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7. All course material except for exams and lecture slides are accessible through this document.
8. The exams and lecture slides are accessible via the "Course Outline" page on iLearn.
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Administrative Information

Instructor: **David Weisbart** **Email:** **weisbart@math.ucr.edu**

All emails sent to me from students in this course should have "MATH 5 Fall 2019 – Section 020: Topic" as the subject line. The word Topic should be replaced with the topic of the email. I cannot guarantee that I will read emails that do not have this label in the subject line. If at all possible, contact me in person before or after class, or in in-person or in online office hours. These are always the best way to reach me and I will have a large number of office hours. Emails to me should be reserved for questions or issues that cannot be resolved by your teaching assistant and for communication that is not possible outside of the above communication methods.

Textbook (not required)

PreCalculus by Miller 1E.

Class Location

Materials Science and Engineering, Room 116 (MSE 116).

Lecture Meeting Time

MWF 11:00am – 11:50am.

Discussion Section Time

TR – see the schedule of classes for your section information.

Office Hours

In-Person: Friday 12:00pm – 1:40pm and by appointment (Surge 233). You are always free to ask me questions after class.

Online: Wednesday 6:30pm-8:30pm at https://www.zoom.us/join where you will need to enter a room number that will be emailed to you prior to the login time.

Academic Honesty

Students caught cheating on an exam will automatically fail the exam and will be reported to the proper authority. You are not required to do your homework in isolation but you must write up your own solutions. Failure to submit your own work may result in disciplinary action by the University as dictated by the University guidelines regarding plagiarism.

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Overview of Graded Materials

Exams

Midterm 1: Monday, October 28th in MSE 116.

Midterm 2: Monday, November 18th in MSE 116.

Final Exam: Tuesday, December 10th, 8:00am - 11:00am in MSE 116.

The above exams consist of three parts each. The various parts test you on various aspects of learning as outlined by Bloom's Taxonomy. The first part comprises about 70% of the overall possible points on the exam. These questions are **Recall** type questions. While some calculation may be involved, these questions essentially test basic skills and knowledge. The second part comprises about 15% of the overall possible points on the exam. These questions are **Analysis** type questions. Significant calculation may be involved and these questions test student's ability to analyze problems that require multiple steps to solve. Simple recall of a fact or a computation technique is not enough to solve such problems. The third part comprises about 15% of the overall possible points on the exam. These questions are **Synthesis** type questions. Significant calculations may be involved. These questions test the student's ability to synthesize multiple ideas in solving a problem and to, possibly, come up with novel techniques based on prior knowledge. You will find a sample of practice exams on the course website and you will be able to familiarize yourself with the exam structure before you must take the exam.

It is expected that you will obtain full credit on your homework and quizzes. Together with these points, getting all Recall type questions correct will roughly correspond to a C+ and getting all Recall and Analysis type questions correct will roughly correspond to a high B+. Thus, learning at the level of Recall is what one would expect for a student scoring in the C range, learning at the level of Analysis is what one would expect for a student scoring in the B range, and learning at the level of Synthesis is what is required of a student to score A range.

Homework

Each week in discussion section, your TA will have you begin working on a worksheet in your discussion section in a collaborative learning group. You will get credit for showing up to the discussion section and participating in the group to solve the worksheet problems. One week later, you will turn in the worksheet completed as a homework assignment. You will earn further credit for your work on the assignment, one point for completion and one point for correctness.

Worksheets are scored out of three points – one for class participation, one for completion a week after the discussion section when it is given, and one for correctness. Correctness is given if approximately 75% of your questions are correct, so if about three out of every four problems are correctly answered. You do not have to do your work in isolation and we encourage you strongly to work in your groups. Your write-ups, however, must be your own.

You have a long list of examples and exercises in this information packet. You are expected to do all of these problems, but will not be directly graded on their completion.

Quizzes

We will administer a large number of quizzes during lecture. The quiz questions will come from the examples and exercises given in this packet. You will earn full credit for attempting a quiz question and extra credit for correct answers. We will be using the crowdmark system to administer quizzes and you will need to learn the system. You will need to bring your phone to class and have it charged as you will use your phone to scan your own quiz. Please let me know immediately if you do not have a smart phone or if you are unable to use a phone to take a picture of your work and upload it in class.

Extra Credit Opportunities

You will earn extra credit for correctly answering quiz questions. Quizzes questions are worth one point each, but you will obtain 1.2 points on the quiz instead of 1 point if your answer is correct. Quizzes are worth 10% of your grade, so you can earn up to 2% extra credit points in the class. You also have the opportunity to turn in two extra worksheets, Worksheet 11 and Worksheet 12, for extra credit. They will each count for one percentage point of credit towards your final course grade. Worksheet 11 contains material we cover in class while Worksheet 12 covers material that is slightly beyond the course content. Example questions for this worksheet are included in this packet under the exercises for IV.4.4.

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Grading Policy

Students must take the final exam in order to pass the course. Regardless of your performance in the course, you must earn a passing grade on the Final exam in order to pass the class. A passing grade on the final exam is 55%. There are 1000 possible points in the class. Barring unforeseen issues and up to minor modification, we will use the standard 90-80-70 grading scale with which most of you are well acquainted, with ± cutoffs at the ±3% levels.

Scale 1:

Worksheets: 200 points.
Quizzes: 100 points.
Midterm 1: 125 points.
Midterm 2: 125 points.
Final Examination: 450 points.

Scale 2:

Worksheets: 200 points.
Quizzes: 100 points.
Best of 2 Midterms: 125 points.
Final Examination: 575 points.

score $\geq 97\%$	is an A+
$93\% \leq$ score $< 97\%$	is an A
$90\% \leq$ score $< 93\%$	is an A-
$87\% \leq$ score $< 90\%$	is an B+
$83\% \leq$ score $< 87\%$	is an B
$80\% \leq$ score $< 83\%$	is an B-
$77\% \leq$ score $< 80\%$	is an C+
$73\% \leq$ score $< 77\%$	is an C
$70\% \leq$ score $< 73\%$	is an C-
$67\% \leq$ score $< 70\%$	is an D+
$63\% \leq$ score $< 67\%$	is an D
$60\% \leq$ score $< 63\%$	is an D-
score $< 60\%$	is an F.

Students are expected to take both midterms. However, the two grading scales permit students to miss an exam due to unavoidable circumstances. For this reason and for the sake of fairness to other students, make-up exams are not given.

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How to Contest Grades

Contesting grades is a formal process that must be formally documented. No contesting of grades in person is permitted as this potentially leads to errors and misunderstandings that affect both students and instructors negatively.

You have one week from the time the exam or assignment is returned to contest your grade on the exam or assignment. You must write a letter that specifies exactly where you believe you were incorrectly graded, that explains what the correct solution is, and that carefully describes to me what you believe the grading error to be. You must also write in your letter what you believe your correct score should be and why. You must email the letter to me and I will email you a response. I will record, as you should, any such communication for my records.

If you wish to contest your final course grade, you must do so before the end of the first week of the following quarter.

When we review your graded material, we will regrade the entire exam or assignment. If you were graded incorrectly on other problems in such a way that it benefitted your grade, you may lose points on those questions. We will not, however, be punitive when regrading. You are encouraged to come to us if there are grading errors and your total grade on the exam or assignment will not be lowered on submitting a regrade.

If you fail to submit a regrade in a timely manner (one week after the material is scored for all materials other than the final exam and by the end of the first week of the following quarter for final exams), then you have forfeited your right to a regrade.

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Advice

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A Path to Success

Step 1. Understand that no one can teach you mathematics and that we are here only to support and guide you. There is no quick way and no easy path. You must teach yourself, for learning is a deeply personal process and is ultimately the responsibility of the student.

Step 2. Come to class every day. We will review many examples and discuss theory that will help you learn the material. The practice will help you to solve test questions as well.

Step 3. Complete the exercises for a given section immediately after we cover the section in lecture and certainly before the next lecture. This way, you can ask questions in the subsequent lecture. It will also ensure that you are not falling behind.

Step 4. Review the lecture slides before lecture. You need not carefully study the slides, even a cursory review will be very helpful.

Step 5. Attend all of the discussion sections, especially the worksheet sessions.

Step 6. Review the exercises before taking the actual exams.

Step 7. After taking the midterm exams, work through them again carefully so that you know how to do every problem.

Step 8. Attend office hours and make appointments with me or your TA if ever you have questions, would like more practice, or would like to dig deeper into the material.

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Principles of Problem Solving

Principle 1. Know the Problem. Be able to clearly state the problem and name every relevant variable so that you can properly refer to them. You must understand every definition involved in the problem and every fact that arises in the description of the problem. You must also try to see where the problem fits into the flow of ideas of the subject.

Principle 2. Don't Change Anything. If you have an expression that you wish to write in a different way, make sure that no manipulation you perform changes its value. If you have an equation that you wish to solve, make sure that no manipulation you perform violates the equality. If you have an inequality that you wish to solve, make sure that no manipulation you perform violates the inequality.

Principle 3. Experiment. Plug things in and try things out. This helps you understand what is going on in the problem. Simplify the problem if need be and solve the simpler problem instead. This will often guide you to the solution of the more difficult problem.

Principle 4. Find Relationships. Find relationships between the unknown quantities. Try to find useful relationships. Look first for the simplest relationships that you can find.

Principle 5. Be Critical. Look back at your solution and see if it works. If you can't check your solutions then at least try to find holes in your arguments.

Principle 6. Reflect and Remember. Remember the problems you have solved. Reflect on them. Ask yourself how you got them correct, what mistakes you made in trying to solve them, what things you could try next time, what consequences follow from their solutions, and what other problems should have similar solutions.

Principle 7. Create. Create new problems like the one you have just seen. Understand what goes into the creation of the question. Is the question artificial or natural? If artificial, what do you need to do to create other questions like it? If it is natural, where do other such questions come from? Either way, can you generalize the question into a class of similar questions?

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Pitfalls to Avoid

Pitfall 1. Rush. Start trying to solve a problem without knowing exactly the statement or idea behind the problem. This is not a race. Your goal should not be to solve the problem instantly, it should be to take the smallest steps that you can possibly take until the solution reveals itself to you. You should view solving a problem like taking a journey and make sure to see the sights along the way. The destination is always less important than the exploration.

Pitfall 2. Trivialize. Confuse knowing an algorithm, a trick, a formula, or even a definition for understanding an idea. Knowing how to get an answer has nothing to do with understanding an idea and often little to do with knowing how to get the right answer. All formulas and algorithms can be broken and if you do not understand the underlying idea, you cannot see when or why they will break.

Pitfall 3. Attempt to Please. Do not let the goal of pleasing your instructor guide your problem solving. Avoid asking the question: What am I supposed to do? You are not doing this to make anyone happy or follow any rule, you are trying to solve a problem.

Pitfall 4. Accept Prematurely. Fail to check to see if your solution is reasonable or to check your solution for correctness when you can. You must understand why your answer is reasonable.

Pitfall 5. Remain a Student. When you understand an idea, you are no longer a student, you are a teacher. If you cannot act as the teacher and explain the idea to someone else, you have not understood. You must never be satisfied with your answer if you cannot clearly and succinctly explain why your answer is correct to someone who knows *less* than you know.

Pitfall 6. Become discouraged. You try to learn something and find it takes a long time. You think you understand and then find yourself unable to explain the idea to someone else. You work a problem that you believe you know how to solve and then find you cannot actually solve it, or worse, you believe that you have solved it and find that your answer is incorrect. These are terribly discouraging "setbacks." The mistake is to view these as setbacks. They are not. They are simply a part of the learning process, a difficult process that takes a lot of time and a lot of patience. Work hard, try hard, judge yourselves honestly, but also remember that we're all just human and the troubles we face are not unique, they are shared by everyone who embarks on this path. Mathematics is a very difficult subject, but you can learn it well. It takes hard work and patience, honesty and humility, and always a good measure of curiosity to drive yourself forward. There is no point in beating yourself up for taking the time you need to learn a difficult idea. Just relax and enjoy the process of discovery!

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Topics

See the course website for the posted lecture notes for each section.

IV.2.7. Survey Problems with Implicit Information
IV.2.8. Modeling the Superposition of Waves – Not covered in the course.

IV.3. Exponential and Logarithmic Functions
IV.3.1. Properties of Exponential Functions
IV.3.2. The Logarithm and Its Properties
IV.3.3. Symmetric Change
IV.3.4. Tuning a Guitar – Not covered in the course.
IV.3.5. The Natural Exponential and Logarithm
IV.3.6. Exponential Growth and Decay

IV.4. Tangency
IV.4.1. Intersections of Lines and Quadratic Curves
IV.4.2. The Telescope – Covered in extra credit worksheet but omitted from class lectures.
IV.4.3. Intersections of Lines and Higher Order Polynomial Curves
IV.4.4. Intersections of Lines with Rational Curves – Covered in extra credit worksheet but omitted from course.
IV.4.5. Symmetry of Tangency under Reflection – Covered in extra credit worksheet but omitted from course.

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Calendar

The date given to the right of the week is the date of the first Monday of the week.

- Week 0 September 23

- Week 1 September 30
- Week 2 October 7
- Week 3 October 14
- Week 4 October 21
- Week 5 October 28
- Week 6 November 4
- Week 7 November 11
- Week 8 November 18
- Week 9 November 25
- Week 10 December 2

- Finals Week December 9

Week 0 – Week of Monday, September 23

Monday	Tuesday	Wednesday	Thursday	Friday
September 23	September 24	September 25	September 26	September 27
				○ I.1.1 ○ I.1.2 ○ I.2.1

Week 1 – Week of Monday, September 30

Monday	Tuesday	Wednesday	Thursday	Friday
September 30	October 1	October 2	October 3	October 4
○ I.2.2 ○ I.2.3		○ I.2.4 ○ I.2.5		○ I.2.6 ○ I.2.7

First discussion section: ○ Worksheet 1

Week 2 – Week of Monday, October 7

Monday	Tuesday	Wednesday	Thursday	Friday
October 7	October 8	October 9	October 10	October 11
○ I.3.1 Leeway		○ II.1.1 ○ II.1.2		○ II.1.3 ○ II.1.4 ○ II.2.1

First discussion section: ○ Worksheet 2

Week 3 – Week of Monday, October 14

Monday	Tuesday	Wednesday	Thursday	Friday
October 14	October 15	October 16	October 17	October 18
○ II.2.2 ○ II.2.3		○ II.2.4 ○ II.2.5		○ II.2.6 ○ II.2.7

First discussion section: ○ Worksheet 3

Week 4 – Week of Monday, October 21

Monday	Tuesday	Wednesday	Thursday	Friday
October 21	October 22	October 23	October 24	October 25
○ II.3.1 ○ II.3.2		○ II.3.3 ○ II.4.1 ○ II.4.2		○ II.4.3 ○ II.4.4

First discussion section: ○ Worksheet 4

Week 5 – Week of Monday, October 28

Monday	Tuesday	Wednesday	Thursday	Friday
October 28	October 29	October 30	October 31	November 1
Midterm 1		○ II.5.1 ○ II.5.2 ○ III.1.1		○ III.1.2 ○ III.1.3

First discussion section: ○ Worksheet 5

Week 6 – Week of Monday, November 4

Monday	Tuesday	Wednesday	Thursday	Friday
November 4	November 5	November 6	November 7	November 8
○ III.1.4 ○ III.2.1		○ III.2.2 ○ III.2.3		○ III.2.4 ○ IV.1.1

First discussion section: ○ Worksheet 6

Week 7 – Week of Monday, November 11

Monday	Tuesday	Wednesday	Thursday	Friday
November 11	November 12	November 13	November 14	November 15
Veteran's Day		○ IV.2.1 ○ IV.2.2		○ IV.2.3 ○ IV.2.4

First discussion section: ○ Worksheet 7

Week 8 – Week of Monday, November 18

Monday	Tuesday	Wednesday	Thursday	Friday
November 18	November 19	November 20	November 21	November 22
Midterm 2		○ IV.2.5 ○ IV.2.6		○ IV.2.7 ○ IV.3.1

First discussion section: ○ Worksheet 8

Week 9 – Week of Monday, November 25

Monday	Tuesday	Wednesday	Thursday	Friday
November 25	November 26	November 27	November 28	November 29
○ IV.3.2 ○ IV.3.3		○ IV.3.5 ○ IV.3.6	Thanksgiving	Thanksgiving

First discussion section: ○ Worksheet 9

Week 10 – Week of Monday, December 2

Monday	Tuesday	Wednesday	Thursday	Friday
December 2	December 3	December 4	December 5	December 6
○ IV.4.1 ○ IV.4.3		Review		Review

First discussion section: ○ Worksheet 10

Finals Week – Week of Monday, December 9

Monday	Tuesday	Wednesday	Thursday	Friday
December 9	December 10	December 11	December 12	December 13
	Final Exam 8:00am–11:00am			

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Exercises and Examples

● I. Decomposition

II. Transformations

- IV. Symmetries

IV.1 Parity
- IV.1.1 Odd and Even Functions

IV.2 Trigonometric Functions
- IV.2.1 Periodicity
- IV.2.2 The Sine, Cosine, and Tangent Functions
- IV.2.3 Angle Addition Formulae for Trigonometric Functions
- IV.2.4 Transformations of Trigonometric Functions
- IV.2.5 Inverse Trigonometric Functions
- IV.2.6 Equations Involving Trigonometric Functions
- IV.2.7 Survey Problems with Implicit Information
- IV.2.8 Modeling the Superposition of Waves

IV.3 Exponential and Logarithmic Functions
- IV.3.1 Properties of Exponential Functions
- IV.3.2 The Logarithm and Its Properties
- IV.3.3 Symmetric Change
- IV.3.4 Tuning a Guitar
- IV.3.5 The Natural Exponential and Logarithm
- IV.3.6 Exponential Growth and Decay

IV.4 Tangency
- IV.4.1 Intersections of Lines and Quadratic Curves
- IV.4.2 The Telescope
- IV.4.3 Intersections of Lines and Higher Order Polynomial Curves
- IV.4.4 Intersections of Lines with Rational Curves
- IV.4.5 Symmetry of Tangency under Reflection

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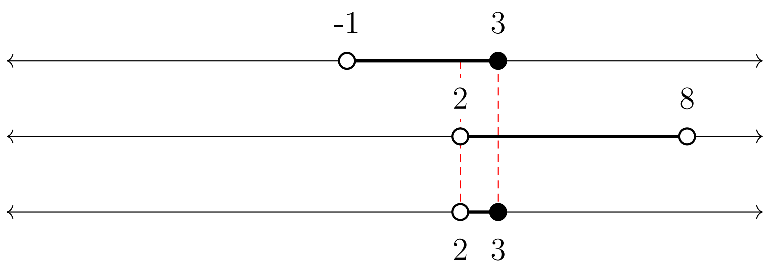
I.1.1 Unions and Intersections of Intervals

Example 1. Write in interval notation the set $(-1,3] \cap (2,8)$.

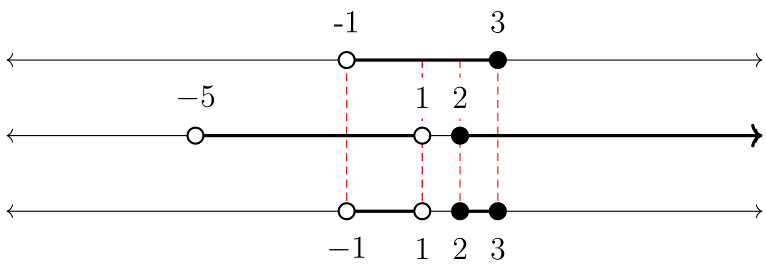
Solution



The intersection is $(2,3]$, the overlap of the shaded segments of the top two lines if the lines are placed on top of each other.

Example 2. Write in interval notation the set $(-1,3] \cap \Big((-5,1) \cup [2,\infty)\Big)$.

Solution



The intersection is the union of intervals $(-1,1) \cup [2,3]$, the overlap of the shaded segments of the top two lines if the lines are placed on top of each other.

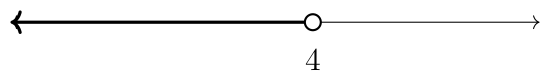
Problem 1. Graph on a number line the following sets and describe them using interval notation:

(a) $(-1,7] \cap (3,10)$
(b) $[1,3] \cup (5,7]$.
(c) $(-2,7] \cup (5,\infty)$
(d) $(-\infty,5) \cap (2,\infty)$
(e) $(-2,7] \cap \Big([-5,0) \cup [3,10)\Big)$.

Example 3. Graph on a number line the solutions to the inequality

$$x < 4.$$

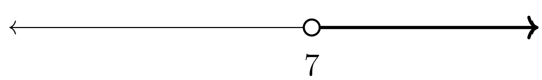
Solution



Example 4. Graph on a number line the solutions to the inequality

$$x > 7.$$

Solution



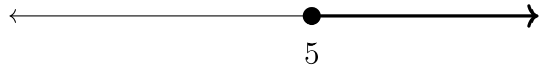
Problem 2. Graph on the real line all points x such that:

(a) $x \geq 3$
(b) $x < 2$.

Example 5. Graph on a number line the set of all points satisfying the inequalities

$$x > 3 \quad \text{and} \quad x \geq 5.$$

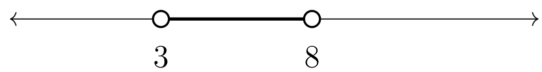
<u>SOLUTION</u>



Example 6. Graph on a number line the set of all points satisfying the inequalities

$$x > 3 \quad \text{and} \quad x < 8.$$

<u>SOLUTION</u>



Problem 3. Graph on the real line all points x such that

$$x \geq 2 \ \text{ and } \ x < 5.$$

Problem 4. Graph on the real line all points x such that

$$x \geq 1 \ \text{ and } \ x > 5.$$

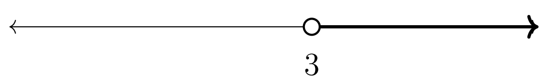
Problem 5. Graph on the real line all points x such that

$$x \geq 2 \ \text{ and } \ x < -7.$$

Example 7. Graph on a number line the set of all points satisfying the inequalities

$$x > 3 \quad \text{or} \quad x \geq 5.$$

<u>SOLUTION</u>



Example 8. Graph on a number line the set of all points satisfying the inequalities

$$x > 3 \quad \text{or} \quad x \leq -4.$$

Solution

Problem 6. Graph on the real line all points x such that

$$x \geq 1 \text{ or } x > 5.$$

Problem 7. Graph on the real line all points x such that

$$x \geq 2 \text{ or } x < -7.$$

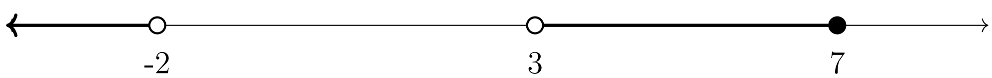
Example 9. Graph on a number line all real values x with the property that either

$$x < -2 \quad \textbf{and} \quad x \leq 0,$$

OR

$$x > 3 \quad \textbf{and} \quad x \leq 7.$$

Solution



Problem 8. Graph on a number line all real values x with the property that either

$$x > -3 \quad \textbf{and} \quad x \leq 1,$$

OR

$$x > 5 \quad \textbf{and} \quad x \leq 8.$$

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I.1.2 Multiple Linear Inequalities

Example 1. Suppose that $a < 0$. Find all x so that

$$ax + 1 < 4.$$

SOLUTION

Since a is negative, we have that

$$ax + 1 < 4 \quad \text{if and only if} \quad ax < 3 \quad \text{if and only if} \quad x > \frac{3}{a}.$$

Problem 1. Graph on the real line all points x such that
(a) $2x - 1 > -5$,
(b) $3x + 2 \leq 20$

Example 2. Write in interval notation the set of all x satisfying

$$2x < 5 \quad \text{and} \quad 3x + 1 \geq -4.$$

SOLUTION

Solving both inequalities, we have that

$$x < \frac{5}{2} \quad \text{and} \quad x \geq -\frac{5}{3}.$$

So the solution set is the set $\left[-\frac{5}{3}, \frac{5}{2}\right)$.

Problem 2. Graph on the real line all points x such that
(a) $2x + 1 > 7$ and $x \geq 5$,
(b) $x + 3 < 5$ or $x - 1 \geq 6$.

Problem 3. Write in interval notation the set of all x satisfying

$$2x < 5 \quad \text{and} \quad 3x + 1 \geq -4.$$

Problem 4. Graph on the real line all points x such that

$$(1)\ \ 4x - 2 \leq 6 \ \text{ and } \ 3 - x \leq 7,$$

or

$$(2)\ \ 3x > 18 \ \text{ and } \ 2x - 9 \geq 11.$$

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I.2.1 Relations, Functions, and Cartesian Products

Example 1. Suppose that

$$X = \{1, 2, 3, 4\} \quad \text{and} \quad Y = \{a, b, c\}.$$

Write down all elements of the set $X \times Y$. If r is a relation in $X \times Y$ given by

$$r = \{(1, a), (1, b), (2, b)\},$$

then what is the natural domain of r, the co-domain of r, the domain of r, and the range of r? Is r a function?

Solution

The cartesian product $X \times Y$ is the set

$$X \times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}.$$

The natural domain of r is X. The co-domain of r is Y. The domain of r is $\{1, 2\}$. The range of r is $\{a, b\}$. The relation r is not a function because 1 appears as the first element of two distinct ordered pairs in r.

Problem 1. Suppose that

$$X = \{x, y, z\} \quad \text{and} \quad Y = \{a, b\}.$$

Write down all elements of the set $X \times Y$.

Problem 2. Suppose that

$$X = \{a, b, c, d, e, f, g\}.$$

Suppose that the relation r is a subset of $\mathbb{N} \times X$ and is defined by

$$r = \{(1, a), (1, c), (2, b), (10, a), (10, e), (10, f), (12, b)\}.$$

What is the natural domain of r, the co-domain of r, the domain of r, and the range of r?

Problem 3. Suppose that

$$X = \{a, b, c, d, e, f, g, h\} \quad \text{and} \quad Y = \{1, 2, 3, 4, 5\}.$$

Suppose that

$$f = \{(a, 1), (b, 4), (c, 5), (a, 3)\}.$$

Is f a function from X to Y? If so, what is its domain and its range?

Problem 4. Suppose that

$$X = \{a, b, c, d, e, f, g, h\} \quad \text{and} \quad Y = \{1, 2, 3, 4, 5\}.$$

Suppose that

$$f = \{(a, 1), (b, 4), (c, 4), (d, 3)\}.$$

Is f a function from X to Y? If so, what is its domain and its range?

