ODEs - Week 6 - ~~Mon~~ Tuesday (1)

Laplace Transforms

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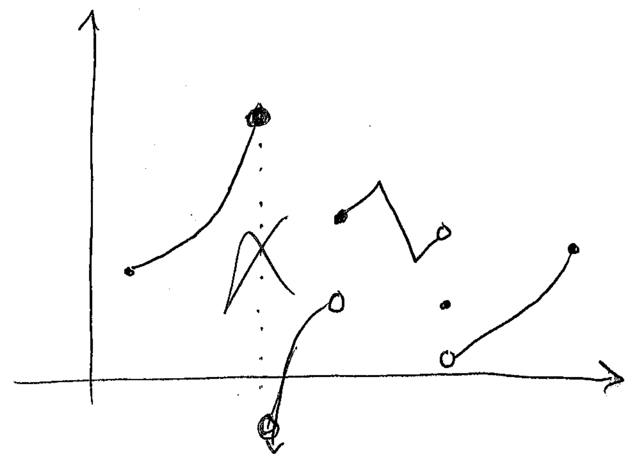
office hours / sample final ... sols on frid.

First recall that a piecewise-continuous function is a function that (on its domain) has only finitely many places where it's not continuous. These are functions we can take integrals of on an subinterval of their domain.



Now the Laplace Transform of a piecewise-continuous function $f(t)$, usually defined only on $(0,\infty)$, is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt .$$

This is now a function of s rather than of t. Also, note that sometimes we have to consider the domain of s, since the integral won't always necessarily converge.

ODEs - Week 6 - Tuesday (2)

The catchphrase to keep in mind while we're talking about Laplace transforms is that they "turn DE problems into algebra problems." Let's compute some Laplace transforms.

"Compute $\mathcal{L}\{1\}$."

The Laplace transform of the constant function 1 will be

$$\int_0^\infty e^{-st} 1 \, \partial t = \lim_{R\to\infty} \left(-\frac{1}{s}\right) e^{-st}\Big|_0^R$$

$$= -\frac{1}{s} \lim_{R\to\infty} e^{-sR} + \frac{1}{s} e^{-s(0)}$$

$$= 0 + \frac{1}{s} \quad \text{if } s > 0 \quad //$$

So $\mathcal{L}\{1\} = \frac{1}{s}$ on the domain $(0,\infty)$.

"Computer $\mathcal{L}\{e^{at}\}$."

$$\int_0^\infty e^{-st} e^{at} \, \partial t = \int_0^\infty e^{(a-s)t} \, \partial t = \frac{1}{a-s}\left(\lim_{R\to\infty} e^{(a-s)t}\Big|_0^R\right)$$

$$= \frac{1}{s-a} \quad \text{for } s > a.$$

only converges for $s > a$

ODEs – Week 6 – Tuesday (3)

In general, computing Laplace transforms is tough (because computing integrals can be tough). So we usually can rely on a table of common Laplace transforms to help us. The fact the Laplace transform is a linear function helps too

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

"Compute $\mathcal{L}\{2\cosh(2t) - 3\sqrt{2t}\}$."

From the table $\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2}$ and $\mathcal{L}(\sqrt{t}) = \frac{1}{2}\sqrt{\pi}\, s^{-3/2}$,

So we get $\frac{2s}{s^2-4} - 3\sqrt{2}\,\frac{1}{2}\sqrt{\pi}\, s^{-3/2} = \frac{2s}{s^2-4} - 3\sqrt{\frac{\pi}{2}}\, s^{-3/2}$.

Recall $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ $\quad(\cos(\theta) = \cosh(i\theta))$

$\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ $\quad(i\sin(\theta) = \sinh(i\theta))$

"Compute $\mathcal{L}\{t^2\cos(2t)\}$."

