Pop Quiz

Ordinary Differential Equations

UCR Math-046-E01, Summer 2018

1. Solve the following differential equation for $x > 0$ and with the initial condition that $y(\mathrm{e}) = \mathrm{e}^{\mathrm{e}+2}$.

$$xy' = y\ln\left(\frac{y}{x}\right)$$

Notice that this differential equation is homogeneous. So let's use the substitution $v = y/x$ where

$$y \mapsto xv \qquad y' \mapsto xv' + v,$$

and our differential equation becomes $x(xv' + v) = xv\ln(v)$ which should be separable:

$$\begin{aligned} x(xv' + v) &= xv\ln(v) \\ x^2 v' &= xv\ln(v) - xv \\ \frac{1}{v\,(\ln(v) - 1)} v' &= \frac{1}{x}. \end{aligned}$$

Notice that we divided through by $v(\ln(v)-1)$, so we have to consider the cases where $v(\ln(v) - 1) = 0$ separately. If $v = 0$, then $y = 0$ which we can check is *not* a solution to our differential equation. If $\ln(v) - 1 = 0$, then $y = \mathrm{e}x$ which *is a solution* do our differential equation! This solution doesn't satisfy our initial condition condition though, so it's not the one we're looking for. Integrating both sides (do a substitution of $u = \ln(v) - 1$ on the left-hand side) we get

$$\begin{aligned} \int \frac{1}{v\,(\ln(v) - 1)}\,\mathrm{d}v &= \int \frac{1}{x}\,\mathrm{d}x \\ \ln(u) &= \ln(x) + C \\ \ln(\ln(v) - 1) &= \ln(Cx) \\ \ln(v) - 1 &= Cx \\ y &= x\mathrm{e}^{Cx+1}. \end{aligned}$$

Looking at our initial condition where $y(\mathrm{e}) = \mathrm{e}^{\mathrm{e}+2}$, we need to find C such that

$$\mathrm{e}^{\mathrm{e}+2} = (\mathrm{e})\mathrm{e}^{C\mathrm{e}+1}$$

so $C = 1$ and our particular solution is $y = x\mathrm{e}^{x+1}$ on the domain $x > 0$.

Note that there is another question on the back

2. Solve the following differential equation:

$$x^x \dot{y} + x^x \ln(x)\, y = 1$$

Here's a link to a more verbose version of the solution, if you're interested.

We first write this as a first-order linear differential equation

$$\dot{y} + \ln(x)\, y = x^{-x}.$$

Since $x > 0$, x^x is never zero and we can freely divide through by it. Our integrating factor is $e^{\int \ln(x) dx}$. Then after we remember that $\int \ln(x)\, dx = x \ln(x) - x$, this integrating factor is

$$e^{\int \ln(x) dx} = e^{x \ln(x) - x} = e^{\ln(x^x)} e^{-x} = x^x e^{-x},$$

and our differential equation becomes

$$\begin{aligned}
\dot{y} + \ln(x)\, y &= x^{-x} \\
e^{\int \ln(x) dx} \dot{y} + e^{\int \ln(x) dx} \ln(x)\, y &= e^{\int \ln(x) dx} x^{-x} \\
\frac{d}{dx}\left(e^{\int \ln(x)\, dx} y\right) &= e^{\int \ln(x) dx} x^{-x} \\
\frac{d}{dx}\left(x^x e^{-x} y\right) &= x^x e^{-x} x^{-x} \\
\frac{d}{dx}\left(x^x e^{-x} y\right) &= e^{-x}.
\end{aligned}$$

Then by taking the antiderivative of both sides we get

$$\begin{aligned}
\int \frac{d}{dx}\left(x^x e^{-x} y\right) dx &= \int e^{-x}\, dx \\
x^x e^{-x} y &= C - e^{-x} \\
y &= \frac{C - e^{-x}}{x^x e^{-x}} \\
y &= \frac{C e^x - 1}{x^x}
\end{aligned}$$

There's no problem dividing through by e^{-x} since e^{-x} is never zero, so this is our general solution.