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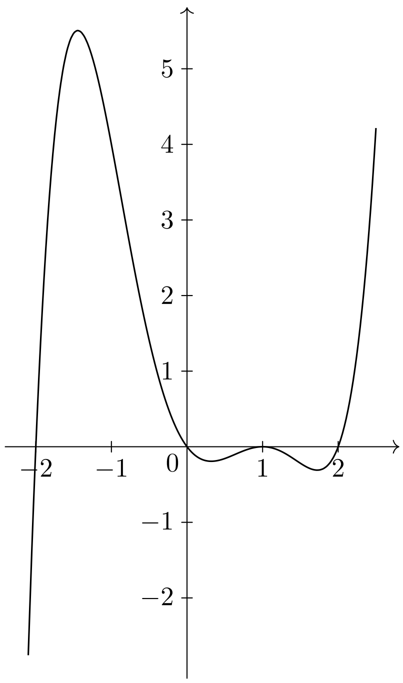
I.2.2 Comparing Functions

Example 1. Let f be the function whose graph is given below.



(a) Write in interval notation and then graph on a number line all x such that $f(x) > 0$.

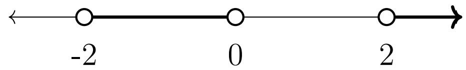
(b) Write in interval notation and then graph on a number line all x such that $f(x) \geq 0$.

(c) Write in interval notation and then graph on a number line all x such that $f(x) < 0$.

(d) Write in interval notation and then graph on a number line all x such that $f(x) \leq 0$.

Solution

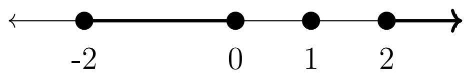
(a) Observe where the graph is above the x-axis to obtain the intervals below.



The set of all x where $f(x)$ is strictly greater than zero is the union of intervals $(-2, 0) \cup (2, \infty)$.

(b) Write in interval notation and then graph on a number line all x such that $f(x) \geq 0$.

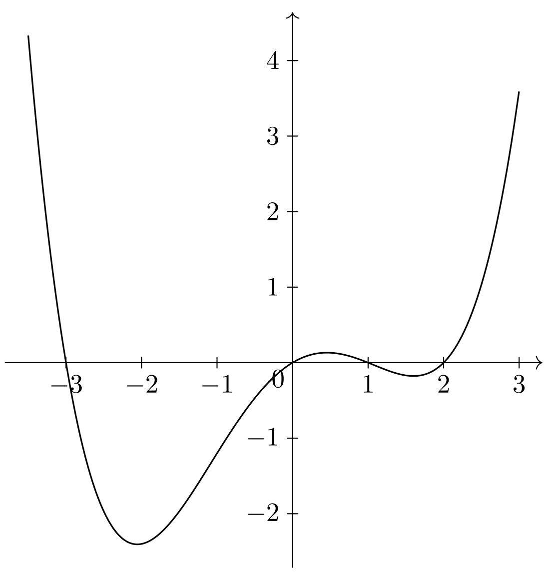
Observe where the graph is above or on the x-axis to obtain the intervals below.



The set of all x where $f(x)$ is strictly greater than zero is the union of intervals $[-2, 0] \cup \{1\} \cup [2, \infty)$. Note that it is perfectly legitimate to write $[1, 1]$ rather than $\{1\}$ as both are sets containing only the number 1.

Solve (c) and (d) in a similar way.

Problem 1. Let f be the function whose graph is given below.

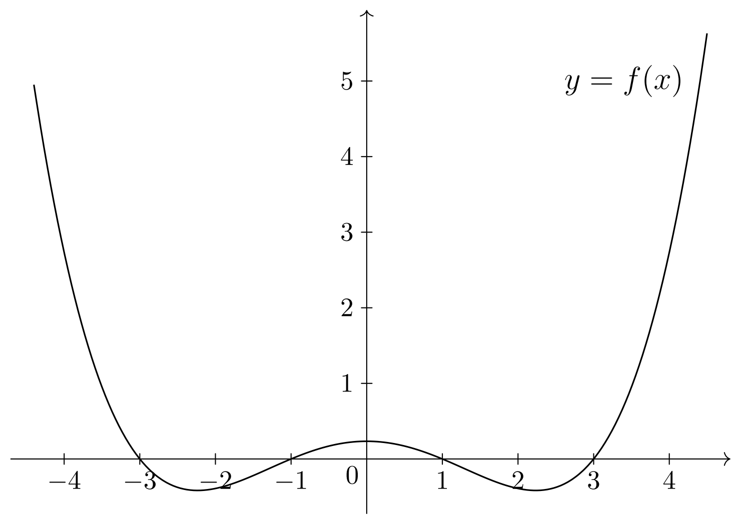


(a) Write in interval notation and then graph on a number line all x such that $f(x) > 0$.

(b) Write in interval notation and then graph on a number line all x such that $f(x) \geq 0$.

(c) Write in interval notation and then graph on a number line all x such that $f(x) < 0$.
(d) Write in interval notation and then graph on a number line all x such that $f(x) \leq 0$.

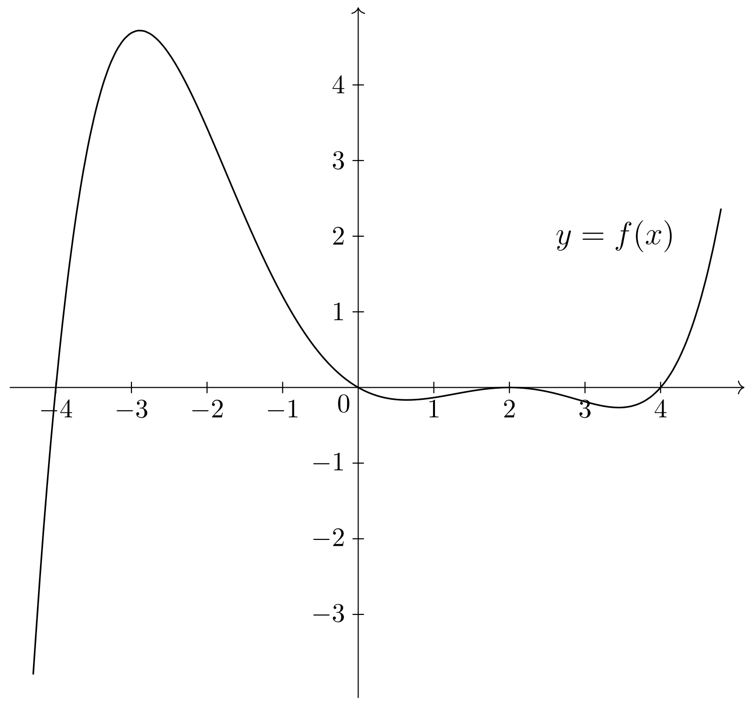
Problem 2. Let f be the function whose graph is given below.



Write in interval notation and then graph on a number line all x such that

(a) $f(x) > 0$,
(b) $f(x) \geq 0$,
(c) $f(x) < 0$,
(d) $f(x) \leq 0$.

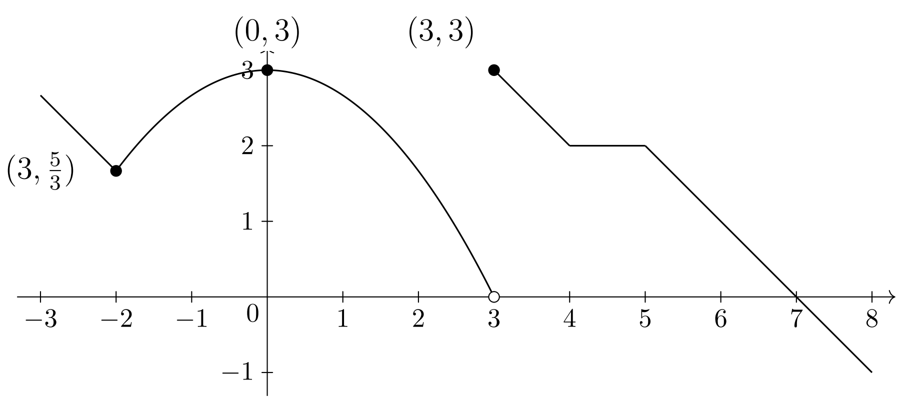
Problem 3. Let f be the function whose graph is given below.



Write in interval notation and then graph on a number line all x such that

(a) $f(x) > 0$,
(b) $f(x) \geq 0$,
(c) $f(x) < 0$,
(d) $f(x) \leq 0$.

Problem 4. Consider the graph of the function, f, below.



(a) Find all monotone intervals, all strictly monotone intervals, and say respectively whether the function is increasing or decreasing, or strictly increasing or strictly decreasing on these intervals.

(b) Find all local maximums and local minimums.

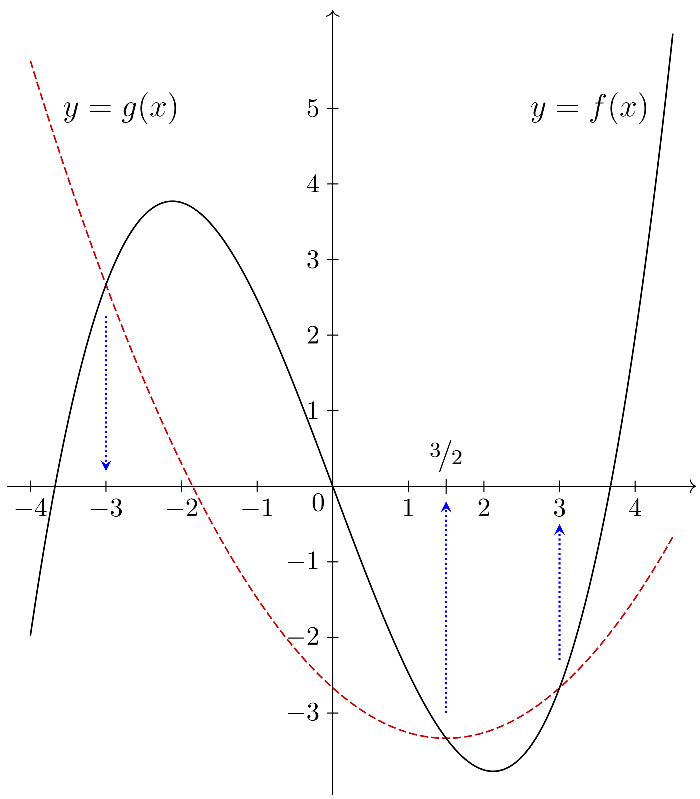
(c) Write in interval notation and then graph on a number line all x such that $f(x) \geq 0$.

(d) Write in interval notation and then graph on a number line all x such that $f(x) > 0$.

(e) Write in interval notation and then graph on a number line all x such that $f(x) \leq 0$.

(f) Write in interval notation and then graph on a number line all x such that $f(x) < 0$.

Problem 5. Consider the graphs of the functions f and g below.



Write in interval notation and then graph on a number line all x such that

(a) $f(x) > g(x)$,

(b) $f(x) \geq g(x)$,

(c) $f(x) < g(x)$,

(d) $f(x) \leq g(x)$.

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I.2.3 An Elementary Library of Functions

Example 1. What is the slope of the line

$$y = 2x + 1?$$

SOLUTION

$$2$$

Example 2. What is the y-intercept of the line

$$y = 7x - 4?$$

SOLUTION

Set x equal to zero to obtain the value for y, namely, -4. The line crosses the y-axis at the height -4.

Example 3. What is the x-intercept of the line

$$y = 4x - 2?$$

SOLUTION

Set

$$0 = 4x - 2$$

and solve for x to obtain

$$x = \frac{1}{2}.$$

The x intercept is therefore $\frac{1}{2}$.

Example 4. Line L passes through points $(1, 3)$ and $(4, 12)$. What is the slope of L?

SOLUTION

Denote by m the slope. Then

$$m = \frac{12-3}{4-1} = \frac{9}{3} = 3.$$

Example 5. Let L be the line containing points $(2,5)$ and $(6,7)$. Find an equation for L.

SOLUTION

The slope of L is

$$m = \frac{7-5}{6-2} = \frac{2}{4} = \frac{1}{2}.$$

Using the point/slope form of the line, we have that

$$y - 5 = \frac{1}{2}(x-2).$$

Problem 1. Find an equation for the line that intersects $(4,10)$ and $(7,2)$.

Problem 2. Find an equation for the line that intersects $(1,2)$ and $(5,8)$.

Problem 3. Find an equation for the line with slope 2 that intersects $(3,2)$. What is the y-intercept and x-intercept of the line?

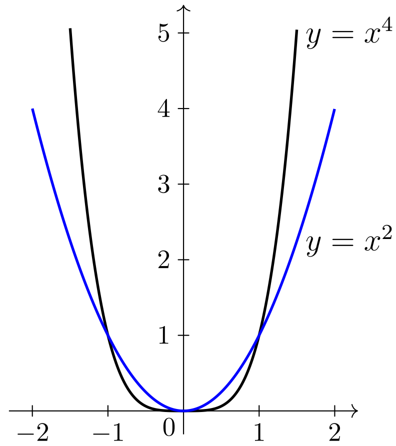
Problem 4. Suppose that f, g, and h are functions defined by

$$f(x) = 3x+2, \quad g(x) = \frac{x^3+2x+1}{x^2-x+2}, \quad \text{and} \quad h(x) = |x^2-10|.$$

Evaluate each function when x is two and then when x is four.

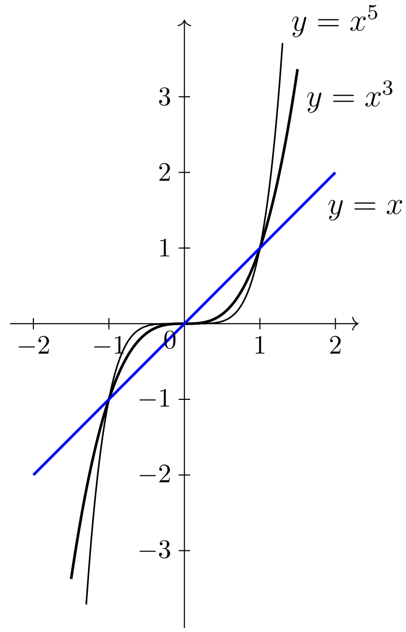
Example 6. Sketch the graphs given by $y = x^2$ and $y = x^4$ on the same coordinate axis.

Solution



Example 7. Sketch the graphs given by $y = x$, $y = x^3$, and $y = x^5$ on the same coordinate axis.

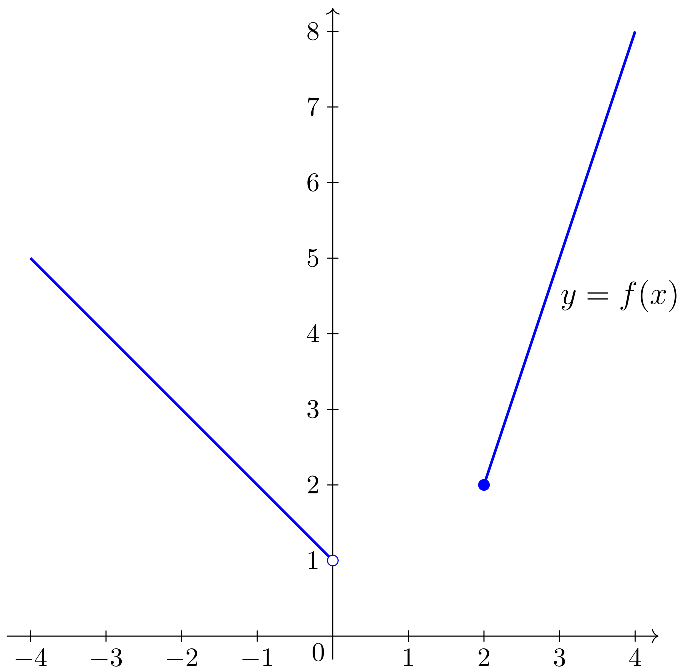
Solution



Example 8. Sketch the graph of f where f is given by

$$f(x) = \begin{cases} -x+1 & \text{if } x < 0 \\ 3x-4 & \text{if } x \geq 2. \end{cases}$$

SOLUTION



Problem 5. What is the domain and range of the function f given by

$$f(x) = \begin{cases} -2 & \text{if } x < -5 \\ x+1 & \text{if } 0 < x < 3? \end{cases}$$

Problem 6. Sketch the graph of f where f is given by

$$f(x) = \begin{cases} 2x-1 & \text{if } x < -1 \\ x^2 & \text{if } 0 < x \leq 2 \\ 3 & \text{if } 2 < x < 4. \end{cases}$$

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I.2.4 The Algebra of Functions

Example 1. Suppose that

$$f(2) = 4 \quad \text{and} \quad g(2) = 7.$$

Calculate (a) $(f+g)(2)$, (b) $(f \cdot g)(2)$, (c) $\dfrac{f}{g}(2)$, and (d) $\dfrac{g}{f}(2)$.

Solution

$$(f+g)(2) = f(2) + g(2) = 4 + 7 = 11.$$

$$(f \cdot g)(2) = f(2) \cdot g(2) = 4 \cdot 7 = 28.$$

$$\frac{f}{g}(2) = \frac{f(2)}{g(2)} = \frac{7}{4}.$$

$$\frac{g}{f}(2) = \frac{g(2)}{f(2)} = \frac{4}{7}.$$

Example 2. Suppose that f and g are given by

$$f(x) = 2x^2 - x + 1 \quad \text{and} \quad g(x) = x + h.$$

Compute $(f \cdot g)(x)$ and $\dfrac{f}{g}(x)$.

Solution

$$(f \cdot g)(x) = f(x) \cdot g(x) = (2x^2 - x + 1)(x + h).$$

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{2x^2 - x + 1}{x + h}.$$

Example 3. Suppose that f and g are given by

$$f(x) = x^3 + x + 2 \quad \text{and} \quad g(2) = 3.$$

Calculate $(f \circ g)(2)$.

Solution

$$(f \circ g)(2) = f(g(2)) = f(3) = (3)^3 + (3) + 2 = 32.$$

Example 4. Suppose that f is given by

$$f(x) = x^2 - 3x + 2.$$

Calculate $\dfrac{f(x+h) - f(x)}{h}$.

Solution

If h is not zero, then

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - 3(x+h) + 2 - (x^2 - 3x + 2)}{h} \\
&= \frac{x^2 + 2hx + h^2 - 3x - 3h + 2 - x^2 + 3x - 2}{h} \\
&= \frac{2hx - 3h + h^2}{h} = 2x - 3 + h.
\end{aligned}$$

Example 5. Suppose that f and g are given by

$$f(x) = x^2 + 2x + 1 \quad \text{and} \quad g(x) = \sqrt{x-4}.$$

Compute (a) $(f \cdot g)(x)$, (b) $\dfrac{f}{g}(x)$, (c) $\dfrac{g}{f}(x)$, and (d) $(f \circ g)(x)$.

Solution

Assume that all expressions are valid for the given x value. In this case, we have that

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) \\ &= \left(x^2 + 2x + 1\right) \cdot \sqrt{x-4},\end{aligned}$$

$$\begin{aligned}\frac{f}{g}(x) &= \frac{f(x)}{g(x)} \\ &= \frac{x^2 + 2x + 1}{\sqrt{x-4}},\end{aligned}$$

$$\begin{aligned}\frac{g}{f}(x) &= \frac{g(x)}{f(x)} \\ &= \frac{\sqrt{x-4}}{x^2 + 2x + 1}, \quad \text{and}\end{aligned}$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{x-4}) \\ &= (\sqrt{x-4})^2 + 2(\sqrt{x-4}) + 1 \\ &= x - 4 + 2(\sqrt{x-4}) + 1.\end{aligned}$$

Problem 1. Suppose that

$$f(x) = x^2 + 1 \quad \text{and} \quad g(x) = 3x - 1.$$

Write a formula for the following

(a) $(f + g)(x)$,

(b) $(f - g)(x)$,

(c) $(f \cdot g)(x)$,

(d) $\left(\frac{f}{g}\right)(x)$,

(e) $(f \circ g)(x)$,

(f) $(g \circ f)(x)$.

Problem 2. Suppose that

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = \frac{x-1}{x^2+1}.$$

Write a formula for the following

(a) $(f+g)(x)$,

(b) $(f-g)(x)$,

(c) $(f \cdot g)(x)$,

(d) $\left(\frac{f}{g}\right)(x)$,

(e) $(f \circ g)(x)$,

(f) $(g \circ f)(x)$.

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I.2.5 Decomposing Function

Example 1. Suppose that f and g are given by

$$f(x) = x^2 \quad \text{and} \quad g(x) = \sqrt{x-2}.$$

What is the domain of $(f \circ g)(x)$?

Solution

$$x \geq 2$$

Example 2. Suppose that f and g are given by

$$f(x) = \frac{1}{x-1} \quad \text{and} \quad g(x) = \sqrt{x}.$$

What is the domain of $(f \circ g)(x)$?

Solution

For $g(x)$ to be defined,

$$x \geq 0.$$

For f to be defined, $g(x)$ must not equal one. Therefore, $f \circ g$ is defined on $[0, 1) \cup (1, \infty)$.

Example 3. Suppose that f and g are given by

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = x.$$

What is the domain of $(f \cdot g)(x)$?

Solution

$$(-\infty, 0) \cup (0, \infty)$$

Example 4. Suppose that f and g are given by

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = x.$$

What is the domain of $(f \cdot g)(x)$?

Solution

$$(-\infty, 0) \cup (0, \infty)$$

Problem 1. Suppose that f is given by

$$f(x) = \sqrt{2x - 5}.$$

What is the domain and range of f?

Problem 2. Suppose that f is given by

$$f(x) = \frac{1}{\sqrt{x - 5}}.$$

What is the domain and range of f?

Problem 3. Find the domain of f where $f(x) = \sqrt{x - 2} + \sqrt{9 - x}$.

Problem 4. Find the domain of f where $f(x) = \dfrac{\sqrt{x + 4}}{x - 5}$.

Example 5. Let f be the function defined by

$$f(x) = \frac{2x + 7}{x - 1}.$$

What is the range and domain of f?

Solution

The function f can be evaluated at any real number not equal to 1 and the f cannot be evaluated at 1. Therefore, the domain of f is $(-\infty, 1) \cup (1, \infty)$.

Suppose that a is in the range of f. In this case, there is an x in the domain of f such that

$$a = \frac{2x+7}{x-1}.$$

Multiply both sides of the above equation by $(x-1)$ to obtain the equality

$$a(x-1) = 2x+7,$$

which implies that

$$x = \frac{a+7}{a-2}.$$

As long as a is not equal to 2, there is a value x such that $f(x)$ equals a. The range of f is therefore equal to $(-\infty, 2) \cup (2, \infty)$.

Problem 5. Let f be the function defined by

$$f(x) = \frac{x-5}{2x+1}.$$

What is the range and domain of f?

Problem 6. What is the domain and range of the function f given by

$$f(x) = \frac{2x-1}{x+5}?$$

Example 6. Suppose that f is given by

$$f(x) = x^2\sqrt{4+(x-1)^4} + (x+2)^2.$$

Write $f(x)$ in terms of sums, differences, products and compositions of the functions

$$a(x) = x^2, \quad b(x) = \sqrt{x}, \quad c(x) = x^4, \quad d(x) = x+4, \quad e(x) = x-1, \quad \text{and} \quad g(x) = x+2.$$

Solution

$$f = a \cdot (b \circ (d \circ c \circ e)) + a \circ g.$$

Problem 7. Write the function f given by

$$f(x) = x^3\sqrt{1 + 2\sqrt{x^2+1}} + x^4 + 1$$

as sums, products, and compositions of the functions a, b, c, d, and e given by

$$a(x) = x^3, \quad b(x) = \sqrt{x}, \quad c(x) = x+1, \quad d(x) = 2x, \quad \text{and} \quad e(x) = x^2.$$

For the remaining problems, recall that if a and b are real numbers, then the functions S_a (scale by a) and T_b (shift by b) are defined by

$$S_a(x) = ax \quad \text{and} \quad T_b(x) = x + b.$$

Problem 8. Suppose that f is given by

$$f(x) = x^2.$$

Suppose that $g = T_1 \circ S_3 \circ f \circ S_7 \circ T_4$. Write $g(x)$ in as simplified a form as possible.

Problem 9. Let f be the function given by

$$f(x) = x^2.$$

Let g be the function given by

$$g(x) = 5(x+2)^2 - 7.$$

Find values for a, b, and c so that

$$f = T_a \circ S_b \circ f \circ T_c.$$

Problem 10. Suppose that f is given by

$$f(x) = \frac{1}{x}.$$

Write $g = T_3 \circ S_5 \circ f \circ S_2 \circ T_1$ as a quotient of two functions.

Problem 11. Suppose that f is given by

$$f(x) = \frac{1}{x}.$$

Write $g = T_d \circ S_c \circ f \circ S_b \circ T_a$ as a quotient of two functions. Use this expressions to write

$$h(x) = \frac{10x+1}{5x-3}$$

in the form in which g is written.

Problem 12. Let f be the function given by

$$f(x) = \frac{1}{x}.$$

Let g be the function given by

$$g(x) = 3 + \frac{2}{x-7}.$$

Find values for a, b, and c so that

$$f = T_a \circ S_b \circ f \circ T_c.$$

Problem 13. Find real numbers a, b, c, and d so that the function g given by

$$g(x) = \frac{2x^3 + 3x^2 + x - 3}{x-2}$$

is equivalently given by

$$g(x) = ax^2 + bx + c + \frac{d}{x-2}.$$

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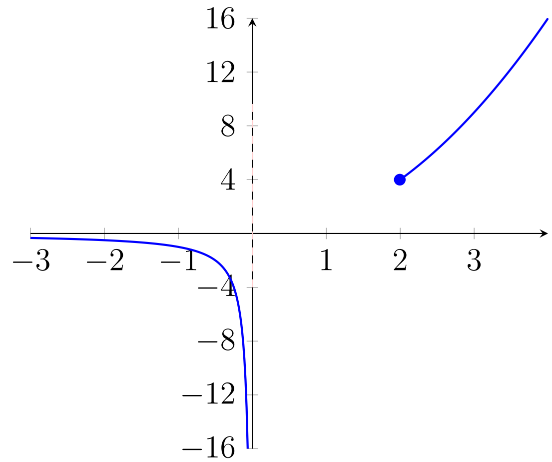
I.2.6 Piecewise and Compound Piecewise Defined Functions

Example 1. Sketch the graph of f where f is given by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 0 \\ x^2 & \text{if } x \geq 2. \end{cases}$$

SOLUTION



Example 2. Write the function f given by

$$f(x) = |2x + 3|$$

as a piecewise defined function.

SOLUTION

$$\begin{aligned} f(x) &= |2x+3| \\ &= \begin{cases} -(2x+3) & \text{if } 2x+3 < 0 \\ (2x+3) & \text{if } 2x+3 \geq 0 \end{cases} \\ &= \begin{cases} -(2x+3) & \text{if } x < -\frac{3}{2} \\ (2x+3) & \text{if } x \geq -\frac{3}{2}. \end{cases} \end{aligned}$$

Problem 1. Let f be the function defined by

$$f(x) = \begin{cases} -2 & \text{if } \ x < -5 \\ x^2 + 1 & \text{if } \ 0 < x < 3 \\ x - 4 & \text{if } \ x \geq 8. \end{cases}$$

Calculate $f(-6)$, $f(1)$, and $f(10)$.