∂s

#30 $\mathcal{L}\{t^2\cos(2t)\} = (-1)^2 \frac{\partial^2}{\partial s^2}\left(\mathcal{L}\{\cos(2t)\}\right)$

$= \frac{\partial^2}{\partial s^2}\left(\frac{s}{s^2+4}\right) = \frac{\partial}{\partial s}\left(\ldots\right.$

ODEs - Week 6 - Tuesday (4)

Now we also need to talk about the opposite problem of inverse Laplace transforms: given a function $F(s)$ how do you find the function $f(t)$ s.t $\mathcal{L}\{f(t)\} = F(s)$?

~~Helpfully, this~~ Use the notation $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Helpfully, this'll be linear too:

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}.$$

We do this using clever algebra and the table:

"Find the inverse Laplace transform of $F(s) = \frac{7}{s} - \frac{4}{s-6}$."

Looking at #1 and #2 in the table, this'll be

$$\mathcal{L}^{-1}\{F(s)\} = 7\mathcal{L}^{-1}\{\tfrac{1}{s}\} - 4\mathcal{L}^{-1}\{\tfrac{1}{s-6}\} = 7(1) - 4e^{6t}.$$

"Find the inverse Laplace transform of

$$F(s) = \frac{2}{3s+5} \qquad G(s) = \frac{4}{s^4} - \frac{3}{s^2+4}."$$

$$\frac{2}{3s+5} = \frac{2}{3}\left(\frac{1}{s+\frac{5}{3}}\right)$$

#3

$$\frac{4}{s^4} = \frac{4}{3!}\left(\frac{3!}{s^{(3+1)}}\right)$$

#7

$$\frac{3}{2}\left(\frac{2}{s^2+2^2}\right)$$

#2 $f(t) = \frac{2}{3}e^{-5/3t}$

$g(t) = \frac{2}{3}t^3$

ODEs - Week 6 - Tuesday (5)

"Find the inverse Laplace Transform of

$$F(s) = \frac{2s+3}{s^2+5}.$$ "

Break is up and use sine & cosine.

"Find the inverse Laplace Transform of

$$F(s) = \frac{2-5s}{(s-6)(s^2+11)}.$$ "

Partial frac it out $\frac{A}{s-6} + \frac{Bs+C}{s^2+11}$.

One that requires completing the square.

"Find $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2-3s+4}\right\}$."

✓

$$\frac{s+2}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2} = \frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2} + \sqrt{7}\,\frac{\frac{\sqrt{7}}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2} \cdots$$

ODEs - Week 6 - Wednesday (1)

You may have seen in the table, but we have formulas for taking Laplace transforms of a derivative. This is pretty important since Differential equations are equations of derivatives.

First, a subtly I've been ignoring that I'll say today, ~~from~~ for a function f, defined on positive real numbers to even exist, the integral $\int_0^\infty e^{-st} f(t)\,dt$ has to converge, right. Well, this converges, and so $\mathcal{L}\{f(t)\}$ exists ~~is f is of exponential order α on its domain, which means that~~ for $s > \alpha$ if f is of <u>exponential</u> <u>order</u> α on its domain, that is there exists some max M and cutoff x_0 s.t. for all $x \geq x_0$ we have $|f(x)| \leq Me^{\alpha x}$.

Don't worry about this too much. It just means f doesn't increase too quickly.

Now, the nifty theorem:

ODEs - Week 6 - Wednesday (2)

Theorem

If f is continuous and f' is piecewise continuous on any bounded interval in $(0,\infty)$, and if for some α we have that f is of exponential order α, then $\mathcal{L}\{f'(t)\}$ exists for $s>\alpha$, and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

We can prove this! Start with $\mathcal{L}\{f'(t)\}$ and perform integration by parts.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt \qquad \begin{cases} u = e^{-st} & v = f(t) \\ du = -se^{-st}dt & dv = f'(t)dt \end{cases}$$

$$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt$$

$$= \underbrace{\lim_{R\to\infty} e^{-sR}f(R)}_{(*)} - \underbrace{e^{s(0)}f(0)}_{= f(0)} + s\mathcal{L}\{f(t)\}$$

