1 and u_2 where y_1 and y_2 are linearly independent solutions to the corresponding homogeneous DE (i.e. $y_c = c_1y_1 + c_2y_2$).

Taking derivatives of our guess, we get

$$Y_p' = u_1'y_1 + u_1y_1' + u_2y_2' + u_2'y_2$$

$$= u_1y_1' + u_2y_2' \quad \text{(by (*))}$$

$$\Rightarrow \quad Y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

(our (*) doesn't help here)

Then plugging these in

$$(\dot{u}_1\dot{y}_1 + u_1\ddot{y}_1 + \dot{u}_2\dot{y}_2 + u_2\ddot{y}_2) + q(t)(u_1\dot{y}_1 + u_2\dot{y}_2) + r(t)(u_1y_1 + u_2y_2) = g(t)$$

If we cleverly rearrange this...

$$(\dot{u}_1\dot{y}_1 + \dot{u}_2\dot{y}_2) + u_1(\ddot{y}_1 + q(t)\dot{y}_1 + r(t)y_1) + u_2(\ddot{y}_2 + q(t)\dot{y}_2 + r(t)y_2) = g(t)$$

But...

ODEs - Week 5 - Thursday (4)

But since y_1 and y_2 are solutions to the homogeneous analogue of the DE, these last two terms are zero! And we're left with

$$\boxed{\dot{u}_1 \dot{y}_1 + \dot{u}_2 \dot{y}_2 = g(t)}$$

Recalling that y_1 and y_2 are allegedly known, and our dream (*), we get a system of two equations of unknowns $\dot{u}_1$ and $\dot{u}_2$ that we may solve for!

$$\boxed{\begin{cases} \dot{u}_1 y_1 + \dot{u}_2 y_2 = 0 \\ \dot{u}_1 \dot{y}_1 + \dot{u}_2 \dot{y}_2 = g(t) \end{cases}}\ ! \quad y_p = u_1 y_1 + u_2 y_2$$

And we can find our solution from there after we find (and integrate) $\dot{u}_1$ and $\dot{u}_2$, since our guessed solution was $y_p = u_1 y_1 + u_2 y_2$.

Questions?

This was for a degree 2 linear DE. For degree n, we go through a similar bit to get a system like:

$$\begin{cases} \dot{u}_1 y_1 + \cdots + \dot{u}_n y_n = 0 \\ \dot{u}_1 y_1' + \cdots + \dot{u}_n y_n' = 0 \\ \vdots \\ \dot{u}_1 y_1^{(n-2)} + \cdots + \dot{u}_n y_n^{(n-2)} = 0 \\ \dot{u}_1 y_1^{(n-1)} + \cdots + \dot{u}_n y_n^{(n-1)} = g(t) \end{cases}$$

ODEs - Week 5 - Thursday (5)

Let's do an example :)

"Find a general solution to $y'' + 9y = 3\tan(3t)$."

First our guess is going to be $Y_p = \mu_1 y_1 + \mu_2 y_2$ where $y_c = c_1 y_1 + c_2 y_2$, so we're gotta find the complementary solution y_c. Since $\lambda^2 + 9$ has roots $\pm 3i$, $y_c = c_1 \cos(3t) + c_2 \sin(3t)$. So our guess is

$$Y_p = \mu_1 \cos(3t) + \mu_2 \sin(3t).$$

The system of equations we found to solve for $\dot{\mu}_1$ and $\dot{\mu}_2$ were

$$\begin{cases} \dot{\mu}_1 y_1 + \dot{\mu}_2 y_2 = 0 \\ \dot{\mu}_1 \dot{y}_1 + \dot{\mu}_2 \dot{y}_2 = g(t) \end{cases} \Rightarrow \begin{cases} \dot{\mu}_1 \cos(3t) + \dot{\mu}_2 \sin(3t) = 0 \\ -\cancel{3}\dot{\mu}_1 \sin(3t) + \cancel{3}\dot{\mu}_2 \cos(3t) = \cancel{3}\tan(3t). \end{cases}$$

The first eq says $\dot{\mu}_1 = -\dot{\mu}_2 \dfrac{\sin(3t)}{\cos(3t)}$. Plugging this into the other eq we get

$$-\left(-\dot{\mu}_2 \frac{\sin(3t)}{\cos(3t)}\right)\sin(3t) + \dot{\mu}_2 \cos(3t) = \tan(3t)$$

$$\rightarrow \quad \dot{\mu}_2 \left(\sin^2(3t) + \cos^2(3t)\right) = \tan(3t)\cos(3t)$$

$$\Rightarrow \quad \dot{\mu}_2 = \sin(3t) \quad \& \quad \dot{\mu}_1 = -\frac{\sin^2(3t)}{\cos(3t)}.$$

ODEs - Week 5 - Thursday (6)

Remember we're looking for u_1 and u_2 though, so we need to integrate. u_2 ain't bad. u_1 is.

$$u_2 = \int \dot{u}_2 \, dt = \int \sin(3t)\, dt = -\frac{1}{3}\cos(3t)$$

(Don't worry about $+C$) ;)

$$u_1 = \int \dot{u}_1 \, dt = -\int \frac{\sin^2(3t)}{\cos(3t)}\, dt = -\int \frac{1}{\cos(3t)}\, dt - \int \cos(3t)\, dt = \cdots$$

$$\cdots = -\int \sec(3t)\,dt + \frac{1}{3}\sin(3t) = -\frac{1}{3}\ln\left|\sec(3t)+\tan(3t)\right| + \frac{1}{3}\sin(3t)$$

But that's all we need. Looking back at our guess,

$$y_p = -\frac{1}{3}\cos(3t)\cos(3t) + \left(-\frac{1}{3}\ln\left|\sec(3t)+\tan(3t)\right| + \frac{1}{3}\sin(3t)\right)\sin(3t)$$

$$= -\frac{1}{3} - \frac{1}{3}\sin(3t)\ln\left(\sec(3t)+\tan(3t)\right)$$

Adding back our complementary solution we get

$$y = c_1\cos(3t) + c_2\sin(3t) - \frac{1}{3}\left(1 + \sin(3t)\ln\left|\sec(3t)+\tan(3t)\right|\right)$$

How ~~[illegible]~~ was that?