Problem 2. Sketch the graph of the function f given by

$$f(x) = \begin{cases} 1 & \text{if } \ x < -2 \\ x^3 & \text{if } \ -2 \leq x \leq 2 \\ x + 1 & \text{if } \ x \geq 3. \end{cases}$$

Problem 3. Suppose that

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4x - 7 & \text{if } x \geq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 3x + 5 & \text{if } x \geq 3. \end{cases}$$

Write $f + g$ as a piecewise defined function.

Problem 4. Write the function f given by

$$f(x) = |x + 2|$$

as a piecewise defined function.

Example 3. Write the function f given by

$$f(x) = |x + 3| + |x + 1|$$

as a piecewise defined function.

Solution

$$f(x) = |x+3| = \begin{cases} -(x+3) & \text{if } x+3 < 0 \\ (x+3) & \text{if } x+3 \geq 0 \end{cases} = \begin{cases} -(x+3) & \text{if } x < -3 \\ (x+3) & \text{if } x \geq -3. \end{cases}$$

$$g(x) = |x+1| = \begin{cases} -(x+1) & \text{if } x+1 < 0 \\ (x+1) & \text{if } x+1 \geq 0 \end{cases} = \begin{cases} -(x+1) & \text{if } x < -1 \\ (x+1) & \text{if } x \geq -1. \end{cases}$$

If

$$x < -3 \quad \text{and} \quad x < -1, \quad \text{then} \quad x < -3.$$

There are no real numbers x with

$$x < -3 \quad \text{and} \quad x \geq -1.$$

If

$$x \geq -3 \quad \text{and} \quad x < -1, \quad \text{then} \quad -3 \leq x < -1.$$

If

$$x \geq -3 \quad \text{and} \quad x \geq -1, \quad \text{then} \quad x \geq -1.$$

By considering the functions f and g on the intersection of the intervals where each is determined by a single formula, we see that

$$f(x) = \begin{cases} -(x+3) & \text{if } x < -3 \\ (x+3) & \text{if } -3 \leq x < -1 \\ (x+3) & \text{if } x \geq -1. \end{cases} \qquad g(x) = \begin{cases} -(x+1) & \text{if } x < -3 \\ -(x+1) & \text{if } -3 \leq x < -1 \\ (x+1) & \text{if } x \geq -1. \end{cases}$$

We therefore have that

$$f(x) + g(x) = \begin{cases} -(x+3) - (x+1) & \text{if } x < -3 \\ (x+3) - (x+1) & \text{if } -3 \leq x < -1 \\ (x+3) + (x+1) & \text{if } x \geq -1 \end{cases}$$
$$= \begin{cases} -2x-4 & \text{if } x < -3 \\ 2 & \text{if } -3 \leq x < -1 \\ 2x+4 & \text{if } x \geq -1. \end{cases}$$

Problem 5. Write the function f given by

$$f(x) = |x+5| - |x+3|$$

as a piecewise defined function.

Example 4. Define f by

$$f(x) = ||x| - 2|.$$

Write $f(x)$ as a piecewise defined function.

Solution

If g by

$$g(x) = |x| - 2,$$

then $g(x)$ is greater than or equal to zero if x is in $(-\infty, -2] \cup [2, \infty)$ and less than zero otherwise. Write $f(x)$ as $|g(x)|$ to obtain the equality

$$f(x) = \begin{cases} g(x) & \text{if } x \in (-\infty, -2] \cup [2, \infty) \\ -g(x) & \text{if } x \in (-2, 2). \end{cases}$$

The function g is given as a piecewise defined function by

$$g(x) = \begin{cases} -x - 2 & \text{if } x < 0 \\ x - 2 & \text{if } x \geq 0 \end{cases}$$

Refine the expression for f to obtain

$$f(x) = \begin{cases} g(x) & \text{if } x \in (-\infty, -2] \\ -g(x) & \text{if } x \in (-2, 0) \\ -g(x) & \text{if } x \in [0, 2) \\ g(x) & \text{if } x \in [2, \infty). \end{cases} = \begin{cases} -x - 2 & \text{if } x \in (-\infty, -2] \\ x + 2 & \text{if } x \in (-2, 0) \\ -x + 2 & \text{if } x \in [0, 2) \\ x - 2 & \text{if } x \in [2, \infty). \end{cases}$$

Problem 6. Define f by

$$f(x) = ||3x + 1| - 7| + 2x + 1.$$

Write $f(x)$ as a piecewise defined function.

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I.2.7 Inequalities Involving Piecewise Defined Functions

Example 1. Find all x so that
$$|3x+1| > 4.$$

Solution

We have that
$$|3x+1| = \begin{cases} -(3x+1) & \text{if } x < -\frac{1}{3} \\ (3x+1) & \text{if } x \geq \frac{1}{3}. \end{cases}$$
In the first case, when
$$x < -\frac{1}{3},$$
we have that
$$|3x+1| = -(3x+1) > 4 \quad \text{if and only if} \quad 3x+1 < -4.$$
In this case,
$$x < -\frac{1}{3} \quad \text{and} \quad x < -\frac{5}{3}.$$
Therefore,
$$x < -\frac{5}{3}.$$
In the second case, when
$$x \geq -\frac{1}{3},$$
we have that
$$|3x+1| = 3x+1 > 4 \quad \text{if and only if} \quad x > 1.$$
In this case,
$$x \geq -\frac{1}{3} \quad \text{and} \quad x > 1.$$
Therefore,
$$x > 1.$$
The solution set is therefore the set of all x in
$$\left(-\infty, -\frac{5}{3}\right) \cup (1, \infty).$$

Problem 1. Suppose that

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4x - 7 & \text{if } x \geq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 3x + 5 & \text{if } x \geq 3. \end{cases}$$

Write using interval notation and then graph on a real number line all solutions to the inequality

$$f(x) \geq g(x).$$

Problem 2. Let f and g be the function given respectively by

$$f(x) = \begin{cases} 2x + 12 & \text{if } x < 0 \\ 5 & \text{if } 0 \leq x \leq 6 \\ x - 7 & \text{if } x > 6 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x + 4 & \text{if } x < 2 \\ 8 - x & \text{if } x \geq 2. \end{cases}$$

Graph on a real number line all solutions to the inequality

$$f(x) > g(x).$$

Problem 3. Write using interval notation and then graph on a real number line all solutions to the inequality $|x + 1| > 2$.

Problem 4. Write using interval notation and then graph on a real number line all solutions to the inequality $|2x - 5| \leq 4$.

Problem 5. Write using interval notation and then graph on a real number line all solutions to the inequality $|x - 12| < 2x + 1$.

Example 2. Find all x so that

$$|2x - 3| \geq |x + 1|.$$

<u>SOLUTION</u>

$$|2x - 3| = \begin{cases} -(2x - 3) & \text{if } x < \frac{3}{2} \\ 2x - 3 & \text{if } x \geq \frac{3}{2}. \end{cases} \qquad |x + 1| = \begin{cases} -(x + 1) & \text{if } x < -1 \\ (x + 1) & \text{if } x \geq -1. \end{cases}$$

If

$$x < \frac{3}{2} \quad \text{and} \quad x < -1, \quad \text{then} \quad x < -1.$$

If

$$x < \frac{3}{2} \quad \text{and} \quad x \geq -1, \quad \text{then} \quad -1 \leq x < \frac{3}{2}.$$

Solution

There are no real numbers x with

$$x \geq \frac{3}{2} \quad \text{and} \quad x < -1.$$

If

$$x \geq \frac{3}{2} \quad \text{and} \quad x \geq -1, \quad \text{then} \quad x \geq \frac{3}{2}.$$

By considering the two functions on the intersection of the intervals where each is determined by a single formula, we see that

$$|2x-3| = \begin{cases} -(2x-3) & \text{if } x < -1 \\ -(2x-3) & \text{if } -1 \leq x < \frac{3}{2} \\ 2x-3 & \text{if } x \geq \frac{3}{2}. \end{cases}$$

$$|x+1| = \begin{cases} -(x+1) & \text{if } x < -1 \\ x+1 & \text{if } -1 \leq x < \frac{3}{2} \\ x+1 & \text{if } x \geq \frac{3}{2}. \end{cases}$$

We now consider the solutions to the inequality

$$|2x-3| \geq |x+1|$$

when x is restricted to the intervals

$$(-\infty, -1), \quad [-1, 3/2), \quad [3/2, \text{ and } \infty).$$

Solution

In the first case, when x is in $(-\infty, -1)$, we have that

$$|2x-3| \geq |x+1| \quad \text{if and only if} \quad -(2x-3) \geq -(x+1).$$

This is to say that

$$2x-3 \leq x+1, \quad \text{that is} \quad x \leq 4.$$

Therefore,

$$x < -1 \quad \text{and} \quad x \leq 4, \quad \text{implying that} \quad x < -1.$$

In the second case, when x is in $[-1, 3/2)$, we have that

$$|2x-3| \geq |x+1| \quad \text{if and only if} \quad -(2x-3) \geq x+1.$$

This is to say that

$$-2x+3 \geq x+1, \quad \text{that is} \quad x \leq 2/3.$$

Therefore,

$$-1 \leq x < 3/2 \quad \text{and} \quad x \leq 2/3, \quad \text{implying that} \quad -1 \leq x \leq 2/3.$$

In the third case, when x is in $[3/2, \infty)$, we have that

$$|2x-3| \geq |x+1| \quad \text{if and only if} \quad 2x-3 \geq x+1.$$

Therefore,

$$x \geq 3/2 \quad \text{and} \quad x \geq 4, \quad \text{implying that} \quad x \geq 4.$$

The set of solutions to the inequality is therefore the set

$$(-\infty, 2/3] \cup [4, \infty).$$

Problem 6. Write using interval notation and then graph on a real number line all solutions to the inequality $|2x+1| \geq |x-3|$.

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I.3.1 Systems of Linear Equations and Inequalities

Example 1. Line L_1 is given by the equation

$$y = 2x + 1$$

and L_2 is given by the equation

$$y = 7x - 9.$$

Where do L_1 and L_2 intersect?

SOLUTION

If (x, y) is the point of intersection, then

$$2x + 1 = 7x - 9 \quad \text{that is} \quad 5x = 10.$$

Therefore,

$$x = 2 \quad \text{and} \quad y = 2(2) + 1 = 5.$$

The point of intersection is $(2, 5)$.

Problem 1. Find all solutions to the system of linear equations

$$\begin{cases} x + y = 1 \\ x - 3y = -7. \end{cases}$$

Problem 2. Find a point (x, y) in the plane that is a solution to the linear system

$$\begin{cases} 4x + 2y = 9 \\ x + 3y = 2. \end{cases}$$

Solve for this point in two ways, first using substitution and then by adding a multiple of one equation with the other. Finally, check that your solution is correct.

Problem 3. Is it possible that a can be chosen so that

$$\begin{cases} x + 2y = 9 \\ 3x + 6y = a \end{cases}$$

has exactly one solution?

Problem 4. Find a real number a so that the linear system

$$\begin{cases} x - 2y = 9 \\ 3x + ay = 2 \end{cases}$$

has no solutions. How many choices of a are there so that the system has no solutions?

Problem 5. Find all solutions to the system of linear equations

$$\begin{cases} x + y + z = 1 \\ x - y + z = 3 \\ 2x + 3z = -5. \end{cases}$$

Problem 6. You have 14 coins in nickels, dimes, and quarters. You have twice the number of quarters as you have nickels and \$2.40 in total. How many nickels, dimes, and quarters do you have?

Example 2. Graph in the plane the solutions (feasible set) of the system of inequalities

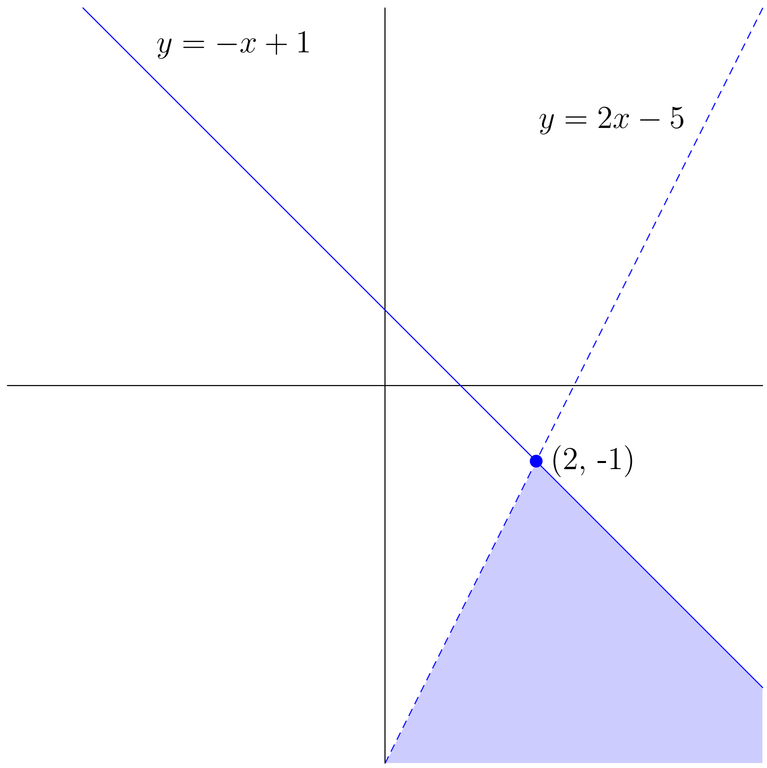
$$\begin{cases} x + y \leq 1 \\ 2x - y > 5. \end{cases}$$

Label with coordinates the vertices of the feasible set.

Solution

Simplify the two equations to obtain the system of inequalities

$$\begin{cases} y \leq -x + 1 \\ y < 2x - 5. \end{cases}$$



Example 3. Graph in the plane the solutions of the system of inequalities

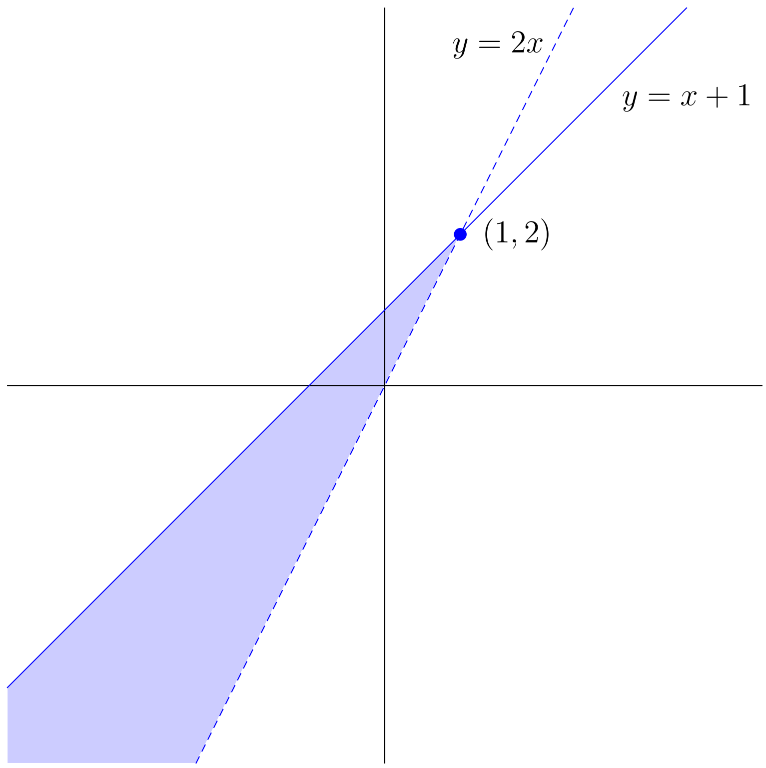
$$\begin{cases} x - y \geq -1 \\ 2x - y < 0. \end{cases}$$

Label with coordinates the vertices of the feasible set.

<u>SOLUTION</u>

Simplify the two equations to obtain the system of inequalities

$$\begin{cases} y \leq x + 1 \\ y > 2x. \end{cases}$$



Problem 7. Sketch the region in the plane that is the solution set to the system of linear inequalities

$$\begin{cases} x - y < 2 \\ 2x + 3y \leq 0. \end{cases}$$

Be sure to label all intersections.

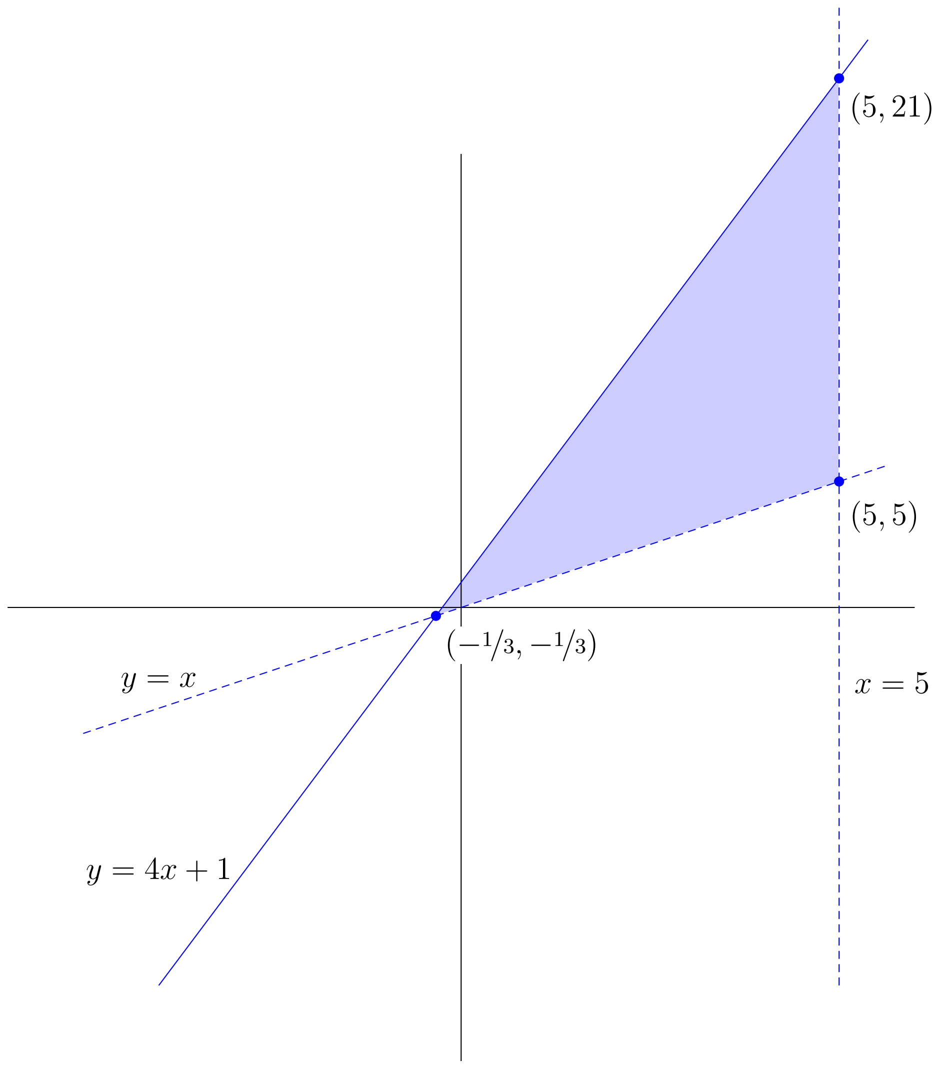
Example 4. Graph in the plane the solutions of the system of inequalities

$$\begin{cases} y \leq 4x + 1 \\ y > x \\ x < 5. \end{cases}$$

Label with coordinates the vertices of the feasible set.

Solution

Note that the y-axis has been scaled by a factor of three in order to facilitate graphing.



Problem 8. Sketch the region in the plane that is the solution set to the system of linear inequalities

$$\begin{cases} y > 0 \\ x \leq 2 \\ x + y > 3. \end{cases}$$

Be sure to label all intersections.

Problem 9. Sketch the region in the plane that is the solution set to the system of linear inequalities

$$\begin{cases} y > 0 \\ x \leq 2 \\ x + y > 3. \end{cases}$$

Be sure to label all intersections.

Problem 10. Graph in the plane the solutions to the system of inequalities

$$\begin{cases} y \leq 4x + 1 \\ y > x \\ x < 5. \end{cases}$$

Describe in set notation the boundary of the region that satisfies the system of inequalities.

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II.1.1 Units

Example 1. One yard is three feet. Write 5 yards in units of feet.

Solution

$$5\,\text{yards} = 5\,\text{yards}\left(\frac{3\,\text{feet}}{1\,\text{yards}}\right) = 15\,\text{feet}$$

Example 2. One yard is three feet and one minute is sixty seconds. Write $10\,\frac{\text{feet}}{\text{second}}$ in units of yards and minutes.

Solution

$$10\,\frac{\text{feet}}{\text{second}} = 10\,\frac{\text{feet}}{\text{second}}\left(\frac{1\,\text{yards}}{3\,\text{feet}}\right)\left(\frac{60\,\text{second}}{1\,\text{minute}}\right) = 200\,\frac{\text{yards}}{\text{minute}}$$

Example 3. A hose can spray water at a rate of two cubic feet per second. Recall that three feet is one yard and that sixty seconds is one minute. At what rate does the hose spray water in units of yards and minutes?

Solution

$$2\,\frac{\text{ft}^3}{\text{s}} = 2\,\frac{\text{ft}^3}{\text{s}}\left(\frac{1\,\text{yard}}{3\,\text{ft}}\right)^3\left(\frac{60\,\text{s}}{1\,\text{min}}\right) = 2\frac{60}{3^3}\,\frac{\text{yard}^3}{\text{min}} = \frac{40}{9}\,\frac{\text{yard}^3}{\text{min}}.$$

Problem 1. A hose can spray water at a rate of two cubic feet per second. Recall that three feet is one yard and that sixty seconds is one minute. At what rate does the hose spray water in units of yards and minutes?

Problem 2. A container carries 3 cubic feet of fluid. How many cubic inches is this?

Problem 3. Alice can run at 15 miles per hour, how many feet per second is this?

Problem 4. A drain drains water at 100 cubic centimeters per minute. How fast does it drain in units of cubic meters per hour?

Problem 5. The formula for the kinetic energy of an object of mass m traveling at a speed v is

$$KE = \frac{1}{2}mv^2.$$

What are the fundamental units of kinetic energy?

Problem 6. What is the fundamental unit of the conversion factor that converts the quantity 200 square feet per second to the same quantity in units of square meters per hour?

Problem 7. A force, F, on a body of mass m causes the body to accelerate. Denote the acceleration of the body by a. The relationship between force, mass, and acceleration is

$$F = ma.$$

If we take the unit of mass to be a kilogram, the unit of distance to be a meter, and the unit of time to be a second, then what are the units of force?

Problem 8.

(a) The force, F, between two bodies of mass m_1 kilograms and m_2 kilograms whose centers of mass are a distance of r meters apart is given by the formula

$$F = \frac{Gm_1m_2}{r^2},$$

where G is the gravitational constant of the universe. What are the units of the gravitational constant G?

(b) The radius of the earth is approximately 4,000 miles. The acceleration due to gravity is approximately 32 feet per square second. The mass of the earth is approximately 6×10^{24} kilograms. Calculate the value of G in units of kilograms, meters, and seconds.

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II.1.2 Linear Scaling

Example 1. Alice can paint a 100 square foot wall in an hour. How many minutes does it take her to paint a 100 square yard wall?

Solution

There is a constant k so that the number of square feet, A, that Alice paints in t minutes is equal to

$$A = kt.$$

We therefore have that

$$100\,\text{feet}^2 = k \cdot 60\,\text{min}.$$

Furthermore,

$$100\,\text{yard}^2 = 900\,\text{feet}^2 = 9 \cdot 100\,\text{feet}^2 = 9 \cdot k \cdot 60\,\text{min} = k \cdot 9 \cdot 60\,\text{min} = k \cdot 540\,\text{min}.$$

So, it takes Alice 540 minutes.

Example 2. Bob can paint two square meters of wall in three minutes. How many square centimeters of wall can he paint in six seconds?

Solution

Since 100 centimeters is one meter and 60 seconds is one minute,

$$\begin{aligned}
\frac{2\,\mathrm{m}^2}{3\,\mathrm{min}} &= \frac{2\,\mathrm{m}^2}{3\,\mathrm{min}}\left(\frac{100\,\mathrm{cm}}{1\,\mathrm{m}}\right)^2\left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right)\\
&= \frac{20{,}000\,\mathrm{cm}^2}{3\,\mathrm{min}}\left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right)\\
&= \frac{20{,}000\,\mathrm{cm}^2}{3\,\mathrm{min}}\left(\frac{1\,\mathrm{min}}{2\cdot 30\,\mathrm{s}}\right)\\
&= \frac{20{,}000\,\mathrm{cm}^2}{3\cdot 2\cdot 30\,\mathrm{s}}\\
&= \frac{20{,}000\,\mathrm{cm}^2}{6\cdot 30\,\mathrm{s}}\\
&= \frac{\frac{20{,}000}{30}\,\mathrm{cm}^2}{6\,\mathrm{s}} = \frac{\frac{2{,}000}{3}\,\mathrm{cm}^2}{6\,\mathrm{s}}
\end{aligned}$$

So, Bob can paint $\frac{2{,}000}{3}$ square centimeters of wall in six seconds.

Example 3. Assume that all workers work at the same constant pace. It takes three workers 4 hours lay the brick for a 50 square foot raised rectangular patio. The owner decides to scale the planar dimensions of the patio by a factor of three (the height remains unchanged) and hires instead six workers. How many hours will it take to complete the project?

Solution

We have that

$$50\,\mathrm{ft}^2 = k\,(3\,\text{workers})\,(4\,\text{hours})\,.$$

Therefore,

$$\begin{aligned}
450\,\mathrm{ft}^2 &= 9 \cdot 50\,\mathrm{ft}^2 \\
&= 9 \cdot k\,(3\,\text{workers})\,(4\,\text{hours}) \\
&= 9 \cdot k \left(3 \cdot \frac{6}{6}\,\text{workers}\right)(4\,\text{hours}) \\
&= 9 \cdot \frac{3}{6} \cdot k\,(6\,\text{workers})\,(4\,\text{hours}) \\
&= \cdot k\,(6\,\text{workers}) \left(9 \cdot \frac{3}{6} \cdot 4\,\text{hours}\right) = \cdot k\,(6\,\text{workers})\,(9 \cdot 2\,\text{hours})
\end{aligned}$$

It therefore takes

$$18\,\text{hours}.$$

Example 4. Assume that all workers work at the same constant pace. It takes three workers four hours lay the brick for a patio. How many hours will it take seven workers to complete the project?