(*) for some $R>\alpha$ $e^{-sR}f(R) < e^{-sR}|f(R)| \le e^{-sR}Me^{s\alpha} = Me^{-s(R-\alpha)}$ which goes to zero as $R\to\infty$. ◻

ODEs – Week 6 – Wednesday (3)

So we have (sans details) $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$:
We can do the same technique again for f'':

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - \underline{sf(0) - f'(0)}$$

And this generalizes to higher derivatives too, if each derivative obeys the appropriate continuity conditions

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - \sum_{i=0}^{n-1} s^i \underline{f^{(n-1-i)}(0)}.$$

Now these formula have a bunch of lower derivatives evaluated at zero. These can be like your initial conditions of an IVP!

This is nifty! Let's do a problem to see how we can use Laplace transforms to turn IVP into algebra problems.

ODEs - Week 6 - Wednesday (4)

"Solve the differential equation $y'' - 10y' + 9y = 5t$ where $y(0) = -1$ $y'(0) = 2$."

Honestly, it'd be easier to use Undetermined Coefficients, but this'll be illustrative. $\mathcal{L}$ is linear, so

$$\mathcal{L}\{y'' - 10y' + 9y\} = \mathcal{L}\{5t\}$$

$$\Rightarrow \quad \underline{\mathcal{L}\{y''\}} - \underline{10\mathcal{L}\{y'\}} + 9\mathcal{L}\{y\} = \frac{5}{s^2}$$

$$\Rightarrow \quad \underline{s^2\mathcal{L}\{y\} - sy(0) - y'(0)} \underline{- 10\left(s\mathcal{L}\{y\} - f(0)\right)} + 9\mathcal{L}\{y\} = \frac{5}{s^2}$$

$$\Rightarrow \quad (s^2 - 10s + 9)\mathcal{L}\{y\} + \overbrace{s - 2 + (-10)}^{s-12} = \frac{5}{s^2}$$

$$\Rightarrow \quad (s^2 - 10s + 9)\mathcal{L}\{y\} = \frac{5}{s^2} + 12 - s$$

$$\Rightarrow \quad \mathcal{L}\{y\} = \frac{5 + 12s^2 - s^3}{s^2(s^2 - 10s + 9)} \quad \cancel{\frac{5}{s^2(s-9)(s-1)}} + \cancel{\frac{12-s}{(s-9)(s-1)}}$$

Cool! Now we're just gotta find the inverse Laplace transform of that expression on the right.

ODEs - Week 6 - Wednesday (5)

We're gotta partial-frac-decompose it because it doesn't immediately match anything in our table.

$$\mathcal{L}\{y\} = \frac{5}{s^2(s-9)(s-1)} + \frac{12s^2 - s^3}{s^2(s-9)(s-1)}$$

DREAM

$$= \frac{As+B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1} = \frac{\overbrace{(As+B)(s^2-10s+9) + C(s^3-s^2) + D(s^3-9s^2)}^{}}{}$$

$$\frac{\underbrace{(As+B)(s-9)(s-1) + C(s^2)(s-1) + D(s^2)(s-9)}}{s^2(s-9)(s-1)}$$

~~ADE~~ ... ~~then~~ Solve for A, B, C, D and

you get to $\mathcal{L}\{y\} = \dfrac{\frac{50}{81}\cancel{A}s + \frac{5}{9}\cancel{B}}{s^2} + \dfrac{\frac{31}{81}}{s-9} + \dfrac{-2}{s-1}$,

So $y = \dfrac{50}{81} + \dfrac{5}{9}t + \dfrac{31}{81}e^{9t} - 2e^{t}$ ✓

Questions?

ODEs – Week 6 – Wednesday (6)

What if the initial conditions you're given aren't at $t=0$ though? Then we're just gotta do a quick substitution.

"Solve the IVP $y'' + 4y' = \cos(t-3) + 4t$ where $y(3) = 0$ and $y'(3) = 7$."