Solution

The area, A, of the patio is proportionate to the number of workers, W, working and the number of hours, H, that the workers work. Therefore, there is a constant k so that

$$A = kWH.$$

Since the same work is accomplished in both scenarios, we have that

$$\begin{aligned}(3\,\text{workers}) \cdot (4\,\text{hours}) &= (7\,\text{workers}) \cdot (H\,\text{hours}) \\ &= \left(7 \cdot \frac{3}{3}\,\text{workers}\right) \cdot (H\,\text{hours}) \\ &= (3\,\text{workers}) \cdot \left(H \cdot \frac{7}{3}\,\text{hours}\right).\end{aligned}$$

So

$$4\,\text{hours} = H \cdot \frac{7}{3}\,\text{hours}$$

and therefore,

$$H = \frac{12}{7}.$$

Problem 1. Five munchkins working together can lay 300 pounds of yellow brick in an hour. How many hours will it take seven munchkins to lay 400 pounds of brick?

Problem 2. It takes three workers eight hours to shovel 200 cubic feet of sand. How many workers are required to shovel 100 cubic yards of sand in seven hours?

Problem 3. You have two cylindrical buckets, Bucket A and Bucket B. You pour a certain amount of water into Bucket A. The water fills the bucket to a height of 20 centimeters. You then pour the water into Bucket B and it fills it to a height of 15 centimeters. You now fill Bucket B to a height of 12 centimeters. You pour the water into Bucket A. How high does the water fill the bucket?

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II.1.3 Nonlinear Scaling of Units

Example 1. It takes 20 cubic feet of concrete make a 80 square foot patio. How many cubic yards of concrete does it take to make a 80 square yard patio? The thickness of the slab of concrete will be the same in both cases.

<u>Solution</u>

The volume, V, of the patio is a constant, k, times the area, A, of the patio. We therefore have that

$$V = kA.$$

We are given that

$$20\,\mathrm{ft}^3 = k\left(80\,\mathrm{ft}^2\right).$$

Therefore,

$$\begin{aligned} k\left(80\,\mathrm{yard}^2\right) &= k\left(9 \cdot 80\,\mathrm{ft}^2\right) \\ &= 9k\left(80\,\mathrm{ft}^2\right) \\ &= 9 \cdot 20\,\mathrm{ft}^3 \\ &= 9 \cdot 20\,\mathrm{ft}^3 \left(\frac{1\,\mathrm{yard}}{3\,\mathrm{ft}}\right)^3 \\ &= \frac{9 \cdot 20}{3^3}\,\mathrm{yard}^3 = \frac{20}{3}\,\mathrm{yard}^3. \end{aligned}$$

Example 2. It takes three cubic feet of paint to paint a 100 square foot wall. It takes five painters three hours to paint this wall. How long will it take ten painters to paint a wall that requires three cubic yards of paint to cover?

Solution

Notice first that

$$3\,\text{yard}^3 = 27 \cdot 3\,\text{ft}^3.$$

The area, A, of the wall is proportionate to the number of painters, P, working and the number of hours, H, that the painters paint. Therefore, there is a constant k so that

$$A = kPH.$$

The area of wall painted with a certain volume of paint is proportionate to the volume of paint used. Therefore, the new quantity of paint will cover a 2700 square foot wall. We have that

$$\begin{aligned} 27 \cdot 100\,\text{ft}^2 &= 27 \cdot k \cdot (5\,\text{Painters}) \cdot (3\,\text{hours}) \\ &= 27 \cdot k \cdot \left(5 \cdot \frac{10}{10}\,\text{Painters}\right) \cdot (3\,\text{hours}) \\ &= 27 \cdot \frac{5}{10} \cdot k \cdot (10\,\text{Painters}) \cdot (3\,\text{hours}) \\ &= k \cdot (10\,\text{Painters}) \cdot \left(27 \cdot \frac{5}{10} \cdot 3\,\text{hours}\right) = k \cdot (10\,\text{Painters}) \cdot \left(\frac{81}{2}\,\text{hours}\right). \end{aligned}$$

It will take

$$\frac{81}{2}\,\text{hours}.$$

Example 3. A sculptor sculpts a scaled down version of a subject out of 2 pounds of clay. She covers it with 3 square feet of gold foil. The gold foil has uniform thickness, it is expensive and so she does not double layer any part. The full scale version of the subject requires 48 square feet of gold foil. How many pounds of clay will she need?

Solution

The area of the full size sculpture is 16 times the area of the scaled down version. Therefore, the linear dimensions of the full size version is four times that of the scaled down version since the area scales with the square of the linear dimension. Since the volume scales with the cube of the linear dimension, the volume of the full size version is 64 ($= 4^3$) times the volume of the scaled down version. The volume is proportionate to the weight of the clay and so the weight of the clay scales by the same factor as the volume. She will therefore need

$$64 \cdot 2\,\text{lbs} = 128\,\text{lbs}$$

of clay.

Example 4. Bob can paint two square meters of wall in three minutes. How many square centimeters of wall can he paint in six seconds?

Solution

$$\begin{aligned}
\frac{2\,\text{m}^2}{3\,\text{min}} &= \frac{2\,\text{m}^2}{3\,\text{min}}\left(\frac{100\,\text{cm}}{1\,\text{m}}\right)^2\left(\frac{1\,\text{min}}{60\,\text{s}}\right) \\
&= \frac{20{,}000\,\text{cm}^2}{3\,\text{min}}\left(\frac{1\,\text{min}}{60\,\text{s}}\right) \\
&= \frac{20{,}000\,\text{cm}^2}{3\,\text{min}}\left(\frac{1\,\text{min}}{2\cdot 30\,\text{s}}\right) \\
&= \frac{20{,}000\,\text{cm}^2}{3\cdot 2\cdot 30\,\text{s}} \\
&= \frac{20{,}000\,\text{cm}^2}{6\cdot 30\,\text{s}} \\
&= \frac{\frac{20{,}000}{30}\,\text{cm}^2}{6\,\text{s}} = \frac{\frac{2{,}000}{3}\,\text{cm}^2}{6\,\text{s}}
\end{aligned}$$

So, Bob can paint $\frac{2{,}000}{3}$ square centimeters of wall in six seconds.

Problem 1. The weight of a solid steel ball is proportionate to its volume. If the weight of a particular steel ball is 100 pounds, how much will a solid steel ball with twice its radius weigh?

Problem 2. It takes three cubic feet of paint to paint a 100 square foot wall. It takes five painters three hours to paint this wall. How long will it take ten painters to paint a wall that requires three cubic yards of paint to cover?

Problem 3. The force applied to a body is F. The mass of the body is m. The acceleration that the body experiences as a result of the force is a. We then have that $F = ma$. If you apply twice the force on a new body and it experiences half the acceleration of the original, what is the mass of the new body compared with m?

Problem 4. The intensity of light is proportionate to the inverse square of the distance to the light source. At the same distance, Lightbulb A shines at 9 times the intensity as Lightbulb B. How many feet do you need to stand from Lightbulb B so it appears to have the same intensity as Lightbulb A at 12 feet away.

Problem 5. The weight of paint is proportionate to its volume. Paint is always applied at the same thickness. It takes 10 ounces of paint to paint a ball that has a volume of 3 cubic feet. How many ounces of paint does it take to paint a ball that has a volume of 24 cubic feet.

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II.1.4 Scaling Graphs

Problem 1. Use the graph of the function f given by

$$f(x) = x$$

in order to graph the function g given by

$$g(x) = 2x.$$

Problem 2. Use the graph of the function f given by

$$f(x) = x^2$$

in order to graph the functions g and h given by

$$g(x) = 9x^2 \quad \text{and} \quad h(x) = \left(\frac{1}{3}x\right)^2.$$

Problem 3. Use the graph of the function f given by

$$f(x) = x^2$$

in order to graph the functions g and h given by

$$g(x) = 3x^2 \quad \text{and} \quad h(x) = (3x)^2.$$

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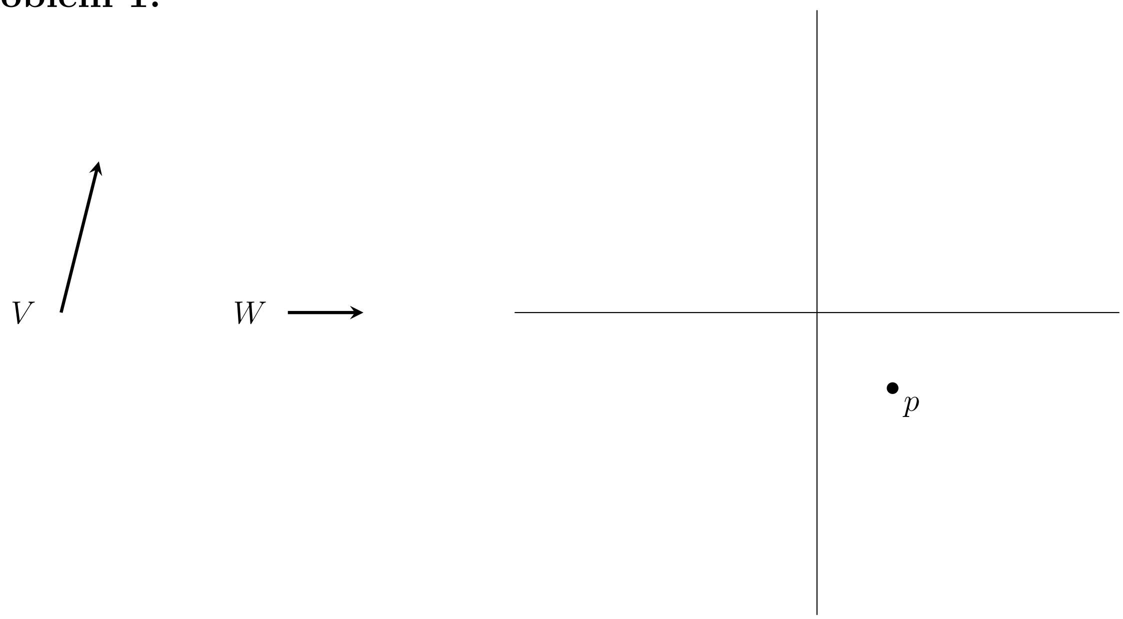
II.2.1 Abstract Translations of the Plane

Problem 1.



On the graph above, draw and label the following points. You should sketch the arrows as well to illustrate that you know how the arrows are acting.

(a) $V + p$,

(b) $W + p$,

(c) $2W + p$,

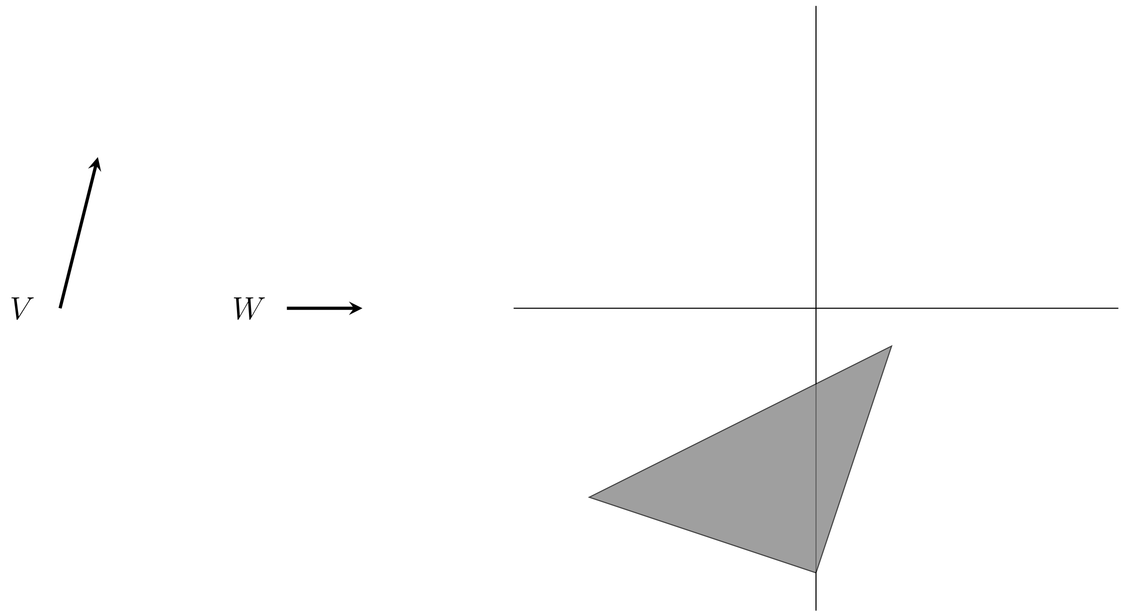
(d) $(V + W) + p$,

(e) $(V - W) + p$,

(f) $(W - V) + p$.

Problem 2.

Let $\triangle$ be the triangle pictured below. Let V and W be the vectors given below. What is $\frac{3}{2}V + \triangle$? What is $(V + W) + \triangle$?



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II.2.2 Vectors and the Method of Coordinates on a Plane

Example 1. Calculate $\langle 2,3\rangle + (1,5)$.

SOLUTION

$$(3,8)$$

Example 2. What is the difference $(6,4) - (1,5)$ equal to?

SOLUTION

$$\langle 5,-1\rangle$$

Problem 1. Fill in the blank:

(a) $(4,1) - (2,6) =$ ______.

(b) $\langle 1,5\rangle + (3,4) =$ ______.

(c) $3\langle -2,1\rangle + (1,6) =$ ______.

Problem 2. Write in coordinates the vector that translates $(2,1)$ to $(7,12)$. Where does this vector translate $(-4,3)$?

Example 3. What is the distance between $(2,5)$ and $(1,8)$?

SOLUTION

$$\sqrt{(1-2)^2+(8-5)^2} = \sqrt{10}.$$

Example 4. What is the length of the vector $\langle 1,4\rangle$?

<u>SOLUTION</u> $\sqrt{17}$

Problem 3. What is the length of the vector $\langle 1, 4 \rangle$?

Problem 4. What is the length of

(a) the line segment with endpoints $(2, 6)$ and $(5, 10)$,

(b) the vector that translates $(2, 6)$ to $(5, 10)$,

(c) the vector that translates $(5, 10)$ to $(2, 6)$?

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II.2.3 Movement Along Lines

Example 1. A boat moves with constant velocity. The velocity vector of a boat is $\langle 3, 4\rangle$ with units in miles per hour. How fast is the boat moving?

<u>Solution</u>

$$5\,\frac{\text{miles}}{\text{hour}}$$

Example 2. Suppose that the point one mile north of the origin is described by the point $(0, 1)$. A boat travels with a velocity of five miles per hour north and one mile per hour east. What is the boat's velocity vector? Include units in your vector notation.

<u>Solution</u>

$$\left\langle 1\,\frac{\text{mile}}{\text{hour}}, 5\,\frac{\text{mile}}{\text{hour}}\right\rangle$$

Example 3. Let L be the line that contains $(1, 4)$ and $(3, 7)$. Find a point on L that is to the right of and a distance of two from the point $(3, 7)$.

<u>Solution</u>

$$2\left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\rangle + (3, 7).$$

Example 4. A boat travels with a velocity of three miles per hour north and two mile per hour east. The boat travels in this way for two hours. What is the boat's displacement vector? Include units in your vector notation.

Solution

$$(2\,\text{hour}) \cdot \left\langle 2\,\frac{\text{mile}}{\text{hour}}, 3\,\frac{\text{mile}}{\text{hour}} \right\rangle = \langle 4\,\text{mile}, 6\,\text{mile} \rangle$$

Example 5. Find the midpoint of the line segment with endpoints $(2, 5)$ and $(4, 12)$.

Solution

If m is the midpoint of the line segment, then

$$m = \frac{1}{2}\left((4, 12) - (2, 5)\right) + (2, 5) = \frac{1}{2}\langle 2, 7\rangle + (2, 5) = \left\langle 1, \frac{7}{2} \right\rangle + (2, 5) = \left(3, \frac{17}{2}\right).$$

Problem 1. Find the midpoint of the line segment with endpoints $(3, 1)$ and $(5, 8)$.

Example 6. Find a point on the line segment with endpoints $(2, 5)$ and $(4, 12)$ that is three times the distance from $(2, 5)$ as it is from $(4, 12)$.

Solution

$$\frac{3}{4}\langle 2, 7\rangle + (2, 5).$$

Example 7. Let L be the line segment with endpoints $(2, 5)$ and $(7, 9)$. Find a point on L that is one seventh of the distance from $(2, 5)$ to $(7, 9)$ away from $(2, 5)$.

Solution

$$\frac{1}{7}\langle 5, 4\rangle + (2, 5)$$

Problem 2. Find a point on the line segment with endpoints $(1, 4)$ and $(7, 9)$ that is four times the distance from $(1, 4)$ as it is from $(7, 9)$.

Problem 3. Let L be the line segment with endpoints $(-1, 3)$ and $(7, 2)$. Find a point on L that is one third of the distance from $(-1, 3)$ to $(7, 2)$ away from $(-1, 3)$.

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II.2.4 Descriptions of Lines

Example 1. A line L is given by the vector equation

$$L(t) = t\langle 3,5\rangle + (1,8).$$

What is the slope of L?

Solution

$$\frac{5}{3}.$$

Example 2. A line L is given by the vector equation

$$L(t) = t\langle 1,7\rangle + (2,3).$$

Calculate $L(0)$ and $L(1)$.

Solution

$$L(0) = (2,3) \quad \text{and} \quad L(1) = (3,10).$$

Example 3. Find the position of a particle moving at a constant velocity that is at $(2,3)$ at time zero and $(3,7)$ at time one.

Solution

$$L(t) = t((3,7) - (2,3)) + (2,3) = t\langle 1,4\rangle + (2,3).$$

So

$$L(t) = (t+2, 4t+3).$$

Example 4. A line L intersects the points $(1,4)$ and $(5,12)$. Find a vector equation for L with
$$L(0)=(1,4) \quad \text{and} \quad L(1)=(5,12).$$

SOLUTION

$$L(t)=t\langle 4,8\rangle+(1,4).$$

Example 5. Find the position of a particle moving at a constant velocity that is at $(2,3)$ at time zero and $(3,7)$ at time three.

SOLUTION

$$L(t)=\frac{t}{3}\langle 1,4\rangle+(2,3)=\left(\frac{t}{3}+2,\frac{4t}{3}+3\right).$$

Example 6. A line L intersects the points $(2,-3)$ and $(5,7)$. Find a vector equation for L with
$$L(0)=(5,7) \quad \text{and} \quad L(4)=(2,-3).$$

SOLUTION

$$L(t)=\frac{t}{4}\langle -3,-10\rangle+(5,7).$$

Example 7. A boat moves with constant velocity. The velocity vector of the boat is $\langle 3,9\rangle$, where units are in miles per hour. The path of motion of this boat is a line in the plane. What is the slope of this line?

Solution

$$\frac{9}{3} = 3.$$

Example 8. A boat moves with constant velocity. The velocity vector of a boat is $\langle 3, 9 \rangle$, where units are in miles per hour. The path of motion of this boat is a line. How fast is the boat moving?

Solution

$$\sqrt{3^2 + 9^2}\,\frac{\text{miles}}{\text{hour}} = 3\sqrt{10}\,\frac{\text{miles}}{\text{hour}}.$$

Example 9. A position of $(1, 0)$ indicates a position one mile due East of a port and a position of $(0, 1)$ indicates a position one mile due North of a port. At time $t = 0$, a certain boat is at position $(3, 2)$. The boat is traveling at a constant velocity of two miles per hour east and one mile per hour north. What is the position of the boat at time t?

Solution

$$B(t) = t\langle 2, 1 \rangle + (3, 2).$$

Example 10. Boat A is initially at position $(2, 5)$ and moves at a constant velocity of $\langle 4, 1 \rangle$. Boat B is at position $(8, 3)$ and moves at a constant velocity of $\langle 2, 10 \rangle$. Do the paths of the boats ever cross? If so where?

Solution

We consider only the paths of motion of the boats, which are both lines. Boat A moves along the line given by

$$y = \frac{1}{4}(x-2) + 5 = \frac{1}{4}x + \frac{9}{2}$$

and Boat B moves along the line given by

$$y = 5(x-8) + 3 = 5x - 37.$$

If (x, y) is a point on intersection of the paths of motion, then

$$\frac{1}{4}x + \frac{9}{2} = 5x - 37$$

and so

$$\frac{19}{4}x = \frac{83}{2}.$$

Therefore

$$x = \frac{83}{2} \cdot \frac{4}{19} = \frac{166}{19} \quad \text{and} \quad y = \frac{1}{4} \cdot \frac{166}{19} + \frac{9}{2}.$$

The paths of motion of the boats indeed intersect at the point determined by these x and y coordinates.

Example 11. See the problem above. Boat A is initially at position $(2, 1)$ and moves at a constant velocity of $\langle 4, 1\rangle$. Boat B is at position $(8, 3)$ and moves at a constant velocity of $\langle 2, 10\rangle$. Do the boats collide?

Solution

The position of Boat A at time s is

$$A(s) = s\langle 4, 1\rangle + (2, 1) = (4s + 2, s + 1).$$

The position of Boat B at time t is

$$B(t) = t\langle 2, 10\rangle + (8, 3) = (2t + 8, 10t + 3).$$

The question is, is there a time s and a time t when $A(s)$ equal $B(t)$? The answer to the above question demonstrates that this is true, however, s and t will not be equal and so while the paths of motion of the boats intersect, the boats do not collide. In particular, the x coordinate of $A(s)$ is $4s + 2$ and so when A is at the intersection point,

$$4s + 2 = \frac{166}{19}$$

and so

$$s = \frac{1}{4}\left(\frac{166}{19} - 2\right) = \frac{1}{4}\left(\frac{166 - 19}{19}\right) = \frac{1}{4}\left(\frac{147}{19}\right) = \frac{147}{76}.$$

Furthermore, the x coordinate of $B(t)$ is $2t + 8$ and so when B is at the intersection point,

$$2t + 8 = \frac{166}{19}$$

and so

$$t = \frac{1}{2}\left(\frac{166}{19} - 8\right) = \frac{1}{2} \cdot \frac{166 - 152}{19} = \frac{7}{19}.$$

So,

$$s = \frac{147}{76} \neq \frac{7}{19} = t.$$

Problem 1. Write a vector equation of the position of a particle that moves in a line, has constant velocity, is at $(2, 5)$ at time zero and is at $(1, 4)$ at time one. What is the slope of this line?

Problem 2. Write a vector equation of the position of a particle that moves in a line, has constant velocity, is at $(0, -3)$ at time zero and is at $(2, 3)$ at time two. What is the slope of this line?

Problem 3. The line L intersects $(2, 7)$ and $(5, 1)$. Find an equation for the line parallel to L and intersecting the origin. Sketch the graph of the two lines.

Problem 4. Boat A is initially at position $(1, 4)$ and moves at a constant velocity of $\langle 3, 5\rangle$. Boat B is at position $(7, 2)$ and moves at a constant velocity of $\langle 1, 10\rangle$.

(a) Do the paths of the boats ever cross? If so where?

(b) Will the boats collide?

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II.2.5 Translating Graphs

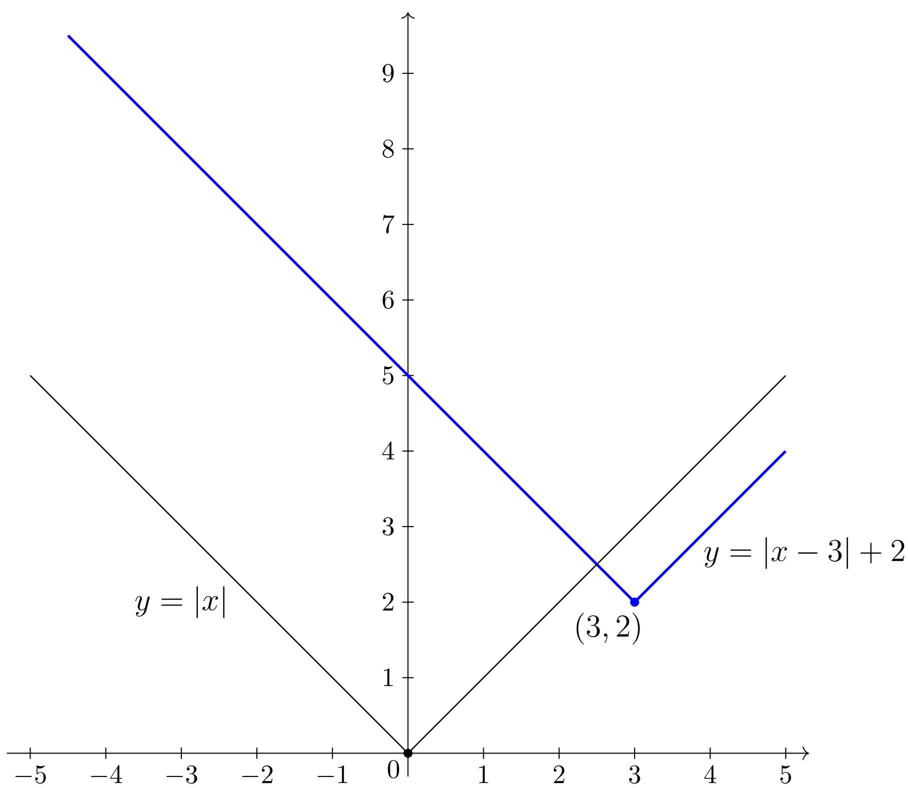
Example 1. Sketch the graph of f given by $f(x) = |x|$. Then sketch the graph of g given by $g(x) = |x-3|+2$. By what vector has the graph of f been translated?

Solution

The function g is given by

$$\begin{aligned} g &= \{(x, |x-3|+2)\colon x \in \mathbb{R}\} \\ &= \{(x+3, |x|+2)\colon x \in \mathbb{R}\} \\ &= \langle 3,2\rangle + \{(x,|x|)\colon x \in \mathbb{R}\} = \langle 3,2\rangle + f. \end{aligned}$$

Therefore, to sketch the graph of g, take the graph of f and shift it to the right by 3 and up by 2, as g is a translate of f by the vector $\langle 3,2\rangle$.



Problem 1. Graph the line L that has slope 2 and that intersects $(0, 0)$. What is the equation for L? Translate L by the vector $\langle 2, 5\rangle$ and sketch the graph of $V + L$. The set $V + L$ is itself the graph of a function that is given by a formula. What is the formula that gives rise to the graph given by $V + L$?

Problem 2. Let f be the function given by set of points in the plane satisfying the equation $y = x^2$. Translate f by the vector $\langle 3, 1\rangle$ and sketch the graph of $V + f$. The set $V + f$ is itself the graph of a function that is given by a formula. What is the formula that gives rise to the graph given by $V + f$?

Problem 3. Let f be the function given by set of points in the plane satisfying the equation $y = x^4$. Translate f by the vector $\langle 2, 3\rangle$ and sketch the graph of $V + f$. The set $V + f$ is itself the graph of a function that is given by a formula. What is the formula that gives rise to the graph given by $V + f$?

Problem 4. Let f be the function given by set of points in the plane satisfying the equation $y = \frac{1}{x}$. Translate f by the vector $\langle -1, 6\rangle$. The set $V + f$ is itself the graph of a function that is given by a formula. What is the formula that gives rise to the graph given by $V + f$?

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II.2.6 Orthogonality of Vectors and Lines

Example 1. Line L_1 is given by the equation $y = 3x - 5$. Line L_2 is perpendicular to L_1. What is the slope of L_2?

SOLUTION

$$-\frac{1}{3}.$$

Example 2. Let L be the line containing points $(2, 5)$ and $(6, 7)$. What is the equation, in the slope/y-intercept form, of a line perpendicular to this line that intersects the point $(2, 3)$?

SOLUTION

The slope of L is $\frac{1}{2}$. So, the equation for the line perpendicular to L, written in slope/y-intercept form is

$$y = -2(x - 2) + 5 \quad \text{that is} \quad y = -2x + 9.$$

Example 3. Let L be the line containing points $(2, 5)$ and $(6, 7)$. What is a parametric equation of a line perpendicular to this line that intersects the point $(2, 3)$?

Solution

The vector V given by

$$V = (6,7) - (2,5) = \langle 4, 2 \rangle$$

acts to translate points along L, so the vector $V_\perp$ given by

$$V_\perp = \langle -2, 4 \rangle$$

acts to translate points along $L_\perp$. We therefore have that

$$L_\perp(t) = t\langle -2, 4 \rangle + (2,3).$$

Example 4. Let L_1 be the line passing through $(3,2)$ and $(4,5)$. Let L_2 be the line passing through $(1,5)$ and perpendicular to L_2. Where do L_1 and L_2 intersect?

Solution

If (x,y) is a point on L_1, then

$$y = 3(x-3) + 2 = 3x - 7.$$

If (x,y) is a point on L_2, then

$$y = -\frac{1}{3}(x-1) + 5 = -\frac{1}{3}x + \frac{16}{3}.$$

Therefore, the point (x,y) lies on both lines if and only if

$$\frac{10}{3}x = \frac{37}{3} \quad \text{that is} \quad x = \frac{37}{10}.$$

Therefore, the two lines intersect at

$$\left(\frac{37}{10}, 3 \cdot \frac{37}{10} - 7\right).$$

Example 5. Let p be the point $(-2,5)$. Let L be the line that pass through $(-2,1)$ and $(0,-3)$. Find the point on L that is closest to p.

Solution

The line L is give by the equation

$$y = -2(x+2) + 1 = -2x - 3.$$

The line $L_\perp$ that is perpendicular to L and that intersects $(-2, 5)$ is given by the equation

$$y = \frac{1}{2}(x+2) + 5 = \frac{1}{2}x + 6.$$

These lines intersect at the desired point p and so if the coordinates of p are (x, y), then

$$\frac{5}{2}x = -9 \quad \text{and so} \quad x = -\frac{18}{5}.$$

Therefore,

$$p = \left(-\frac{18}{5}, \frac{21}{5}\right).$$

Problem 1. Describe all vectors perpendicular to $\langle 5, 12\rangle$. Draw examples of three of these vectors in relation to each other and $\langle 5, 12\rangle$ where each has its origin drawn at a common point.

Problem 2. The line L has slope equal to 3. The line $L_\perp$ is perpendicular to L. What is the slope of $L_\perp$?

Problem 3. Let L be the line containing points $(2, 5)$ and $(6, 7)$.

(a) Find an equation for L.

(b) What is the equation of a line perpendicular to this line that intersects the point $(2, 3)$?

Problem 4. The line L intersects $(2, 7)$ and $(5, 1)$. Find an equation for the line perpendicular to L and intersecting $(2, 1)$. Sketch the graph of the two lines. Where do they intersect?

Problem 5. Find an equation for the line intersecting $(5, 3)$ and perpendicular to the line that intersects $(3, 2)$ and $(8, 2)$.

Problem 6. Suppose that a line, L, is given by the equation $y = 2x - 1$. Find the point on L that is closest to $(2, -1)$.

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II.2.7 Circles and the Polar Form of a Vector

Example 1. Write the polar form of the vector $\langle 5, 3\rangle$?

SOLUTION

$$\sqrt{34}\left\langle \frac{5}{\sqrt{34}}, \frac{3}{\sqrt{34}} \right\rangle$$

Example 2. A boat travels at a constant velocity from $(1, 5)$ to $(4, -3)$. The boat travels at a speed of two miles per hour. Where will it be in 3 hours?

SOLUTION

We will suppress the units in the problem. The velocity vector of the boat, V, is given by

$$V = \langle 3, -8\rangle.$$

Therefore, the direction of motion is $\hat{V}$ where

$$\hat{V} = \left\langle \frac{3}{\sqrt{73}}, -\frac{8}{\sqrt{73}} \right\rangle.$$

The speed of the boat is two miles per hour, and so the displacement vector at time t is

$$2t\hat{V}.$$

Therefore, the position of the boat at time t is $B(t)$ where

$$B(t) = 2t\hat{V} + (1, 5) = 2t\left\langle \frac{3}{\sqrt{73}}, -\frac{8}{\sqrt{73}} \right\rangle + (1, 5).$$

Therefore, the position of the boat when time is three hours is $B(3)$ where

$$B(3) = 6\left\langle \frac{3}{\sqrt{73}}, -\frac{8}{\sqrt{73}} \right\rangle + (1, 5).$$

Example 3. At time zero, Boat 1 is at the position $(2,5)$. At time zero, Boat 2 is at the position $(7,1)$. The boats are moving on a line and move in such a way that they eventually collide head on. Boat 1 is moving twice as fast as Boat 2. Where do the boats collide?

Solution

The boats move along the line segment between $(2,5)$ and $(7,1)$. Boat 1 moves in the direction $\langle 5,-4\rangle$ and Boat 2 moves in the direction $\langle -5,4\rangle$. Boat 1 goes twice the distance as Boat 2 and so moves two thirds the distance from $(2,5)$ to $(7,1)$. Therefore, the boats meet at

$$\frac{2}{3}\langle 5,-4\rangle + (2,5) = \left(\frac{16}{3},\frac{7}{3}\right).$$

Example 4. At time zero, Boat 1 is at the position $(2,5)$. At time zero, Boat 2 is at the position $(7,1)$. The boats are moving on a line and move in such a way that they eventually collide head on. Boat 1 is moving at 2 miles per hour. Where will Boat 1 be in 5 hours?

Solution

Boat 1 moves in the direction

$$\hat{V} = \left\langle \frac{5}{\sqrt{41}}, -\frac{4}{\sqrt{41}} \right\rangle.$$

In five hours, Boat 1 will move a distance of 10 miles and will be at the position

$$10\left\langle \frac{5}{\sqrt{41}}, -\frac{4}{\sqrt{41}} \right\rangle + (2,5).$$

Example 5. Let L be the line given by the equation

$$y = 4x + 1.$$

The point $(2,9)$ is the lower left vertex of a square with side length 5 that has an edge along L. What are the other vertices of this square?

Solution

The vector $\langle 1,4\rangle$ translates to the right points on L to other points on L. Let V be the unit vector that points in this direction, that is

$$V = \frac{1}{\sqrt{17}}\langle 1,4\rangle.$$

The point of the square, S, that lies on L to the right of $(2,9)$ is

$$5V + (2,9).$$

The vector $\langle -4,1\rangle$ translates points on L upwards and in a direction perpendicular to L. Let W be the unit vector that points perpendicular to V and upwards, that is

$$W = \frac{1}{\sqrt{17}}\langle -4,1\rangle.$$

The two other vertices of S are

$$W + (2,9) \quad \text{and} \quad 5W + 5V + (2,9).$$

You can now explicitly calculate the coordinates of these vertices.

Example 6. Let C be the circle given by equation

$$(x-2)^2 + (y+3)^2 = 16.$$

What is the center of C and what is its radius?

Solution

Center is at $(2,-3)$, radius is 4.

Example 7. What is the equation of the circle centered at $(1,5)$ that contains the point $(4,7)$?

Solution

$$(x-1)^2+(y-5)^2=(4-1)^2+(7-5)^2 \quad \text{so} \quad (x-1)^2+(y-5)^2=13.$$

Example 8. The set of points satisfying the equation $x^2+y^2-4x+y=52$ is a circle. What is its center and what is its radius?

Solution

We have that

$$(x+h)^2+(y+k)^2=x^2+2hx+h^2+y^2+2ky+k^2.$$

So,

$$x^2+2hx+y^2+2ky=(x+h)^2+(y+k)^2-h^2-k^2.$$

Since

$$x^2+2hx+y^2+2ky=x^2+y^2-4x+y,$$

we have that

$$h=-2 \quad \text{and} \quad k=\frac{1}{2}.$$

Therefore,

$$x^2+y^2-4x+y=(x-2)^2+(y+{}^1\!/_2)^2-4-\frac{1}{4}=(x-2)^2+(y+{}^1\!/_2)^2-\frac{17}{4}=52.$$

Therefore,

$$(x-2)^2+(y+{}^1\!/_2)^2=\frac{225}{4} \quad \text{and so} \quad (x-2)^2+(y+{}^1\!/_2)^2=\left(\frac{15}{2}\right)^2.$$

The center of the circle is $(2,-{}^1\!/_2)$ and the radius is $\frac{15}{2}$.

Example 9. Does the line $y=2$ intersect the unit circle?

Solution

No. The point $(0,2)$ is the point closest to the origin and its distance to the origin is 2. The unit circle has radius 1.

Example 10. A point p lies on the unit circle and is in Quadrant IV. The x-coordinate of p is $\frac{1}{7}$. What is the y-coordinate of p?

Solution

Since

$$x^2 + y^2 = 1,$$

we have that

$$y^2 = 1 - x^2 \quad \text{and so} \quad |y| = \sqrt{1-x^2}.$$

Therefore,

$$y = \sqrt{1 - \left(\frac{1}{7}\right)^2} = -\frac{\sqrt{48}}{7} = -\frac{4\sqrt{3}}{7}.$$

Example 11. Suppose that L is a line of slope 4 that intersects the origin. Where does L intersects the unit circle?

Solution

Note that if (x, y) lies on L, then

$$y = 4x.$$

The points where L intersects the unit circle are all of the points $(x, 4x)$ with

$$x^2 + (4x)^2 = 17x^2 = 1 \quad \text{and so} \quad |x| = \frac{1}{\sqrt{17}}.$$

The points are therefore

$$\left(-\frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right).$$

One can also solve this problem using vectors. In particular, the vector $\langle 1, 4\rangle$ translates points along this line and the point $(0, 0)$ is on this line. The vector

$$\hat{V} = \frac{1}{\sqrt{17}}\langle 1, 4\rangle$$

is a unit vector that point along this line. Therefore,

$$\hat{V} + (0, 0) \quad \text{and} \quad -\hat{V} + (0, 0)$$

both lie on the unit circle and on L. Of course, these are the points given above.

Problem 1. The line L intersects $(1, 5)$ and $(3, 9)$. Find all points on L that lie a distance of three away from the point $(1, 5)$.

Problem 2. The line L has slope 3 and intersects $(2, 1)$. Find all points on L that lie a distance of two away from the point $(2, 1)$.

Problem 3. The line segment L has endpoints $(1, 4)$ and $(2, 7)$. The ray $L_\perp$ is perpendicular to L, points in the positive y direction and intersects L at $(2, 7)$. Find the vertices of a square with side length $\frac{1}{4}$ that has $(2, 7)$ as a vertex and that has two of its edges lying along L and $L_\perp$. Assume all units are in inches. You are to program a computer to trace the square given above. Find an equation for the path that starts at the vertex of the square that lies on L and that is not $(2, 7)$ and that moves along the given square in the counter-clockwise direction at a rate of one inch per minute.

Problem 4. The center of the circle C is at $(2,5)$ and the circle intersects the point $(5,9)$. What is the equation for the circle?

Problem 5. Using only the definition of a circle, find the center and radius of a circle that intersects the points $(-2,1)$, $(3,2)$, and $(2,4)$.

Problem 6. Find all points on the circle that have x coordinate equal to $\frac{1}{4}$. Sketch these points on the unit circle.

Problem 7. Find all points on the circle that have y coordinate equal to $-\frac{2}{5}$. Sketch these points on the unit circle.

Problem 8. Find the projection of the point $(-4,2)$ onto the unit circle.

Problem 9. Where does the line given by $y = 3x$ intersect the unit circle?

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II.3.1 Reflecting Sets Across Arbitrary Lines

Example 1. Let L_1 be the line passing through $(3,1)$ and $(5,4)$. Let L_2 be the line passing through $(1,5)$ and $(2,8)$. What is the equation that describes the reflection of L_2 across L_1?

Solution

Denote by R the reflection of L_2 across L_1. The line L_1 is given by the equation

$$y = \frac{3}{2}(x-3) + 1 = \frac{3}{2}x - \frac{7}{2}.$$

The line L_2 is given by the equation

$$y = 3(x-1) + 5 = 3x + 2.$$

The lines L_1 and L_2 intersect at the point (x,y) where

$$\frac{3}{2}x - \frac{7}{2} = 3x + 2.$$

Therefore,

$$\frac{3}{2}x = -\frac{11}{2} \quad \text{and so} \quad x = -\frac{11}{3}.$$

Solving for y, we obtain

$$y = -9.$$

Therefore, L_1 and L_2 intersect at $\left(-\frac{11}{3}, -9\right)$. This point is on R. We need only find one more point on R. To find such a point, notice that the point $(1,5)$ is on L_2. We will reflect this point across L_1. The line L given by

$$y = -\frac{2}{3}(x-1) + 5 = -\frac{2}{3}x + \frac{17}{3}$$

is perpendicular to L_1 and intersects $(1,5)$. Therefore, L_1 and L intersect the point (x,y) if and only if

$$\frac{3}{2}x - \frac{7}{2} = -\frac{2}{3}x + \frac{17}{3}$$

and so

$$\left(\frac{3}{2} + \frac{2}{3}\right)x = \frac{17}{3} + \frac{7}{2}.$$

From this we get that

$$\frac{13}{6}x = \frac{55}{6} \quad \text{and so} \quad x = \frac{55}{13}.$$

We therefore have that

$$y = \frac{3}{2} \cdot \frac{55}{13} - \frac{7}{2}.$$

Let m be the intersection

$$m = \left(\frac{55}{13}, \frac{3}{2} \cdot \frac{55}{13} - \frac{7}{2}\right).$$

Then the point q given by

$$q = (m - (1,5)) + m$$

is the reflection of $(1,5)$ across L_2 and therefore R is the line intersecting $\left(-\frac{11}{3}, -9\right)$ and q. You can calculate the equation of this line using, for example, the point slope formula for a line, but these calculations are getting a little nasty and so we'll leave it at this. On the exam, I would give you nicer numbers and probably tell you where the original two lines intersect.

Problem 1. What are the coordinates of the reflection of $(2,3)$ across the line given by $y = 0$?

Problem 2. What are the coordinates of the reflection of $(2,3)$ across the line given by $x = 0$?

Problem 3. Let L be the line intersecting $(2,1)$ and $(7,2)$. What are the coordinates for the reflection of the point $(3,4)$ across L?

Problem 4. The line L_1 intersects $(2,5)$ and $(3,6)$. Line L_2 intersects $(2,7)$ and $(3,10)$. What is the equation of the reflection of L_1 across the line L_2?

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II.3.2 Inverse Functions

Problem 1. Suppose that $f = \{(a,1),(b,1),(c,0),(d,5),(e,3),(f,1),(g,5),(h,4),(i,5)\}$.

(a) What is f^{-1}?

(b) What is $f^{-1}(0)$?

(c) What is $f^{-1}(1)$?

(d) What is $f^{-1}(2)$?

(e) What is $f^{-1}(3)$?

(f) What is $f^{-1}(4)$?

(g) What is $f^{-1}(5)$?

Example 1. The range of an invertible function h is $(2,7) \cup [12,20)$. What is the domain of the inverse of h?

SOLUTION

The domain of the inverse of a function is the range of the original function, so the domain of h^{-1} is $(2,7) \cup [12,20)$

Example 2. The graph of the invertible function f is a line with slope 5. The graph of its inverse function, f^{-1}, is also a line. What is its slope?

SOLUTION

Inverting f interchanges the x and y coordinates, and so the slope of f^{-1} is the reciprocal of the slope of f, that is, $\frac{1}{5}$.

Example 3. The graph of the invertible function f is a line with slope 3 that intersects $(1,2)$. Find an equation for f^{-1}.

Solution

The line given by f^{-1} has slope $\frac{1}{3}$ and intersects $(2,1)$. Therefore,

$$f^{-1}(x) = \frac{1}{3}(x-2)+1.$$

Problem 2. Let f be the the function whose graph is a line with slope 3. The graph of the inverse function f^{-1} is also a line. What is the slope of this line?

Problem 3. Let f be the the function given by $f(x) = 2x+1$. Calculate $f^{-1}(x)$.

Problem 4. Let f be the the function given by

$$f(x) = (x-4)^3 + 1.$$

Calculate $f^{-1}(x)$.

Example 4. The one-to-one function f is defined by

$$f(x) = \frac{3x}{x-2}.$$

What is its inverse function and what is the domain and range of its inverse function? Graph both f and f^{-1}.

Solution

The domain of f is $(-\infty, 2) \cup (2, \infty)$ and the range of f is $(-\infty, 3) \cup (3, \infty)$. So, the range of f^{-1} is $(-\infty, 2) \cup (2, \infty)$ and the domain of f^{-1} is $(-\infty, 3) \cup (3, \infty)$. We now calculate the inverse function. If (x, y) is on the graph of f^{-1}, then

$$x = \frac{3y}{y-2}$$

and so

$$x(y-2) = xy - 2x = 3y$$

and so

$$xy - 3y = (x-3)y = 2x.$$

Therefore,

$$y = \frac{2x}{x-3}.$$

Use a graphing calculator or application to graph the two functions and check your solution.

Problem 5. Let f be the the function given by

$$f(x) = \frac{3x-1}{x+2}.$$

Calculate $f^{-1}(x)$. What are the domains and ranges of f and f^{-1}? What do you notice about this?

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II.3.3 Restricting the Domain

Problem 1. Suppose that f is the function given by

$$f = \{(1,2),(1,5),(1,7),(2,-1),(2,3),(5,-1),(5,0),(6,3),(7,10)\}.$$

Find at least two different restrictions of f so that f becomes invertible on this restricted domain and then write down f^{-1} for each of these restrictions.

Problem 2. Find a domain consisting of exactly three disjoint intervals so that the function given by $f(x) = x^2$ is invertible on this domain, the domain of the inverse function is $[0,\infty)$, and the range of the inverse function contains at least one negative number and one positive number. Sketch the graph both f and f^{-1}.

Problem 3. Calculate the inverse of the function f given by $f(x) = 2(x-1)^3 + 4$.

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II.4.1 Abstract Motions on a Circle

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II.4.2 Circle Actions and the Method of Coordinates on a Circle

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Example 1. The point $(2,4)$ is rotated around $(0,0)$ by the point $\left(\frac{5}{13},\frac{12}{13}\right)$ to the point p. Find the coordinates for p.

SOLUTION

$$p=\left(2\cdot\frac{5}{13}-4\cdot\frac{12}{13},2\cdot\frac{12}{13}+4\cdot\frac{5}{13}\right).$$

Problem 1. Rotate the point $(4,-10)$ around the origin by $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. What are the coordinates of the resulting point?

Problem 2. The points $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(\frac{5}{7},\frac{2\sqrt{6}}{7}\right)$ both lie on the unit circle. Verify this and then compute $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)\star\left(\frac{5}{7},\frac{2\sqrt{6}}{7}\right)$.

Problem 3. Let p be the point $\left(\frac{1}{3},\frac{\sqrt{8}}{3}\right)$ and q be the point $\left(\frac{2}{9},\frac{\sqrt{77}}{9}\right)$. Calculate $\angle pOq$ and $\angle qOp$.

Example 2. A line, L, passes through the points $(2,5)$ and $(4,9)$. We rotate L about $(0,0)$ by the point $\left(\frac{1}{5},\frac{\sqrt{24}}{5}\right)$. Verify that $\left(\frac{1}{5},\frac{\sqrt{24}}{5}\right)$ is a point on the unit circle and write an equation describing the rotated line.

Solution

First verify that $\left(\frac{1}{5}, \frac{\sqrt{24}}{5}\right)$ lies on the unit circle. To do this, we need only show that its distance to the origin is one. We have that

$$\left(\frac{1}{5}\right)^2 + \left(\frac{\sqrt{24}}{5}\right)^2 = \frac{1}{25} + \frac{24}{25} = \frac{25}{25} = 1.$$

All that we need to do is rotate the points $(2, 5)$ and $(4, 9)$. We then calculate the equation for the line that intersects the rotated points. We have that

$$\left(\frac{1}{5}, \frac{\sqrt{24}}{5}\right) \star (2, 5) = \left(\frac{1}{5} \cdot 2 - \frac{\sqrt{24}}{5} \cdot 5, \frac{1}{5} \cdot 5 + \frac{\sqrt{24}}{5} \cdot 2\right) = \left(\frac{2 - 5\sqrt{24}}{5}, \frac{5 + 2\sqrt{24}}{5}\right)$$

and

$$\left(\frac{1}{5}, \frac{\sqrt{24}}{5}\right) \star (4, 9) = \left(\frac{1}{5} \cdot 4 - \frac{\sqrt{24}}{5} \cdot 9, \frac{1}{5} \cdot 9 + \frac{\sqrt{24}}{5} \cdot 4\right) = \left(\frac{4 - 9\sqrt{24}}{5}, \frac{9 + 4\sqrt{24}}{5}\right).$$

An equation for the line that intersects these two rotated point is

$$y = \frac{9 + 4\sqrt{24} - (5 + 2\sqrt{24})}{4 - 9\sqrt{24} - (2 - 5\sqrt{24})} \left(x - \frac{2 - 5\sqrt{24}}{5}\right) + \frac{5 + 2\sqrt{24}}{5}.$$

Therefore, the equation for the rotated line is

$$y = \frac{2 + \sqrt{24}}{1 - 2\sqrt{24}} \left(x - \frac{2 - 5\sqrt{24}}{5}\right) + \frac{5 + 2\sqrt{24}}{5}.$$

Example 3. A line through the origin has a slope equal to two. Rotate this line about the origin by the angle $\left(\frac{4}{5}, \frac{3}{5}\right)$. What is the slope of the resulting line?

Solution

The line of slope two through the origin intersects the unit circle at the point $(x, 2x)$, therefore

$$x^2 + 4x^2 = 5x^2 = 1$$

and so

$$x = \frac{1}{\sqrt{5}}.$$

Therefore, the line intersects the unit circle at the point $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$. Since

$$\begin{aligned}\left(\frac{4}{5}, \frac{3}{5}\right) \star \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) &= \left(\frac{4}{5} \cdot \frac{1}{\sqrt{5}} - \frac{3}{5} \cdot \frac{2}{\sqrt{5}}, \frac{4}{5} \cdot \frac{2}{\sqrt{5}} + \frac{3}{5} \cdot \frac{1}{\sqrt{5}}\right) \\ &= \left(-\frac{2}{5\sqrt{5}}, \frac{11}{5\sqrt{5}}\right),\end{aligned}$$

the rotated line has slope equal to

$$-\frac{11}{2}.$$

Problem 4. A line through the origin has a slope equal to three. Rotate this line about the origin by the angle $\left(\frac{4}{5}, \frac{3}{5}\right)$. What is the slope of the resulting line?

Problem 5. Let L be the line intersecting $(1, 2)$ and $(3, 6)$. Calculate the equation of the line given by the rotation of the line L around the origin by $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. For now, you may assume that rotations take lines to lines.

Example 4. The point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is a point on the unit circle. About what angle must you rotate this point around the origin to rotate it to the point $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$?

Solution

The angle is a point (a, b) on the unit circle. So, we have that

$$(a, b) \star \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(\frac{a\sqrt{3}}{2} - \frac{b}{2}, \frac{a}{2} + \frac{b\sqrt{3}}{2}\right) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right).$$

Therefore,

$$\begin{cases} \frac{a\sqrt{3}}{2} - \frac{b}{2} = \frac{2}{\sqrt{5}} \\ \frac{a}{2} + \frac{b\sqrt{3}}{2} = \frac{1}{\sqrt{5}}, \end{cases}$$

and so

$$\begin{cases} \frac{3a}{2} - \frac{b\sqrt{3}}{2} = \frac{2\sqrt{3}}{\sqrt{5}} \\ \frac{a}{2} + \frac{b\sqrt{3}}{2} = \frac{1}{\sqrt{5}}, \end{cases}$$

Therefore,

$$2a = \frac{2\sqrt{3}+1}{\sqrt{5}} \quad \text{and so} \quad a = \frac{2\sqrt{3}+1}{2\sqrt{5}}.$$

Finally,

$$b = \frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{5}} - \frac{2\sqrt{3}+1}{4\sqrt{5}}\right) = \frac{3-2\sqrt{3}}{2\sqrt{3}\sqrt{5}} = \frac{\sqrt{3}-2}{2\sqrt{5}}.$$

Problem 6. The point $\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right)$ is a point on the unit circle. About what angle must you rotate this point around the origin to rotate it to the point $\left(-\frac{1}{3}, \frac{\sqrt{8}}{3}\right)$?

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II.4.3 Rotating Points About an Arbitrary Point

Example 1. Rotate the point $(2,5)$ around the point $(1,3)$ by the angle $\left(\frac{3}{5}, \frac{4}{5}\right)$.

Solution

Let p be the point obtained by the above rotation. If V is the vector $\langle 1,3\rangle$, then

$$-V+(1,3)=(0,0) \quad \text{and} \quad -V+(2,5)=(1,2).$$

The point p is given by

$$\begin{aligned} p &= V+\left(\left(\frac{3}{5}, \frac{4}{5}\right) \star (1,2)\right) \\ &= V+\left(\frac{3}{5}-\frac{8}{5}, \frac{6}{5}+\frac{4}{5}\right) \\ &= V+(-1,2) \\ &= \langle 1,3\rangle+(-1,2)=(0,5). \end{aligned}$$

Problem 1. Rotate the point $(4,-10)$ around the point $(6,1)$ by $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. What are the coordinates of the resulting point? Draw a picture to show what this looks like.