The initial conditions are at $t=3$. We need to shift the domain over by three. When $t=3$, we want another variable, say η, equal to zero. So take

$$\eta = t-3,$$

and our DE becomes $y''(\eta+3) + 4y'(\eta+3) = \cos(\eta) + \underbrace{4(\eta+3)}_{4\eta+12}$.

Since this is a linear slide, nothing goofy happens with chain rule business; we can just solve the IVP

$$y'' + 4y' = \cos(\eta) + 4\eta + 12$$

and remember to sub back for t later.

$$\left\{ \begin{aligned} \frac{1}{(s+4)(s^2+1)} &= \frac{1/17}{s+4} + \frac{1}{17}\left(\frac{-s+4}{s^2+1}\right) \\ \frac{4+12s+7s^2}{s^3(s+4)} &= \frac{1}{s^3} + \frac{11/4}{s^2} + \frac{17/16}{s} - \frac{17/16}{s+4} \end{aligned} \right.$$

... Gross...

ODEs – Week 6 – Wednesday (7)

Give them this, then present solution Thursday.

"Solve the IVP $y''+3ty'-6y=2$ where

$y(0)=0$ and $y'(0)=0$."

Notice we haven't found <u>any</u> technique to solve this sort of DE (Variation of Parameters only works if we go in knowing the complementary solution.)

ODEs - Week 6 - Thursday (1)

"Solve the IVP $y'' + 3ty' - 6y = 2$
where $y(0) = 0 \quad y'(0) = 0$."

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{ty'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{2\}$$

$$s^2\mathcal{L}\{y\} - s\underset{0}{y(0)} - \underset{0}{y'(0)} + 3\left((-1)\frac{\partial}{\partial s}\left(\mathcal{L}\{y'\}\right)\right) - 6\mathcal{L}\{y\} = \frac{2}{s}$$

$$-\left(\frac{\partial}{\partial s}\left(s\mathcal{L}\{y\} - y(0)\right)\right)$$

$$-\left(s\frac{\partial}{\partial s}\mathcal{L}\{y\} + \mathcal{L}\{y\}\right)$$

$$s^2\mathcal{L}\{y\} - 3s\frac{\partial}{\partial s}\mathcal{L}\{y\} - 3\mathcal{L}\{y\} - 6\mathcal{L}\{y\} = \frac{2}{s}$$

Let $Y = \mathcal{L}\{y\}$ and divide through by $-3s$

$$Y' + \frac{s^2 - 9}{-3s}Y = -\frac{2}{3s^2} \quad \text{Linear DE!}$$

ODEs - Week 6 - Thursday (2)

It's linear with integrating factor

$$e^{-\int \frac{s^2-9}{3s}\,ds} = e^{\int \frac{3}{s}-\frac{s}{3}\,ds} = e^{3\ln(s)-\frac{1}{6}s^2} = s^3 e^{-s^2/6}$$

$$\int \frac{\partial}{\partial s}\left(Y s^3 e^{-s^2/6}\right)ds = \int -\frac{2}{3s^2} s^3 e^{-s^2/6}\,ds$$

$$Y = -s^{-3} e^{s^2/6} \frac{2}{3}\int s e^{-s^2/6}\,ds$$

$$\mu = -s^2/6 \quad d\mu = -\frac{1}{3}s\,ds$$

$$-3\,d\mu = s\,ds$$

$$Y = 2s^{-3} e^{s^2/6}\int e^{\mu}\,d\mu$$

$$= 2s^{-3} e^{s^2/6}\left(e^{-s^2/6} + C\right)$$

$$= \frac{2}{s^3} + \frac{ce^{s^2/6}}{s^3}$$

Now this second term is wierd, right? Its asymptotic to e^{s^2} which is NOT of exponential order. We cannot take the inverse Laplace transform of it since $\lim_{s\to\infty} \frac{e^{s^2/6}}{s^3} \neq 0$, so we must assume $C=0$, and it goes away. So

$$Y = \frac{2}{s^3} \quad \text{and} \quad y = t^2.$$

ODEs - Week 6 - Thursday (3)

Here's a little one, but eh...