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II.4.4 Fractions of a Circle and Measurement of Angles

Example 1. How many radians is $20°$?

Solution

$$2\pi \cdot \frac{20}{360} = \frac{\pi}{9}.$$

Example 2. How many degrees is $\frac{7\pi}{12}$ radians?

Solution

$$360° \cdot \frac{\frac{7\pi}{12}}{2\pi} = 360° \cdot \frac{7}{24} = 15° \cdot 7 = 105°.$$

Example 3. A point p on the circle has the property that the arc between $(1,0)$ and p is $\frac{1}{12}$th of a unit circle. What fraction of the unit circle lies between p and $(1,0)$?

Solution

$$\frac{11}{12}.$$

Problem 1. Convert the following degree measures of angles into radian measures of the angles:

(a) $780°$,
(b) $20°$
(c) $-9°$.

Problem 2. Convert the following radian measures of angles into degree measures of the angles:

(a) $\frac{5\pi}{3}$ radians,
(b) 5 radians,
(c) $\frac{7\pi}{8}$.

Problem 3. Find the principle angles (angles in $[0, 2\pi)$) corresponding to the following angles,

(a) $-73°$,

(b) $560°$,

(c) $-584°$,

(d) $-273°$,

(e) $780°$.

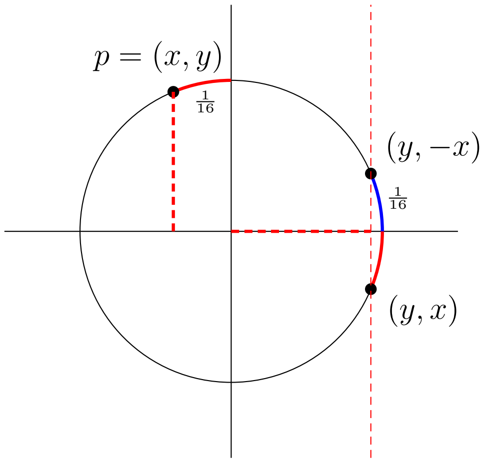
Example 4. A point p on the circle has the property that the arc between $(1,0)$ and p is $\frac{5}{16}$ths of a unit circle. In which quadrant does it lie? What fraction of the unit circle lies between $(0,1)$ and p?

Solution

Since
$$\frac{1}{4} < \frac{5}{16} < \frac{1}{2},$$
p lies in Quadrant II. The fraction of the circle between $(0,1)$ and p is
$$\frac{5}{16} - \frac{1}{4} = \frac{1}{16}.$$

Example 5. A point p on the circle has coordinates (x, y). The arc between $(1,0)$ and p is $\frac{5}{16}$ths of a unit circle. What fraction of a circle is the arc from $(1,0)$ to $(y, -x)$?

Solution



Examine the picture to see that the arc is $\frac{1}{16}$th of the unit circle. You should definitely try to make similar calculations using the points (y, x), $(-x, y)$, and $(x, -y)$.

Problem 4. Suppose that the point p is a point on the unit circle lying in the second quadrant and that the arc between $(1, 0)$ and p is $\frac{5}{11}$ths of the unit circle. Suppose that p has coordinates (a, b). Let q be a point on the unit circle that has b as an x-coordinate. What are the possible choices for q? For each such choice of q, what fraction of the circle is the arc between $(1, 0)$ and q?

Problem 5. Suppose that the point p is a point on the unit circle lying in the third quadrant and that the arc between $(1, 0)$ and p is $\frac{5}{9}$ths of the unit circle. Suppose that p has coordinates (a, b). Let q be a point on the unit circle that has b as an x-coordinate. What are the possible choices for q? For each such choice of q, what fraction of the circle is the arc between $(1, 0)$ and q?

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II.5.1 Reflections and Rotation by Half of a Circle

Problem 1. Let L be the line given by $y = 2x + 7$. Reflect the point $(1, 12)$ across this line to obtain the point p. Find the coordinates for p. Now, reflect the point p across L. What point will you obtain?

Problem 2. Rotate the point $(3, 7)$ by half a circle around the origin to obtain the point p. Find the coordinates for p. Now, rotate the point p by half a circle again around the origin. What point will you obtain?

Problem 3. Find two reflections that, when consecutively applied to any given point in the plane results in a rotation of half a circle around the origin of the given point.

Problem 4. Find a line L such that reflection of the point $(2, 3)$ across L is equivalent to adding the vector $\langle 1, 5\rangle$ to $(2, 3)$. Which points in the plane will have the property that reflection across L will be equivalent to translation by $\langle 1, 5\rangle$? What translation corresponds to the reflection of $\langle 1, 5\rangle + (2, 3)$ across L?

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II.5.2 Inverting the Axes

Problem 1. Graph $f(x) = (x - 2)$ and then graph $g(x) = \frac{1}{(x-2)}$.

Problem 2. Graph $f(x) = (x + 1)^2$ and then graph $g(x) = \frac{1}{(x+1)^2}$.

Problem 3. Graph $f(x) = (x - 5)^3$ and then graph $g(x) = \frac{1}{(x-5)^3}$.

Problem 4. Graph $f(x) = (x + 3)^4$ and then graph $g(x) = \frac{1}{(x+3)^4}$.

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III.1.1 Graphing Quadratic Functions

Example 1. Find the vertex of the parabola given by

$$y = 4(x+3)^2 + 2.$$

SOLUTION

$$(-3, 2).$$

Example 2. Consider the quadratic function

$$p(x) = x^2 + x - 8.$$

Where is the vertex of the graph of p? Does p have a maximum or a minimum value? If it has one, explain where it is and what its value is.

SOLUTION

Suppose that

$$x^2 + x - 8 = A(x+h)^2 + k = Ax^2 + 2Ahx + Ah^2 + k.$$

The we have that

$$A = 1,$$

$$2Ah = 2h = 1 \quad \text{and so} \quad h = \frac{1}{2},$$

and

$$Ah^2 + k = \frac{1}{4} + k = -8 \quad \text{and so} \quad k = -\frac{33}{4}.$$

Therefore,

$$p(x) = \left(x + \frac{1}{4}\right)^2 - \frac{33}{4}$$

and so the vertex of f is at $\left(-\frac{1}{4}, -\frac{33}{4}\right)$ and f has a minimum when x is $-\frac{33}{4}$.

Example 3. Suppose that a is non-zero and that

$$ax^2 + bx + c = 0.$$

Solve for x. Do not use any previously memorized formula, the point is for you to derive this formula.

Solution

We first write the given quadratic polynomial in a suitably scaled and shifted form. Namely, we have that

$$\begin{aligned} ax^2 + bx + c &= A(x+h)^2 + k \\ &= Ax^2 + 2Ahx + Ah^2 + k. \end{aligned}$$

We therefore have that

$$A = a,$$

$$2Ah = 2ah = b \quad \text{and so} \quad h = \frac{b}{2a},$$

and

$$Ah^2 + k = a\left(\frac{b}{2a}\right)^2 + k = \frac{b^2}{4a} + k = c,$$

and so

$$k = c - \frac{b^2}{4a} = \frac{4ac - b^2}{4a}.$$

Finally we have that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0$$

if and only if

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

if and only if

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking square roots of both sides and using the equality

$$\sqrt{z^2} = |z|,$$

we see that

$$ax^2 + bx + c = 0 \quad \text{if and only if} \quad \left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2|a|}.$$

Therefore,

$$ax^2 + bx + c = 0 \quad \text{if and only if} \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{4a}.$$

Subtracting $\frac{b}{2a}$ from each side of the equality, we see that

$$ax^2 + bx + c = 0 \quad \text{if and only if} \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{4a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 4. A parabola intersects the points $(1, 3)$, $(-2, -3)$, and $(2, 13)$. Where is the vertex of the parabola?

Solution

A parabola satisfies the equation

$$y = A(x+h)^2 + k.$$

The requirement that the above points lie on the given parabola gives us equations involving A, h, and k. In particular, we have that

$$\begin{cases} A(1+h)^2 + k = A + 2Ah + Ah^2 + k = 3 \\ A(-2+h)^2 + k = 4A - 4Ah + Ah^2 + k = -3 \\ A(2+h)^2 + k = 4A + 4Ah + Ah^2 + k = 13. \end{cases}$$

Subtracting the first equation from the second and the third gives us the system

$$\begin{cases} 3A - 6Ah = -6 \\ 3A + 2Ah = 10. \end{cases}$$

Adding three times the second equation to the first equation in the system immediately above gives us the equality

$$12A = 24 \quad \text{and so} \quad A = 2.$$

Since

$$3A + 2Ah = 10$$

we have the equality

$$6 + 4h = 10 \quad \text{and so} \quad h = 1.$$

Finally, our first equation in the first system above gives us the equality

$$4A - 4Ah + Ah^2 + k = 8 - 8 + 2 + k = 2 + k = -3, \quad \text{and so} \quad k = -5.$$

Therefore,

$$y = 2(x+1)^2 - 5.$$

The vertex of this parabola is at $(-1, -5)$.

Problem 1. A parabola intersects the points $(-1, 12)$, $(1, 4)$, and $(2, 6)$. Where is the vertex of the parabola?

Problem 2. Let f be the function given by

$$f(x) = x^2.$$

Let g be the quadratic polynomial given by

$$g(x) = x^2 + 4x - 10.$$

Find the vertex of g by using transformations of functions to scale and shift f. Find all points where g is zero, again by only appealing to transformations of functions. Sketch the graph of g.

Problem 3. Let f be the quadratic polynomial given by

$$f(x) = 3x^2 - 2x - 5.$$

Find the vertex f by using transformations of functions. Find all points where f is zero, again by only appealing to transformations of functions. Describe which transformations you have used and what your primitive function is.

Problem 4. Find all real roots of the polynomial function f where:

(a) $f(x) = x^2 - 5x + 6$,
(b) $f(x) = x^2 + 2x - 5$,
(c) $f(x) = 2x^2 + x + 3$.

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III.1.2 Minimizing Quadratic Functions

Example 1. Consider the quadratic function

$$p(x) = x^2 + x - 8.$$

Where is the vertex of the graph of p? Does p have a maximum or a minimum value? If it has one, explain where it is and what its value is.

SOLUTION

Suppose that

$$x^2 + x - 8 = A(x + h)^2 + k = Ax^2 + 2Ahx + Ah^2 + k.$$

The we have that

$$A = 1,$$

$$2Ah = 2h = 1 \quad \text{and so} \quad h = \frac{1}{2},$$

and

$$Ah^2 + k = \frac{1}{4} + k = -8 \quad \text{and so} \quad k = -\frac{33}{4}.$$

Therefore,

$$p(x) = \left(x + \frac{1}{4}\right)^2 - \frac{33}{4}$$

and so the vertex of f is at $\left(-\frac{1}{4}, -\frac{33}{4}\right)$ and f has a minimum when x is $-\frac{33}{4}$.

Problem 1. Find the minimum y-value of a point on the graph of f where

$$f(x) = 3x^2 + 2x + 1.$$

Problem 2. Find the maximum y-value of a point on the graph of f where

$$f(x) = -x^2 + x - 1.$$

Example 2. You enclose a rectangular yard with a fence that boarders a large brick wall on one side. You have 200 feet of fencing. What is the maximum area of yard that you can enclose?

Solution

The rectangular yard has length L and width W. Suppose that the large brick wall lies along a side of length L. Then we have that

$$L + 2W = 200 \quad \text{and so} \quad L = 200 - 2W.$$

The area of the yard is therefore

$$A = LW = (200 - 2W)W = -2W^2 + 200W = -2(W - 50)^2 + 5000.$$

The area is therefore given by a quadratic function of W and has a maximum when W is 50. The yard with maximal area will have length equal to 100 feet and width equal to 50 feet. Its area will be 5000 square feet.

Example 3. At time zero, a boat is two miles west and three miles south of the port, whose coordinates are given to be $(0, 0)$. Therefore the boat's original position is $(-2, -3)$. The boat travels with a velocity of five miles per hour north and one mile per hour east. What is the closest the boat gets to the port and at what time does the boat get this close? Solve this problem without appealing to perpendicularity. It is important to be able to do it in this way because it is the only way to solve the problem involving minimizing the distance between two boats moving at a constant velocity.

Solution

The position of the boat at time t is

$$B(t) = t\langle 1, 5\rangle + (-2, -3) = (t-2, 5t-3).$$

The square of the distance of the boat to the origin is

$$d(t)^2 = (t-2)^2 + (5t-3)^2 = t^2 - 4t + 4 + 25t^2 - 30t + 9 = 26t^2 - 34t + 13.$$

We have that

$$\begin{aligned} d(t)^2 &= 26t^2 - 34t + 13 \\ &= A(t+h)^2 + k = At^2 + 2Aht + Ah^2 + k. \end{aligned}$$

Therefore,

$$A = 26,$$

$$2Ah = 52h = -34 \quad \text{and so} \quad h = -\frac{17}{26},$$

and

$$Ah^2 + k = 26 \cdot \frac{17^2}{26^2} + k = \frac{17^2}{26} + k = 13, \quad \text{and so} \quad k = 13 - \frac{17^2}{26}.$$

Therefore,

$$d(t)^2 = 26\left(t - \frac{17}{26}\right)^2 + 13 - \frac{17^2}{26}.$$

The boat gets closest when

$$t = \frac{17}{26}$$

and at this time, the distance is

$$\sqrt{13 - \frac{17^2}{26}}.$$

Problem 3. A boat is initially one mile east and three miles north of the harbor. The coordinates of the port are taken to be $(0, 0)$ and north is taken to be in the direction of the positive y axis. The boat is moving at a constant velocity of three miles per hour north and one mile per hour east. Find the equation for the boat's position as a function of time, sketch its velocity vector at time zero, and draw the path of motion of the boat on the coordinate plane. What is the boat's speed? In the above problem, what is the closest the boat above gets to the port? At what time does this occur?

Now assume that the boat is initially seven miles east and four miles South of the harbor but with the same constant velocity vector as the boat above. What is the closest the boat above gets to the port? At what time does this occur?

Problem 4. Boat A is initially at position $(2, 5)$ and moves at a constant velocity of $\langle 4, 1\rangle$. Boat B is at position $(8, 3)$ and moves at a constant velocity of $\langle 2, 10\rangle$. What is the minimum distance between the boats and at what time are the boats this minimum distance apart?

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III.1.3 The Factor Theorem

Example 1. Suppose that f is the quadratic polynomial whose roots are 1 and -4, and whose leading coefficient is 3. What does $f(x)$ equal?

Solution

$$f(x) = 3(x-1)(x+4).$$

Example 2. Let

$$p(x) = x(x-2)^3(x-4).$$

What are the zeros of p and what are their respective orders?

Solution

$x = 0$ is a zero of order 1, $x = 2$ is a zero of order 3, and $x = 4$ is a zero of order 1.

Example 3. A degree three polynomial has two distinct real roots. What are the possible orders of the roots?

Solution

Suppose that p is a degree three polynomial with distinct roots a and b. The factor theorem implies that the polynomial p is given by

$$p(x) = (x-a)(x-b)g(x),$$

where g is a degree one polynomial. Since g is a degree one polynomial, it has a real root. The root of g must either be a or b and so one root of p is of order one and the other is of order two.

Example 4. What is the degree of the polynomial $3x^4 + 2x^2 - x - 10$.

Solution

4

Example 5. Suppose that the degree of the polynomial r is less than the degree of q. Find r and q so that
$$x^2 + 3x - 5 = q(x+1) + r.$$

Solution

Since q is a degree one polynomial, r is a constant and so

$$\begin{aligned} x^2 + 3x - 5 &= (ax+b)(x+1) + c \\ &= ax^2 + (a+b)x + (b+c), \end{aligned}$$

and so

$$a = 1, \quad a + b = 1 + b = 3 \implies b = 2, \quad b + c = 2 + c = -5 \implies c = -7.$$

We therefore have that

$$x^2 + 3x - 5 = (x+2)(x+1) - 7.$$

Example 6. Suppose that the degree of the polynomial r is less than the degree of q. Find r and q so that
$$2x^4 + 3x^2 - x + 1 = q(x^2+4) + r.$$

Solution

The polynomial q is a degree two polynomial and r is a degree one polynomial. Therefore,

$$\begin{aligned} 2x^4 + 3x^2 - x + 1 &= (ax^2 + bx + c)(x^2+4) + (dx+e) \\ &= (a)x^4 + (b)x^3 + (4a+c)x^2 + (4b+d)x + (4c+e), \end{aligned}$$

and so
$$a = 2, \quad b = 0, \quad 4a + c = 8 + c = 3 \implies c = -5,$$
$$4b + d = d = 0, \quad 4c + e = -20 + e = 1 \implies e = 21.$$

Example 7. Factor the polynomial $x^2 - 5x + 6$ and then use the factorization to find the roots of the polynomial.

Solution

$x^2 - 5x + 6 = (x-2)(x-3)$ and so is zero if and only if $x = 2$ and $x = 3$.

Example 8. Let

$$p(x) = 2x^{100} + x^{73} + 5x^3 + 3 \quad \text{and} \quad q(x) = x + 1.$$

What is the remainder when $p(x)$ is divided by $q(x)$?

Solution

Since

$$p(x) = (?)(x+1) + \text{constant} \quad \text{and} \quad p(-1) = 0 + \text{constant},$$

the remainder is equal to $p(-1)$ which equals -1.

Example 9. Suppose that f is the quadratic polynomial whose roots are 1 and -4, and whose leading coefficient is 3. What does $f(x)$ equal?

Solution

$$f(x) = 3(x-1)(x+4).$$

Example 10. Can a degree six polynomial have exactly three distinct real roots and each of these roots be of order one?

Solution

No. Factoring out the three factors corresponding to the three real roots yields a degree three polynomial, which must have at least one real root.

Example 11. At least how many roots must a degree three polynomial have? At most how many can it have?

Solution

It must have at least one root. It can have as many as three.

Example 12. At least how many roots can a degree two polynomial have? At most how many can it have?

Solution

An even degree polynomial may have no roots, so zero. It can have at most two.

Problem 1. A degree three polynomial, f, has the property that $f(1) = f(2) = f(3) = 0$. We also have that $f(0) = 12$. What does $f(x)$ equal?

Problem 2. A degree three polynomial has exactly two real roots, $x = 1$ and $x = 2$. What are the possible orders of these roots? If $f(0) = 4$ then what are the possibilities for $f(x)$?

Problem 3. Let a and b be polynomial functions. Match coefficients to find polynomials q and r with the degree of r strictly less than the degree of b so that

$$a = qb + r$$

where

(a) $a = 5x^3 + x - 5$ and $b = x + 3$
(b) $a = 2x^5 + x^2 + 1$ and $b = x^2 - x + 3$,

(c) $a = 3x^4 + 2x^2 - x + 1$ and $b = x^2 - x + 1$.

Problem 4. Solve the problem immediately above, but by using polynomial long division.

Problem 5. Solve for m and b so that $x^2 - x + 3 - (mx + b)$ has exactly one real root at $x = 1$. What is the order of this root?

Problem 6. Solve for m and b so that $x^3 - (mx + b)$ has exactly one real root of order two at $x = 2$.

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III.1.4 Polynomial Graphs

Example 1. Sketch the graph of the polynomial f, where f is given by

$$f(x) = (x+2)(x-1)^2(x-4)^4$$

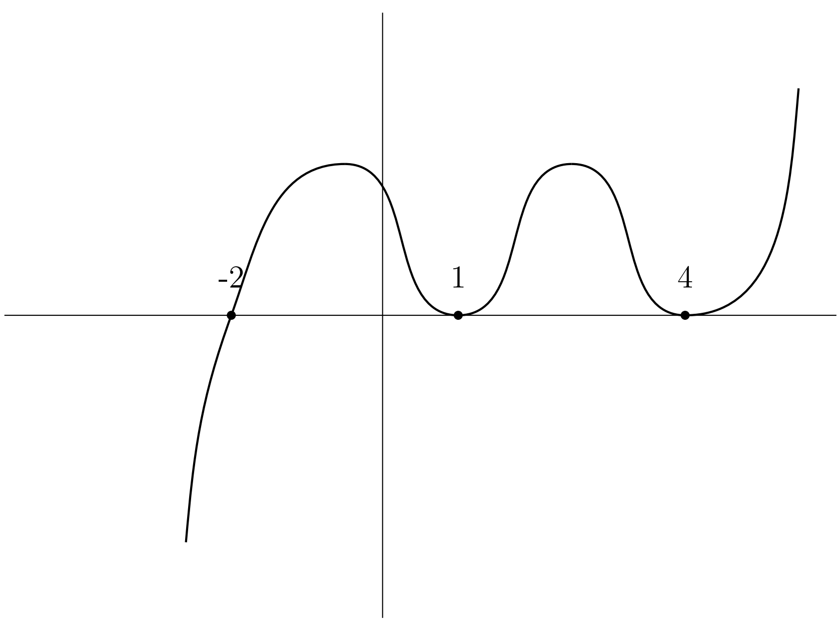
and then graph on a number line the solutions to the equation $f(x) < 0$.

Solution

Note first that

$$f(x) = x^7 + \text{lower degree terms.}$$

So the asymptotic behavior is that of x^7. Furthermore, f has zeros when x is -2, 1, and 4. The graph of f is sketched below.



So,

$$f(x) < 0$$

when

$$x \in (-\infty, -2).$$

Example 2. Sketch the graph of the function f, where f is given by

$$f(x) = (x+3)^3(x-2)^2(x-4)^5$$

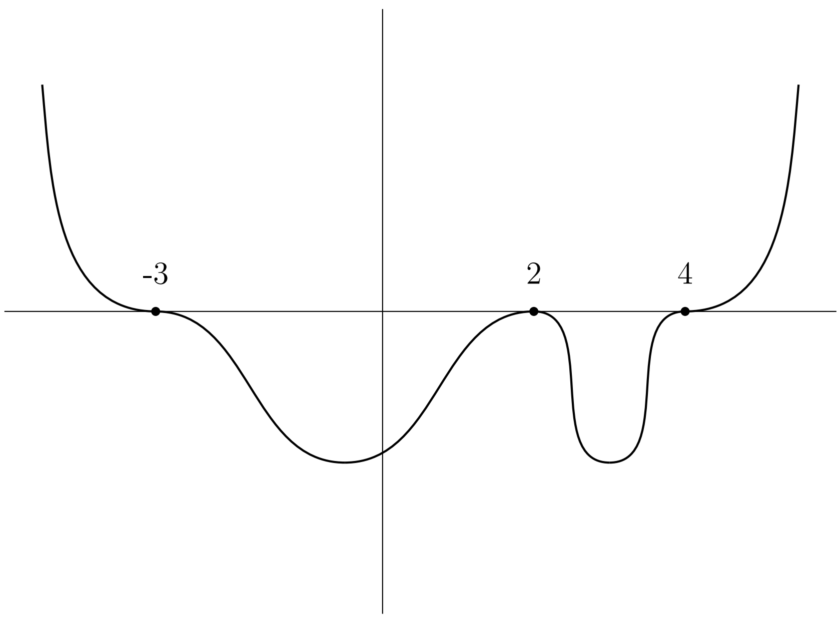
and describe using interval notation the solutions to the inequality $f(x) \geq 0$.

SOLUTION

Note first that

$$f(x) = x^{10} + \text{lower degree terms.}$$

So the asymptotic behavior is that of x^{10}. Furthermore, f has zeros when x is -3, 2, and 4. The graph of f is sketched below.



So,

$$f(x) \geq 0$$

when

$$x \in (-\infty, -3] \cup \{2\} \cup [4, \infty).$$

Problem 1. Let f be the polynomial given by

$$f(x) = x^3(x+3)(x+1)^2(x-4).$$

(a) Sketch the graph of f.
(b) Describe in interval notation the set of all x such that $f(x) > 0$.
(c) Graph on a number line the set of all x such that $f(x) > 0$.

(d) Describe in interval notation the set of all x such that $f(x) \geq 0$.
(e) Graph on a number line the set of all x such that $f(x) \geq 0$.

Problem 2. Let f be the polynomial given by

$$f(x) = (x+6)(x+4)^2(x+2)^3(x-3)(x-5).$$

(a) Sketch the graph of f.
(b) Describe in interval notation the set of all x such that $f(x) > 0$.
(c) Graph on a number line the set of all x such that $f(x) > 0$.
(d) Describe in interval notation the set of all x such that $f(x) \geq 0$.
(e) Graph on a number line the set of all x such that $f(x) \geq 0$.

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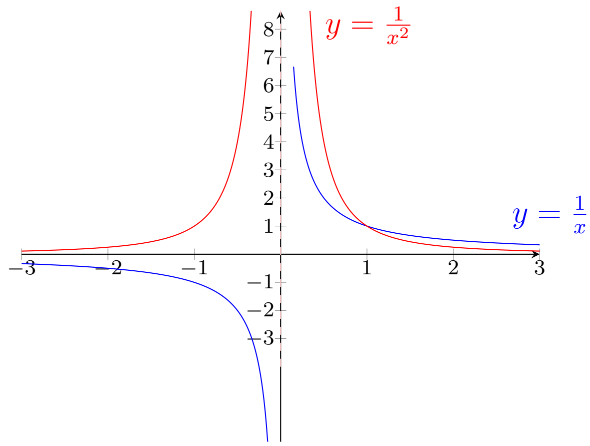
III.2.1 Graphing Reciprocals of Polynomials

Example 1. Sketch the graphs given by $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$.

Solution



Problem 1. Sketch the graph given by

$$y = x + 2$$

and use it to sketch the graph given by

$$y = \frac{1}{x+2}.$$

Problem 2. Sketch the graph given by

$$y = (x+2)(x-1)$$

and use it to sketch the graph of

$$y = \frac{1}{(x+2)(x-1)}.$$

Problem 3. Sketch the graph given by

$$y = (x-2)(x-3)^2(x+5)^3$$

and use it to sketch the graph of

$$y = \frac{1}{(x-2)(x-3)^2(x+5)^3}.$$

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III.2.2 Rational Functions with Linear Denominators

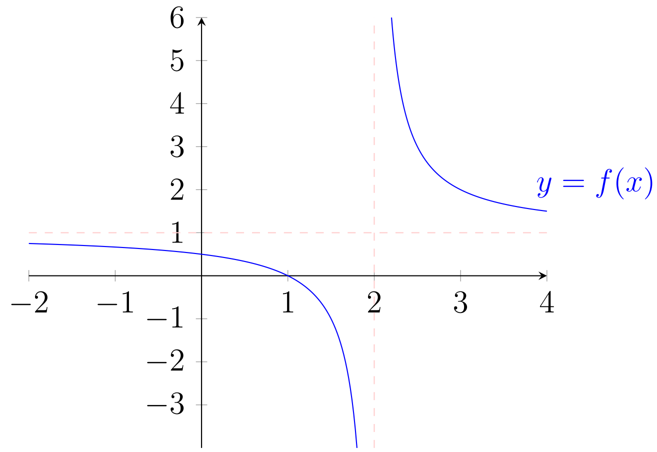
Example 1. Sketch the graph of the function

$$f(x) = \frac{1}{x-2} + 1$$

and graph on the real line the solutions to the inequality $f(x) < 0$.

SOLUTION

Use transformations to graph f below.



Since

$$\frac{1}{x-2} + 1 = \frac{x-1}{x-2} = 0 \quad \text{if and only if} \quad x = 1,$$

we see that the the graph of f intersects the x-axis at $(1, 0)$. From the graph, we see that the inequality is satisfied when x is in $(-\infty, 1) \cup (2, \infty)$.

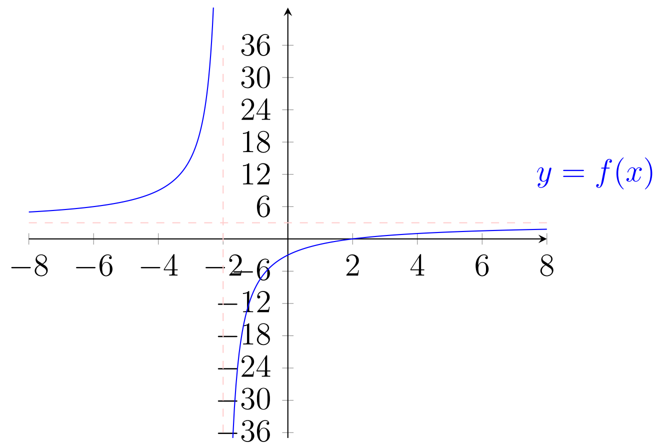
Example 2. Sketch the graph of the function

$$f(x) = \frac{3x-6}{x+4}$$

and graph on the real line the solutions to the inequality $f(x) \geq 0$.

Solution

Use transformations to graph f below.



Since

$$f(x) = \frac{3x-6}{x+4} = 0 \quad \text{if and only if} \quad x = 2,$$

we see that the the graph of f intersects the x-axis at $(2, 0)$. From the graph, we see that the inequality is satisfied when x is in $(-\infty, -2) \cup [2, \infty)$.

Example 3.

(a) Solve the inequality

$$\frac{x-2}{x-4} > 0.$$

(b) Find the solution to the inequality

$$\frac{x-2}{x-4} \geq 0.$$

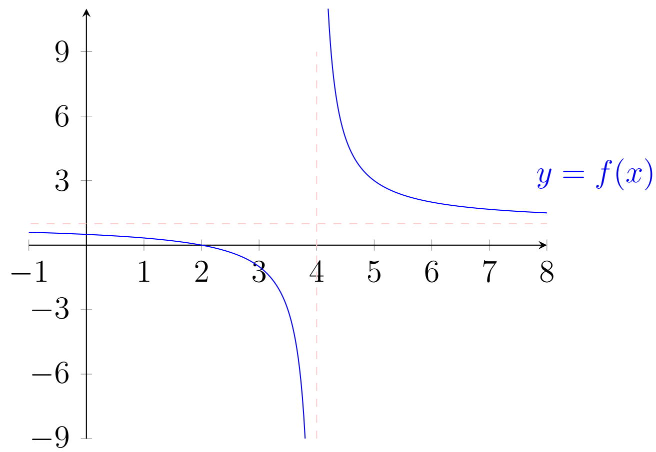
The only difference between this problem and the one above is that we have the inequality $\geq$ instead of $>$.

Solution

Use transformations to graph f below where f is given by

$$f(x) = \frac{x-2}{x-4}.$$

Note that $f(x)$ is zero if and only if x equals 2.



From the graph, we see that the inequality

$$\frac{x-2}{x-4} > 0$$

is satisfied when x is in $(-\infty, 2) \cup (4, \infty)$ and the inequality

$$\frac{x-2}{x-4} \geq 0$$

is satisfied when x is in $(-\infty, 2) \cup [4, \infty)$

Example 4. Sketch the graph the function

$$f(x) = \frac{x^2-1}{x-2}.$$

Find all solutions to the equation

$$\frac{x^2-1}{x-2} < x.$$

Solution

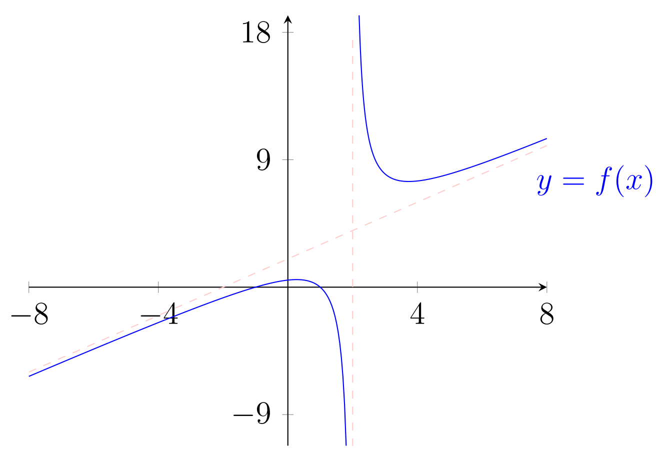
First notice that $f(x)$ is zero if and only if x is either one or negative one. Now, here is a little trick that lets you avoid doing long division. Notice that

$$x = x - 2 + 2.$$

Therefore,

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x - 2} \\ &= \frac{(x - 2 + 2)^2 - 1}{x - 2} \\ &= \frac{(x - 2)^2 + 4(x - 2) + 4 - 1}{x - 2} \\ &= (x - 2) + 4 + \frac{3}{x - 2} = x + 2 + \frac{3}{x - 2}. \end{aligned}$$

We now graph f by using transforming functions and viewing f as a sum of functions.



A real number x is a solution to the inequality

$$\frac{x^2-1}{x-2} < x$$

if and only if

$$\frac{x^2-1}{x-2} - x = \frac{x^2-1}{x-2} - \frac{x(x-2)}{x-2} = \frac{x^2-1-x^2+2x}{x-2} = \frac{2x-1}{x-2} < 0.$$

You can now sketch the graph given by

$$y = \frac{2x-1}{x-2}$$

to solve the inequality graphically or solve the inequality algebraically. Since we have use graphing several times already, let us now approach the problem algebraically for some variety. Note that there are two cases. In the first case,

$$x-2<0, \quad \text{which is to say that} \quad x<2.$$

In this case, we multiply both sides of the inequality to obtain the equivalent inequality

$$\frac{2x-1}{x-2} < 0 \quad \text{if and only if} \quad 2x-1>0.$$

Therefore,

$$x > \frac{1}{2}.$$

Therefore,

$$x > \frac{1}{2} \quad \text{and} \quad x<2, \quad \text{so} \quad \frac{1}{2} < x < 2.$$

In the second case,

$$x-2 \geq 0, \quad \text{which is to say that} \quad x \geq 2.$$

In this case, we multiply both sides of the inequality to obtain the equivalent inequality

$$\frac{2x-1}{x-2} < 0 \quad \text{if and only if} \quad 2x-1<0.$$

Therefore,

$$x < \frac{1}{2} \quad \text{and so} \quad x < \frac{1}{2} \quad \text{and} \quad x \geq 2.$$

Therefore, no solutions can come from this case and so the solution set is the set

$$\left(\frac{1}{2}, 2\right).$$

Example 5. Sketch the graph given by

$$y = \frac{2x^2 + x - 4}{x + 1}.$$

SOLUTION

Use the techniques we used above and check your answer with a graphing tool.

Problem 1. Use the scaling and shifting transformations to sketch the graph given by

$$y = \frac{2x + 1}{x - 5}.$$

Problem 2. Sketch the graph given by

$$y = \frac{2x^2 + x - 1}{x - 1}.$$

Problem 3. Sketch the graph given by

$$y = \frac{x^3 + 5x - 1}{x + 2}.$$

Problem 4. Let f be the rational function given by

$$f(x) = \frac{x - 2}{x - 4}.$$

(a) Find all x with $f(x) > 0$.
(a) Find all x with $f(x) \geq 0$.

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III.2.3 General Rational Functions

Example 1. Write the improper fraction $\frac{x^3+2x-1}{x^2+x-2}$ as a polynomial plus a proper fraction.

Solution

$$\begin{aligned} x^3+2x-1 &= (ax+b)(x^2+x-2)+(cx+d) \\ &= (a)x^3+(b+a)x^2+(b-2a+c)x+(d-2b), \\ \text{so} \quad & a=1, \quad b+a=b+1=0 \implies b=-1, \\ & b-2a+c=-3+c=2 \implies c=5, \quad d-2b=d+2=-1 \implies d=-3. \end{aligned}$$

Therefore,

$$x^3+2x-1=(x-1)(x^2+x-2)+(5x-3)$$

and so

$$\frac{x^3+2x-1}{x^2+x-2}=x-1+\frac{5x-3}{x^2+x-2}.$$

Example 2. Solve for A and B so that

$$\frac{2x+1}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}.$$

Solution

Add the fractions an equate numerators so that

$$B(x-1)+A(x+3)=(A+B)x+(3A-B)=2x+1.$$

We therefore have that

$$\begin{cases} A+B=2 \\ 3A-B=1. \end{cases}$$

Adding the first equation to the second, we obtain the equality

$$4A=3 \implies A=\frac{3}{4}$$

and so

$$A+B=\frac{3}{4}+B=2 \implies B=\frac{5}{4}.$$

Example 3. Solve for A, B, and C so that

$$\frac{x-2}{(x-4)(x^2+3)}=\frac{A}{x-4}+\frac{Bx+C}{x^2+3}.$$

Solution

Adding the fractions together and equating numerators, we obtain the equality

$$A(x^2+3)+(Bx+C)(x-4)=(A+B)x^2+(C-4B)x+(3A-4C)=x-2$$

and so

$$\begin{cases} A+B=0 \\ C-4B=1 \\ 3A-4C=-2. \end{cases}$$

Therefore,

$$A=-B$$

and so

$$\begin{cases} C-4B=1 \\ 3B+4C=2. \end{cases}$$

Adding -4 times the first equation to the second, we obtain the equality

$$19B=-2 \implies B=-\frac{2}{19},\ A=\frac{2}{19},\text{ and } C=1+4B=\frac{11}{19}.$$

Example 4. Let f be the rational function given by

$$f(x)=\frac{(x-5)^2(x+1)^3(x+3)(x+7)^2}{(x-1)^2(x+5)^3}.$$

What are the zeros of f and their orders. What are the poles of f and their orders?

Solution

Zeros: 5 (order 2), -1 (order 3), -3 (order 1), -7 (order 2). Poles: 1 (order 2), -5 (order 3).

Problem 1. List all zeros along with their orders and all poles along with their orders of the function f given by

$$f(x)=\frac{(x-7)^3(x+2)^4(x-1)}{x^5(x+4)^2(x-3)}.$$

Example 5. Sketch the graph of the rational function f, where f is given by

$$f(x) = \frac{(x-4)^2(x-2)^3(x+1)(x+7)^2}{(x-1)^2(x+5)^3}.$$

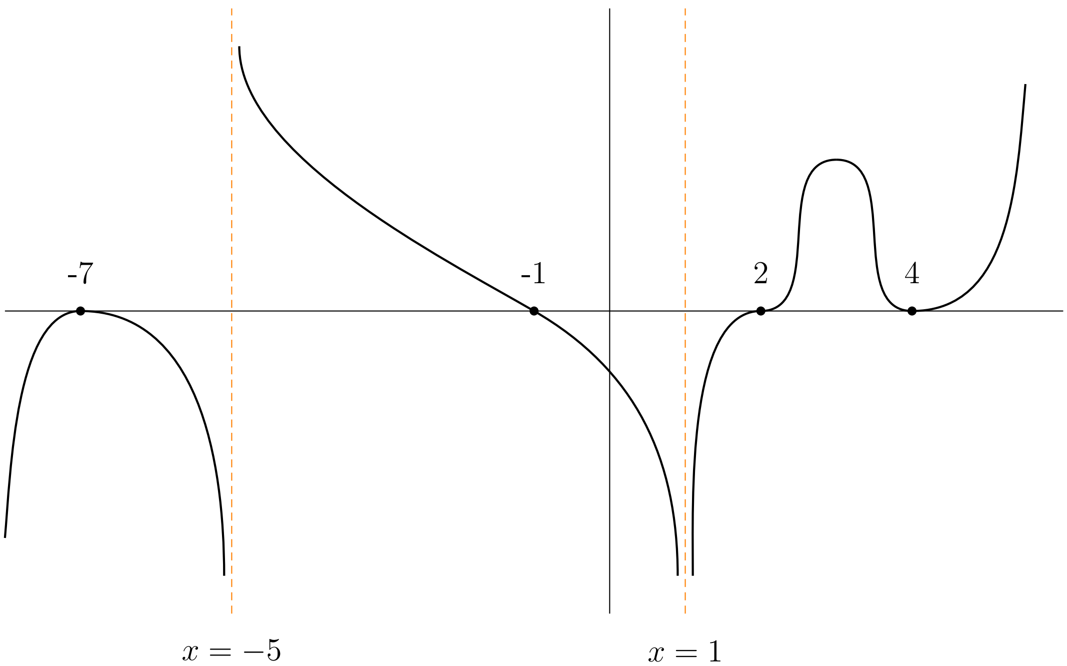
Find all x so that (1) $f(x) < 0$, and (2) $f(x) \leq 0$.

SOLUTION

Note first that

$$f(x) = x^3 + \text{lower degree terms.}$$

So the asymptotic behavior is that of x^3. Furthermore, f has zeros when x is -7, -1, 2, and 4 and vertical asymptotes $x = 1$ and $x = -5$. Also 1 is an even order pole and -5 is an odd order poll of f. The graph of f is sketched below.



We see from inspecting the graph that

$$f(x) < 0 \quad \text{when} \quad x \in (-\infty, -7) \cup (-7, -5) \cup (-1, 1) \cup (1, 2),$$

and that

$$f(x) \leq 0 \quad \text{when} \quad x \in (-\infty, -5) \cup [-1, 1) \cup (1, 2] \cup \{4\}.$$

Note that $\{4\}$ is the set that contains only the number 4, you could also write this as $[4, 4]$.

Example 6. Sketch the graph of the rational function f, where f is given by

$$f(x) = \frac{x^2 - 4}{(x-1)(x+7)}.$$

Find all solutions to the equation

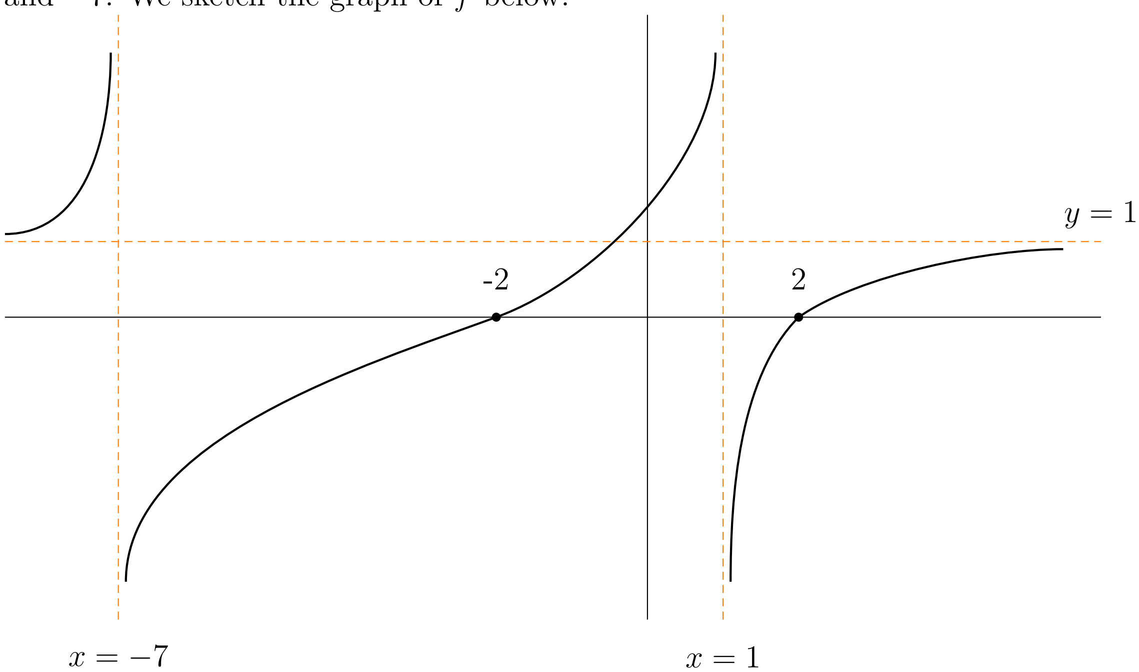
$$f(x) \leq 0.$$

SOLUTION

Note that $f(x) = 1 +$ proper fraction. Therefore, $f(x)$ has the line $y = 1$ as a horizontal asymptote. Furthermore,

$$f(x) = \frac{(x-2)(x+2)}{(x-1)(x+7)}$$

and so has as vertical asymptotes the lines $x = 1$ and $x = -7$ and the points 2 and -2 are zeros of f, each of order 1. The function f has odd order polls at 1 and -7. We sketch the graph of f below.



We see from inspecting the graph that

$$f(x) \leq 0 \quad \text{when} \quad x \in (-7, -2] \cup (1, 2].$$

Problem 2. Sketch the graph of the function $f(x) = \dfrac{(x+1)(x+3)}{x^2(x-1)}$.

Problem 3. Sketch the graph the function

$$f(x) = \frac{(x-4)^2(x-2)^3(x+1)(x+7)^2}{(x-1)^2(x+5)^3}.$$

Problem 4. Sketch the graph of the function f given by

$$f(x) = \frac{(x+1)(x+3)}{x^2(x-1)}$$

and sketch on the real number line the set of all x satisfying $f(x) > 0$.

Problem 5. Find all solutions to the equation

$$\frac{x^2-4}{(x-1)(x+7)} \leq 0.$$

Problem 6. Find all solutions to the equation

$$\frac{(x+5)^2(x+3)^5(x-2)(x-4)^3}{x(x-1)(x-3)^4(x-6)} > 0.$$

Problem 7. Find all solutions to the equation

$$\frac{(x+2)^4(x-3)^3(x-2)}{(x+1)(x+2)^3(x-7)} < 0.$$

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III.2.4 Inequalities Involving Piecewise Rational Functions

Example 1. Graph on a number line the solutions to the inequality

$$|x^2-4| > 2.$$

<u>SOLUTION</u>

We begin by writing the function f given by

$$f(x) = |x^2-4|$$

as a piecewise defined function. Notice first that

$$x^2-4=0 \quad \text{if and only if} \quad x=-2 \quad \text{or} \quad x=2.$$

We can use the graph

$$y = x^2-4$$

to see that

$$x^2-4<0 \quad \text{if and only if} \quad -2<x<2$$

and

$$x^2-4\geq 0 \quad \text{if and only if} \quad x\in(-\infty,-2]\cup[2,\infty).$$

We therefore have that

$$\begin{aligned} |x^2-4| &= \begin{cases} -(x^2-4) & \text{if } x^2-4<0 \\ x^2-4 & \text{if } x^2-4\geq 0 \end{cases} \\ &= \begin{cases} -x^2+4 & \text{if } -2<x<2 \\ x^2-4 & \text{if } x\in(-\infty,-2]\cup[2,\infty). \end{cases} \end{aligned}$$

In the first case, when x is in $(-2,2)$, we have that

$$|x^2-4| = -x^2+4 > 2,$$

that is,

$$x^2<2, \quad \text{hence} \quad x\in(-\sqrt{2},\sqrt{2}).$$

Since we are in the first case, we have that x is in the intersection of $(-2,2)$ and $(-\sqrt{2},\sqrt{2})$, which is $(-\sqrt{2},\sqrt{2})$. So, $(-\sqrt{2},\sqrt{2})$ is the solution set in this case.

In the second case, when x is in $(-\infty, -2] \cup [2, \infty)$, we have that

$$|x^2 - 4| = x^2 - 4 > 2,$$

that is,

$$x^2 - 6 > 0, \quad \text{hence} \quad x \in (-\infty, -\sqrt{6}) \cup (\sqrt{6}, \infty).$$

Since we are in the first case, we have that x is in the intersection of $(-\infty, -2] \cup [2, \infty)$ and $(-\infty, -\sqrt{6}) \cup (\sqrt{6}, \infty)$, which is $(-\infty, -\sqrt{6}) \cup (\sqrt{6}, \infty)$. So, $(-\infty, -\sqrt{6}) \cup (\sqrt{6}, \infty)$ is the solution set in this case. We therefore have that the solution set to the inequality

$$|x^2 - 4| > 2$$

is the set

$$(-\infty, -\sqrt{6}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{6}, \infty).$$

Example 2. Write the function f given by

$$f(x) = |x^2 - 9| - |3x - 7|$$

as a piecewise defined function. When is $f(x) > 0$?

<u>SOLUTION</u>

We begin by writing $|x^2 - 9|$ as a piecewise defined function. Notice first that

$$x^2 - 9 = 0 \quad \text{if and only if} \quad x = -3 \quad \text{or} \quad x = 3.$$

We can use the graph

$$y = x^2 - 9$$

to see that

$$x^2 - 9 < 0 \quad \text{if and only if} \quad -3 < x < 3$$

and

$$x^2 - 9 \geq 0 \quad \text{if and only if} \quad x \in (-\infty, -3] \cup [3, \infty).$$

We therefore have that

$$|x^2 - 9| = \begin{cases} -(x^2-9) & \text{if } x^2 - 9 < 0 \\ x^2 - 9 & \text{if } x^2 - 9 \geq 0 \end{cases}$$
$$= \begin{cases} -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \in (-\infty, -3] \cup [3, \infty). \end{cases}$$

We furthermore have that

$$|3x - 7| = \begin{cases} -3x + 7 & \text{if } x < \frac{7}{3} \\ 3x + 7 & \text{if } x \geq \frac{7}{3}. \end{cases}$$

By considering $|x^2 - 9|$ and $|3x - 7|$ on the intersection of the regions where each is determined by a single formula, we see that

$$|x^2 - 9| = \begin{cases} -x^2 + 9 & \text{if } -3 < x < \frac{7}{3} \\ -x^2 + 9 & \text{if } \frac{7}{3} \leq x < 3 \\ x^2 - 9 & \text{if } x \in (-\infty, -3] \\ x^2 - 9 & \text{if } x \in [3, \infty). \end{cases} = \begin{cases} x^2 - 9 & \text{if } x \in (-\infty, -3] \\ -x^2 + 9 & \text{if } -3 < x < \frac{7}{3} \\ -x^2 + 9 & \text{if } \frac{7}{3} \leq x < 3 \\ x^2 - 9 & \text{if } x \in [3, \infty) \end{cases}$$

and that

$$|3x - 7| = \begin{cases} -3x + 7 & \text{if } x \in (-\infty, -3] \\ -3x + 7 & \text{if } -3 < x < \frac{7}{3} \\ 3x - 7 & \text{if } \frac{7}{3} \leq x < 3 \\ 3x - 7 & \text{if } x \in [3, \infty). \end{cases}$$

Therefore,

$$|x^2 - 9| - |3x - 7| = \begin{cases} x^2 + 3x - 16 & \text{if } x \in (-\infty, -3] \\ -x^2 + 3x + 2 & \text{if } -3 < x < \frac{7}{3} \\ -x^2 - 3x + 16 & \text{if } \frac{7}{3} \leq x < 3 \\ x^2 - 3x - 2 & \text{if } x \in [3, \infty). \end{cases}$$

We now use this description of f given by

$$f(x) = |x^2 - 9| - |3x - 7|$$

to find all x with such that $f(x)$ is greater than zero. Consider separately the following four cases.

<u>Case 1:</u> $x \leq -3$. In this case, $f(x)$ is positive if and only if $x^2 + 3x - 16$ is positive. Therefore, x is in $\left(-\infty, \frac{-3-\sqrt{73}}{2}\right) \cup \left(\frac{-3+\sqrt{73}}{2}, \infty\right)$ and $x \leq -3$. In this case, the solution set is

$$\left(-\infty, \frac{-3 - \sqrt{73}}{2}\right).$$

<u>Case 2:</u> $-3 < x < \frac{7}{3}$. In this case, $f(x)$ is positive if and only if $-x^2 + 3x + 2$ is positive. Therefore, x is in $\left(\frac{3-\sqrt{17}}{2}, \frac{3+\sqrt{17}}{2}\right)$ and $-3 < x < \frac{7}{3}$. In this case, the solution set is

$$\left(\frac{3 - \sqrt{17}}{2}, \frac{7}{3}\right).$$

<u>Case 3:</u> $\frac{7}{3} \leq x < 3$. In this case, $f(x)$ is positive if and only if $-x^2 - 3x + 16$ is positive. Therefore, x is in $\left(\frac{-3-\sqrt{73}}{2}, \frac{-3+\sqrt{73}}{2}\right)$ and $\frac{7}{3} \leq x < 3$. In this case, the solution set is

$$\left[\frac{7}{3}, \frac{-3 + \sqrt{73}}{2}\right).$$

<u>Case 4:</u> $x \geq 3$. In this case, $f(x)$ is positive if and only if $x^2 - 3x - 2$ is positive. Therefore, x is in $\left(-\infty, \frac{3-\sqrt{17}}{2}\right) \cup \left(\frac{3+\sqrt{17}}{2}, \infty\right)$ and $x \geq 3$. In this case, the solution set is

$$\left(\frac{3 + \sqrt{17}}{2}, \infty\right).$$

The solution set is therefore the set

$$\left(-\infty, \frac{-3 - \sqrt{73}}{2}\right) \cup \left(\frac{3 - \sqrt{17}}{2}, \frac{-3 + \sqrt{73}}{2}\right) \cup \left(\frac{3 + \sqrt{17}}{2}, \infty\right).$$

Problem 1. Suppose that

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < -2 \\ 3 & \text{if } -2 \leq x < 1 \\ 3x + 2 & \text{if } x \geq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} -4x + 2 & \text{if } x < 0 \\ 5 & \text{if } x \geq 0. \end{cases}$$

Write using interval notation and then graph on a real number line all solutions to the inequality

$$f(x) > g(x).$$

Problem 2. Write using interval notation and then graph on a real number line all solutions to the inequality $|x^2 - 4| < 3$.

Problem 3. Write using interval notation and then graph on a real number line all solutions to the inequality $|x^2 - 2|x + 1|| > 2$.

Problem 4. Find all solutions to the equation

$$\left|\frac{1}{x} - 9\right| < x.$$

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III.3.1 Some Basic Applications to Modeling

Example 1. A table and chair set costs \$1200. The table costs \$200 less than the chairs. What does the table cost?