"Solve $y' - 5y = e^{5x}$ where $y(0) = 0$."

eh.

$$\left(s\mathcal{L}\{y\} - y(0)\right) - 5\mathcal{L}\{y\} = \mathcal{L}\{e^{5x}\}$$

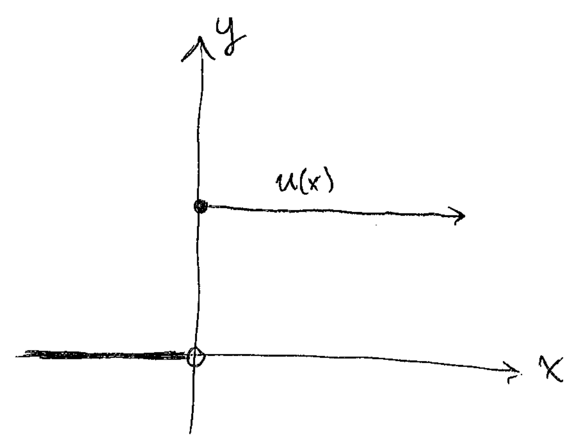
$$\mathcal{L}\{y\} = \frac{1}{(s-5)^2}$$

#23 $$= \mathcal{L}^{-1}\left\{\frac{1!}{(s-5)^{1+1}}\right\} = xe^{5t}$$

Let's start talking about the Heaviside function ~~and convolutions.~~ (instead)

The Heaviside function, also called the unit step function is defined as

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



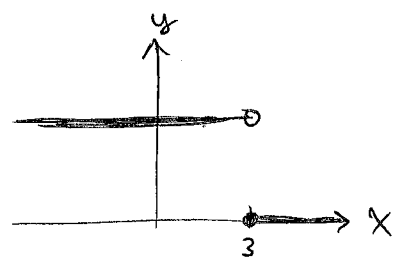
ODEs - Week 6 - Thursday (4)

Then we can immediately define a more general step function

$$u_c(x) = u(x-c) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}.$$

You can think of the Heaviside/step function as an on/off switch: u_c is "on" for $x \geq c$, and "off" for $x < c$. But what if we want the "on" part to come first?

" Graph the function $f(t) = 1 - u_3(t)$. "

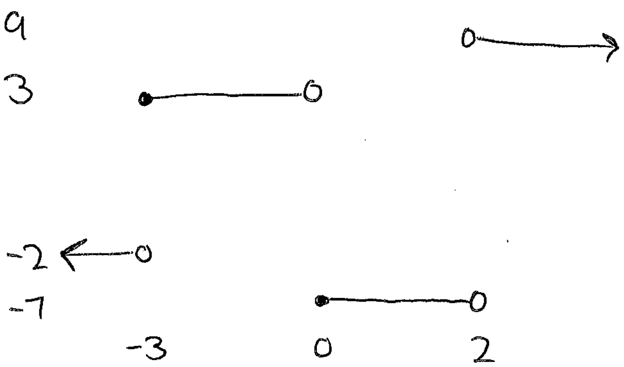


Now we can make things slightly more complicated, and use u_c to represent wierder switches.

ODEs - Week 6 - Thursday (5)

"Write the following function in terms of Heaviside functions:

$$f(x) = \begin{cases} -2 & x < -3 \\ 3 & -3 \le x < 0 \\ -7 & 0 \le 2 \\ 9 & 2 \le x \end{cases}$$ "

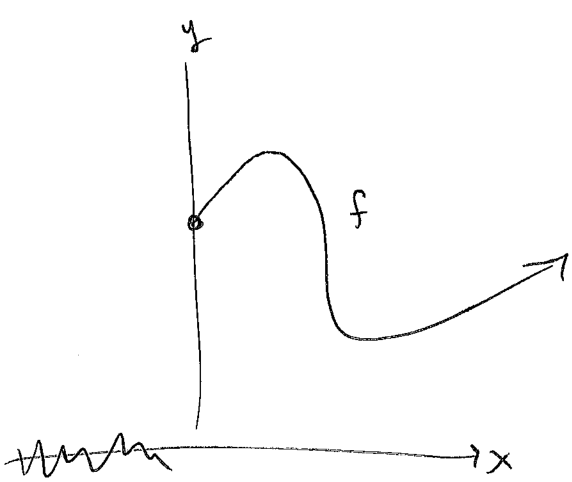


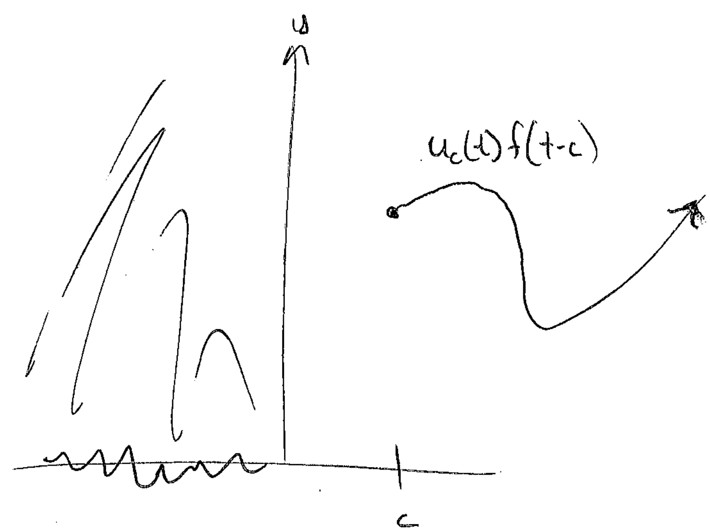
There are three jumps, so we'll need 3 copies of u:

u_{-3} u_0 u_2

$$f(x) = -2 + 5u_5(x) - 10u_0(x) + 16u_2(x) \checkmark$$

Now we usually wanna think about switching on/off real functions though like $u_c(t) f(t-c)$. f on $(0,\infty)$





ODEs - Week 6 - Thursday (6)

"What is the Laplace transform of $u_c(t)f(t-c)$?"

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty u_c(t)f(t-c)e^{-st}\,\partial t$$

$$= \int_c^\infty f(t-c)e^{-st}\,\partial t$$

let $\mu = t-c$
$t \mapsto t+c$

$$= \int_0^\infty f(t)\,e^{-s(t+c)}\,\partial t$$

$$= \int_0^\infty f(t)\,e^{-st}e^{-sc}\,\partial t$$

$$= e^{-sc}\mathcal{L}\{f(t)\} \checkmark$$

In particular if $f=1$, $\mathcal{L}\{u_c(t)\} = \dfrac{1}{se^{sc}}$.

Cool? Let's work through an example now.

ODEs - Week 6 - Thursday (7)

"Take the Laplace transform of the function

$$f(t) = \begin{cases} 2 & 0 \le t < \pi \\ \sin(3t) & \pi < t \end{cases}.$$"

First we write it as a single function

$$f(t) = 2 + u_{\pi}(t)\left(\sin(3t) - 2\right)$$

Then we can just take the Laplace transform of the whole thing.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2\} + \mathcal{L}\{u_{\pi}(t)\left(\sin(3t) - 2\right)\}$$

Recalling $\mathcal{L}\{1\} = \frac{1}{s}$ and $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$, we need to rewrite the second function with

$$\sin(3t) = \sin(3t - 3\pi + 3\pi) = \sin(3(t-\pi) + 3\pi)$$

$$= -\sin(3(t-\pi))$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s} + e^{-\pi s}\left(\mathcal{L}\{-\sin(3t)\} - \mathcal{L}\{2\}\right)$$

$$= \frac{2}{s} - e^{-\pi s}\left(\frac{3}{s^2+9} + \frac{2}{s}\right)$$