Solution

The cost of a table is T dollars and the cost of the chairs is C dollars. We have the system

$$\begin{cases} T + C = 1200 \\ T = C - 200. \end{cases}$$

We therefore have that

$$T + C = (C - 200) + C = 2C - 200 = 1200$$

and so

$$2C = 1400.$$

We solve the system to obtain the equality

$$C = 700$$

and

$$T = C - 200 = 700 - 200 = 500.$$

The cost of the chairs is therefore \$700 and the cost of the table is \$500.

Example 2. Alice has 10 more records than Bob. Bob has twice the number of records that Cindy has. In total, they have 60 records. How many records does each person have?

Solution

Let A be the number of records that Alice has, B be the number of records that Bob has, and C be the number of records that Cindy has. We have the system of equalities

$$\begin{cases} A = B + 10 \\ B = 2C \\ A + B + C = 60. \end{cases}$$

Combining the first and second equations, we obtain the equality

$$A = B + 10 = 2C + 10.$$

Therefore,

$$A + B + C = 2C + 10 + 2C + C = 5C + 10 = 60.$$

Therefore,

$$C = 10, \quad B = 20, \quad \text{and} \quad A = 30.$$

Example 3. Dan bought a couch at a store for \$1600. It was on sale and marked off by 20%. The couch maker sells the couch to the store for half the non-sale price of the couch. How much did the store make off of Dan's purchase?

Solution

We do not know the cost of the couch, denote its cost by C dollars. The couch was marked off by 20%, and so

$$C - 1600 = .2C.$$

Therefore,

$$.8C = \frac{4}{5} \cdot C = 1600 \quad \text{and so} \quad C = 2000.$$

Example 4. Two trains leave stations at the same time and start 300 miles apart. They are traveling on a straight line towards each other. Train A travels at 100 miles per hour and Train B travels at 50 miles per hour. Where do they meet in relation to the train stations?

Solution

A less formulaic approach: Train A travels at twice the rate of Train B and for the same length of time, so it goes twice the distance. Therefore, Train A travels two-thirds of the distance and Train B travels one-third. The sum of the distances traveled is 300 miles. So Train A goes 200 miles and Train B goes 100 miles. The meet 200 miles from Train A's starting station.

A more formulaic approach: The total time duration is unknown, let us denote this duration by t hours. In this time, Train A travels a distance of $100t$ miles and Train B travels a distance of $50t$ miles. Therefore,

$$100t + 50t = 150t = 300,$$

and so

$$t = 2.$$

When the trains meet, Train A is $100 \cdot 2$ miles, or 200 miles from its starting location.

Example 5. Erin invests \$100,000 split between stocks and futures. She earns 20% on the futures and 7% on the stocks in her first year. She ends the year with \$110,000. To the nearest dollar, how much does she invest in futures?

Solution

Denote by S the dollar amount invested in stocks. Denote by F the dollar amount invested in futures. We have the following system

$$\begin{cases} S + F = 100{,}000 \\ .07S + .2F = 10{,}000. \end{cases}$$

Solve the system for S and F.

Example 6. You have a tub with two drains that is filled with 60 cubic feet of water. Drain A drains water at two cubic feet per minute and Drain B drains water at a rate of three cubic feet per minute. You open Drain A and then open Drain B to empty the tub of water. How much earlier must Drain A be opened so that both drains drain the same amount of water?

Solution

Both drains drain the same amount of water, so Drain A and Drain B each drain 30 cubic feet of water. It takes Drain A 15 minutes to do this. It takes Drain B 10 minutes. Open Drain A five minutes before you open Drain B.

Example 7. Frank can paint three square feet per minute and Greg can paint five square feet per minute. How long must Frank and Greg work together to paint 100 square feet?

Solution

We do not know how long they will have to work, denote this time duration by t minutes. The amount painted in this time duration is

$$3t + 5t = 8t = 100.$$

So it takes them $^{100}/_{8}$ minutes, or twelve minutes and 30 seconds.

Example 8. You have two bottles of salty water. Bottle A contains salt water at a concentration of 20 grams per liter and Bottle B contains salt water at a concentration of 5 grams of salt per liter. How many liters of each must you pour into a third bottle in order to obtain a mixture that is 30 liters at a concentration of 10 grams per liter?

Solution

We do not know how many liters of solution from Bottle A and Bottle B we require, denote these respectively by A liters and B liters. After mixing the appropriate amounts, we will obtain a solution of 30 liters of salt water at a concentration of 10 grams of salt per liter. This means that we will have 300 grams of salt. The a,ount of salt that we have in the mixture is the sum of the quantities of salt from A liters of Bottle A and B liters of Bottle B. We therefore have the following system of equations

$$\begin{cases} 20A + 5B = 300 \\ A + B = 30. \end{cases}$$

Solve this system to obtain the equalities

$$A = 10 \quad \text{and} \quad B = 20.$$

Example 9. You have a 10 pound bag of apples and the apples are 80% water by weight. You dehydrate the apples so that they are now only 10% water by weight. How many pounds of dehydrated apples do you have?

Solution

The weight of the apples comes from the weight of the dry apple material and the water in the apples. The dry weight of the apples is 2 pounds, 20% of the 10 pound bag. After dehydration, the dry weight of the apples is unchanged, but now constitutes 90% of the unknown weight of the dehydrated apples. Denote this unknown weight by W pounds to obtain the equality

$$.9W = \frac{9}{10}W = 2.$$

Therefore,

$$W = \frac{20}{9}.$$

Example 10. A jet travels 800 miles against the wind in 2 hours and 1200 miles with the wind in the same amount of time. What is the rate of the jet in still air and what is the rate of the wind?

Solution

Denote by v miles per hour the speed of the jet in still air. Denote by w miles per hour the speed of the wind. We have the system of equations

$$\begin{cases} 2(v+w) = 1200 \\ 2(v-w) = 800 \end{cases}.$$

Equivalently, we have

$$\begin{cases} v+w = 600 \\ v-w = 400 \end{cases}.$$

Solve the system to obtain the equalities

$$v = 500 \quad \text{and} \quad w = 100.$$

Example 11. You have nickels, dimes, and quarters in your pocket. You have 10 coins in total. You have three times the number of nickels as you have dimes. The total value of the coins is one dollar. How many each of nickels, dimes, and quarters do you have?

Solution

Denote by N the number of nickels, D the number of dimes, and Q the number of quarters. The above statements correspond to the system

$$\begin{cases} N + D + Q = 10 \\ N = 3D \\ .05N + .1D + .25Q = 1. \end{cases}$$

Solve this system in any way you like to obtain

$$N = 6, \quad D = 2, \quad \text{and} \quad Q = 2.$$

Example 12. Alice can paint a wall in three hours and Bob can paint the same wall in four hours. How long does it take for Alice and Bob to paint the wall together?

Solution

Alice paints at a rate of $\frac{1\,\text{wall}}{3\,\text{hours}}$ and Bob paints at a rate of $\frac{1\,\text{wall}}{4\,\text{hours}}$. It takes them h hours to paint one wall, so

$$(h\,\text{hours}) \cdot \frac{1\,\text{wall}}{3\,\text{hours}} + (h\,\text{hours}) \cdot \frac{1\,\text{wall}}{4\,\text{hours}} = \frac{h}{3}\,\text{wall} + \frac{h}{4}\,\text{wall} = \frac{7h}{12}\,\text{wall} = 1\,\text{wall}.$$

Therefore,

$$h = \frac{12}{7}\,\text{hours}.$$

Example 13. Hank walks on a 100 foot long moving walkway at a constant rate relative to the motion of the walkway. He turns around at the end and walks back. His round trip takes 40 seconds and the walkway is moving at 2 feet per second. How fast is Hank's motion relative to a stationary observer.

Solution

Hank's speed relative to the walkway is unknown, let us denote this speed by v feet per second. When moving with the walkway's motion, his speed relative to a stationary observer is $v + 2$ feet per second. When moving against the walkways's motion, his speed relative to a stationary observer is $v - 2$ feet per second. His travel time in a direction is the length of the walkway divided by his speed. Therefore,

$$\frac{100}{v-2} + \frac{100}{v+2} = 40.$$

We clear denominators to obtain

$$100(v+2) + 100(v-2) = 200v = 40(v-2)(v+2) = 40\left(v^2 - 4\right).$$

Therefore,

$$5v = v^2 - 4, \quad \text{and so} \quad v^2 - 5v - 4 = 0.$$

Using whatever method you like to solve the quadratic equation given, obtain two potential solutions,

$$v = \frac{5 \pm \sqrt{41}}{2}.$$

Only the positive solution is admissible because v is a positive number, so

$$v = \frac{5 + \sqrt{41}}{2}.$$

Example 14. An unladen swallow flies in a straight line 10 miles against a constant 2 mile per hour head wind. The swallow then turns around and flies back along its path to its starting position. The bird makes the trip in 2 hours. What is the airspeed of an unladen swallow?

Solution

We do not know the airspeed of an unladen swallow, denote it by v miles per hour. Its speed relative to the ground when flying against the wind is $v - 2$ miles per hour. Its speed relative to the ground when flying with the wind is $v + 2$ miles per hour. The time it takes to complete the trip is the sum of the time it takes in each direction, therefore

$$\frac{10}{v+2} + \frac{10}{v-2} = 2.$$

Clear the denominators to obtain

$$10(v-2) + 10(v+2) = 20v = 2(v+2)(v-2) = 2(v^2 - 4).$$

Simplify to obtain the equality

$$v^2 - 10v - 4 = 0.$$

Solving the quadratic equation, we obtain two potential solutions to the problem

$$v = \frac{10 \pm \sqrt{116}}{2}.$$

Since only positive values of v are admissible, we have that

$$v = \frac{10 + \sqrt{116}}{2} = 5 + \sqrt{29}.$$

Problem 1. An airplane flies 2000 miles in a straight line against a constant headwind. The trip takes five hours. The plane returns along the same path at the same speed relative to the wind, the wind's velocity has not changed. The trip now takes four hours. How fast does the airplane fly in still air and how fast is the wind moving?

Problem 2. Bottles A and B contain salty water. Bottle A contains salt at a concentration of 10 grams per liter. Bottle B contains salt at a concentration of 30 grams per liter. How many liters of water from Bottle A and Bottle B should you mix to get 10 liters of salty water with a concentration of 14 grams of salt per liter?

Problem 3. A jet traveling at a constant speed relative to the wind flies 1000 miles against the wind in 2 hours and 1100 miles with the wind in the same amount of time. What is the speed of the jet in still air and what is the speed of the wind?

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IV.1.1 Odd and Even Functions

Example 1. Is the function f given by

$$f(x) = x^6$$

even or odd or neither even nor odd?

<u>SOLUTION</u>

If x is a real number, then

$$f(-x) = (-x)^6 = x^6.$$

Therefore, f is even.

Example 2. Is the function f given by

$$f(x) = x^5$$

even or odd or neither even nor odd?

<u>SOLUTION</u>

If x is a real number, then

$$f(-x) = (-x)^5 = -x^5 = -f(x).$$

Therefore, f is odd.

Example 3. Is the function f given by

$$f(x) = x^4 + |x^3|$$

even or odd or neither even nor odd?

SOLUTION

If x is a real number, then

$$f(-x) = (-x)^4 + |(-x)^3| = x^4 + |-x^3| = x^4 + |x^3| = f(x).$$

Therefore, f is even.

Example 4. Is the function f given by

$$f(x) = -x|x^4 - 2x^2|$$

even or odd or neither even nor odd?

SOLUTION

If x is a real number, then

$$f(-x) = -(-x)|(-x)^4 - 2(-x)^2| = x|x^4 - 2x^2| = -(-x|x^4 - 2x^2|) = -f(x).$$

Therefore, f is odd.

Example 5. Is the function f given by

$$f(x) = x^3 + x^2$$

even or odd or neither even nor odd? Explain your answer with sufficient detail.

SOLUTION

Evaluate f at 1 and -1 to see that

$$f(-1) = (-1)^3 + (-1)^2 = -1 + 1 = 0 \quad \text{and} \quad f(1) = 2.$$

Since

$$f(-1) = 0 \neq 2 = f(1) \quad \text{and} \quad f(-1) = 0 \neq -2 = -f(1),$$

f is neither even nor odd.

Problem 1. Are the functions listed below even, odd, or neither even nor odd?

(a) $a(x) = x^2 + 2|x|^3$,

(b) $b(x) = x - x|x|$,

(c) $c(x) = x^2 + 2x^5$,

(d) $d(x) = x|x^3 - x|$.

Problem 2. Suppose that f and g are functions whose domains have a non-empty intersection. Show that if f is an even function and g is an odd function, then $f \cdot g$ is an odd function.

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IV.2.1 Periodicity

Example 1. The function f is one on every integer and zero everywhere else. What is the principle period of f?

Solution

If x is an integer, then $x+1$ is also an integer and so $f(x+1)$ is again equal to one. If x is not an integer, then $x+1$ is not an integer and so $f(x+1)$ is again equal to zero. Therefore, one is a period. If a is positive and less than one and if x is an integer, then $x+a$ is not an integer and so $f(x)$ does not equal to $f(x+1)$. The principle period is therefore equal to one.

Example 2. A periodic function f that is defined on the entire real line has principle period equal to three. The function g is given by

$$g(x) = 2x + 1.$$

What is the principle period of $f \circ g$?

Solution

Easy way. The function g is the composition of a scaling and a translation. Translations do not change the period of a function. Scaling x by a factor of two scales the graph on the x-axis by a factor of $\frac{1}{2}$. So the period of $f \circ g$ is $\frac{3}{2}$.

Solution

<u>A more algebraic approach</u>. The positive real number a is the principle period of $f \circ g$ if and only if it is the smallest positive number so that for every real number x,

$$(f \circ g)(x + a) = (f \circ g)(x).$$

Notice that

$$(f \circ g)(x + a) = f(g(x + a)) = f(2(x + a) + 1) = f(2x + 1 + 2a)$$

and

$$(f \circ g)(x) = f(g(x)) = f(2x + 1).$$

Take

$$z = 2x + 1$$

and notice that z can attain every real value if x can. Therefore, a is the principle period of $f \circ g$ if and only if it is the smallest positive number such that for every real z,

$$f(z + 2a) = f(z).$$

Since a is the smallest positive number with this property, $2a$ is the smallest positive number such that for every real number z,

$$f(z + 2a) = f(z).$$

Therefore, $2a$ is the principle period of f and this implies that

$$2a = 3.$$

Therefore,

$$a = \frac{3}{2}.$$

Example 3. The function f is one on every integer and zero everywhere else. What is the principle period of f?

Solution

If x is an integer, then $x+1$ is also an integer and so $f(x+1)$ is again equal to one. If x is not an integer, then $x+1$ is not an integer and so $f(x+1)$ is again equal to zero. Therefore, one is a period. If a is positive and less than one and if x is an integer, then $x+a$ is not an integer and so $f(x)$ does not equal to $f(x+1)$. The principle period is therefore equal to one.

Problem 1. Suppose that a function f is periodic. Is it possible that it has only one period?

Problem 2. Give an example of a function that is periodic but that has no principle period.

Problem 3. Suppose that f is a periodic function with principle period equal to five. What is the principle period of $f \circ S_7$? Justify your answer clearly and rigorously.

Problem 4. For any real number x, the floor function, denoted floor, inputs x and outputs the first integer value smaller than or equal to x. We write

$$\text{floor}(x) = \lfloor x \rfloor,$$

where for example,

$$\lfloor 0 \rfloor = 0, \lfloor 1 \rfloor = 1, \lfloor 1.2 \rfloor = 1, \lfloor 1.7 \rfloor = 1, \lfloor 2.3 \rfloor = 2, \lfloor -1.3 \rfloor = -2, \lfloor -.9 \rfloor = -1 \text{ and so on.}$$

What is the principle period of the function f where

$$f(x) = x - \lfloor x \rfloor?$$

Sketch the graph of this function. Now, set

$$g(x) = 3x + \frac{1}{2}.$$

What is the principle period of $f \circ g$?

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IV.2.2 The Sine, Cosine, and Tangent Functions

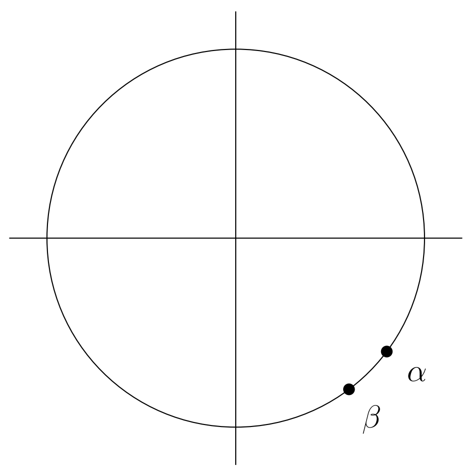
Example 1. For parts (a) through (c), suppose that α and β are angles in quadrant IV, i.e., α and β are in the interval $[{}^{3\pi}\!/_{2}, 2\pi]$. Suppose that $\alpha > \beta$. Put the correct symbol, $>$ or $<$ into the boxes.

(a) $\sin(\alpha)$ $\boxed{}$ $\sin(\beta)$.

(b) $\cos(\alpha)$ $\boxed{}$ $\cos(\beta)$.

(c) $\tan(\alpha)$ $\boxed{}$ $\tan(\beta)$.

Solution



Angle α has a larger x-coordinate since it is farther to the right than β. It has a larger y-coordinate since it is above β. The line from the origin to α and the corresponding line associated to β both have negative slopes, but the first has a smaller magnitude and so is a larger number.
(a) $>$ (b) $>$ (c) $>$

Problem 1. For parts (a) through (c), suppose that α and β are angles in quadrant II, i.e., α and β are in the interval $[{}^{3\pi}\!/_{2}, 2\pi]$. Suppose that $\alpha > \beta$. Put the correct symbol, $>$ or $<$ into the boxes.

(a) $\sin(\alpha)$ $\boxed{}$ $\sin(\beta)$.

(b) $\cos(\alpha)$ $\boxed{}$ $\cos(\beta)$.

(c) $\tan(\alpha)$ $\boxed{}$ $\tan(\beta)$.

Example 2. A point on the unit circle that has angle measure equal to θ has coordinates $\left(\frac{1}{3}, \frac{\sqrt{8}}{3}\right)$. Calculate (a) $\cos(\theta)$ and (b) $\sin(\theA)$.

SOLUTION

$$\cos(\theta) = \frac{1}{3} \quad \text{and} \quad \sin(\theta) = \frac{\sqrt{8}}{3}.$$

Example 3. A point on the unit circle that has angle measure equal to θ has coordinates $\left(\frac{1}{3}, \frac{\sqrt{8}}{3}\right)$. Calculate (a) $\tan(\theta)$, (b) $\sec(\theta)$, (c) $\csc(\theta)$, and (d) $\cot(\theta)$.

SOLUTION

$$\tan(\theta) = \frac{\frac{\sqrt{8}}{3}}{\frac{1}{3}} = \sqrt{8}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} = 3$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{3}{\sqrt{8}}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{\sqrt{8}}$$

Example 4. Calculate $\cos(\theta)$ and $\sin(\theta)$ for each θ in $\{0°, 30°, 45°, 60°, 90°\}$.

SOLUTION

$$\cos(0^\circ) = 1,\ \cos(30^\circ) = \frac{\sqrt{3}}{2},\ \cos(45^\circ) = \frac{\sqrt{2}}{2},\ \cos(60^\circ) = \frac{1}{2},\ \cos(90^\circ) = 0.$$

$$\sin(0^\circ) = 0,\ \sin(30^\circ) = \frac{1}{2},\ \sin(45^\circ) = \frac{\sqrt{2}}{2},\ \sin(60^\circ) = \frac{\sqrt{3}}{2},\ \sin(90^\circ) = 1.$$

Example 5. Calculate $\cos(\theta)$ and $\sin(\theta)$ for each θ in $\left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$.

SOLUTION

$$\cos(0) = 1,\ \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\ \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ \cos\left(\frac{\pi}{2}\right) = 0.$$

$$\sin(0) = 0,\ \sin\left(\frac{\pi}{6}\right) = \frac{1}{2},\ \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},\ \sin\left(\frac{\pi}{2}\right) = 1.$$

Example 6. Calculate $\sin(120^\circ)$.

SOLUTION

$$\frac{\sqrt{3}}{2}$$

Example 7. Calculate $\sec(150^\circ)$.

Solution

$$\sec(150°) = \frac{1}{\cos(150°)} = \frac{2}{\sqrt{3}}.$$

Example 8. Calculate $\tan(240°)$.

Solution

$$\tan(240°) = \tan(60°) = \frac{\sin(60°)}{\cos(60°)} = \sqrt{3}.$$

Example 9. Calculate $\cos\left(-\frac{\pi}{6}\right)$.

Solution

$$\frac{\sqrt{3}}{2}.$$

Problem 2. Show that

$$\tan^2(x) - \sin^2(x) = \tan^2(x)\sin^2(x).$$

Problem 3. Let p be the point in the plane that is a distance of 4 from the origin and that has the property that the ray from the origin to p makes an angle of $\frac{7\pi}{6}$ radians with the ray lying along the positive x-axis. Find the Cartesian coordinates, (x, y) coordinates, for the point p.

Problem 4. Let p be the point in the plane that is a distance of 3 from the origin and that has the property that the ray from the origin to p makes an angle of $\frac{5\pi}{4}$ radians with the ray lying along the positive x-axis. Find the Cartesian coordinates, (x, y) coordinates, for the point p.

Problem 5. The points $(-1, 2)$, $(3, 0)$, and $(2, 5)$ are the vertices of a triangle on a record whose center is at the origin $(0, 0)$. What is the point on the triangle closest to the center of the record? Rotate the record by an angle of $120°$ about its center. Describe the rotated triangle's vertex set and find equations that describe the edges of the rotated triangle.

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IV.2.3 Angle Addition Formulae for Trigonometric Functions

Example 1. Angles A and B are in the first quadrant. We have that $\cos(A) = \frac{1}{3}$ and $\sin(B) = \frac{2}{5}$. Calculate $\sin(A+B)$ and $\cos(A+B)$.

Solution

The pythagorean identities give us the equalities

$$\sin(A) = \sqrt{1-\cos^2(A)} = \sqrt{1-\frac{1}{9}} = \frac{\sqrt{8}}{3}$$

and

$$\cos(B) = \sqrt{1-\sin^2(B)} = \sqrt{1-\frac{4}{25}} = \frac{\sqrt{21}}{5}.$$

Therefore,

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) = \frac{\sqrt{8}}{3}\cdot\frac{\sqrt{21}}{5} + \frac{1}{3}\cdot\frac{2}{5}$$

and

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) = \frac{1}{3}\cdot\frac{\sqrt{21}}{5} - \frac{\sqrt{8}}{3}\cdot\frac{2}{5}.$$

Example 2. Calculate $\cos(15°)$.

Solution

We approach this in two ways. First:

$$\cos(15°) = \cos(45°-30°) = \cos(45°)\cos(30°)+\sin(45°)\sin(30°) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}.$$

Second:

$$\cos(15°) = \cos\left(\frac{30°}{2}\right) = \sqrt{\frac{1+\cos(30°)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

Example 3. Calculate $\sin(22.5°)$.

Solution

We use the half angle formula to obtain the equalities

$$\sin(22.5°) = \sin\left(\frac{45°}{2}\right) = \sqrt{\frac{1-\cos(45°)}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Example 4. Calculate $\cos(67.5°)$.

Solution

We have that

$$\begin{aligned}\cos(67.5°) &= \cos(45°)\cos(22.5°) - \sin(45°)\sin(22.5°)\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2-\sqrt{2}}}{2}.\end{aligned}$$

Note that $\cos(67.5°)$ must equal to $\sin(22.5°)$. Can you see why by drawing points on a circle and considering reflections across the line $y = x$? Can you show numerically that this is the case by simplifying the expression above and the expression in the previous question?

Problem 1. Suppose that A and B are both angles in the first quadrant. If

$$\sin(A) = \frac{1}{4} \quad \text{and} \quad \cos(B) = \frac{1}{5},$$

then compute the following:

(a) $\sin(A+B)$,
(b) $\sin(A-B)$,
(c) $\cos(A+B)$,
(d) $\cos(A-B)$,
(e) $\tan(A+B)$,
(f) $\tan(A-B)$.

Problem 2. Calculate the sine, cosine, and tangent of the following angles

(a) $15°$,
(b) $7.5°$,
(c) $3.25°$,
(d) $142.5°$.

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IV.2.4 Transformations of Trigonometric Functions

Example 1. What is the period, amplitude, maximum value, and minimum value of the function $f(\theta) = 5\cos\left(\pi\left(\theta - \frac{\pi}{3}\right)\right) - 2$? Sketch the graph of this function.

Solution

$$\text{period} = 2, \text{ amplitude} = 5, \text{ maximum value} = 3, \text{ and minimum value} = -7.$$

Sketch it on a graphing application to check your answer.

Example 2. What is the period of the function $f(\theta) = 3\tan\left(\frac{2}{3}\left(\theta - \frac{\pi}{3}\right)\right) - 5$?

Solution

The period is

$$\frac{\pi}{\frac{2}{3}} = \frac{3\pi}{2}.$$

Example 3. What is the period, amplitude, maximum value, and minimum value of the function $f(\theta) = 5\cos\left(\pi\left(\theta - \frac{\pi}{3}\right)\right) - 2$? Sketch the graph of this function.

Solution

$$\text{period} = 2, \text{ amplitude} = 5, \text{ maximum value} = 3, \text{ and minimum value} = -7.$$

Sketch it on a graphing application to check your answer.

Example 4. What is the period of the function $f(\theta) = 3\tan\left(\frac{2}{3}\left(\theta - \frac{\pi}{3}\right)\right) - 5$?

Solution

The period is

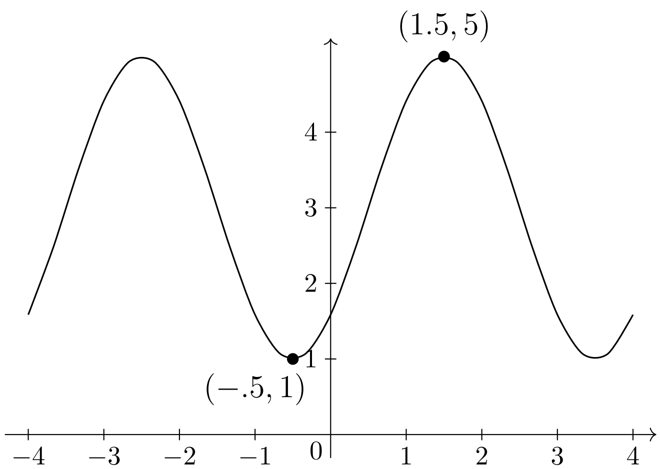
$$\frac{\pi}{\frac{2}{3}} = \frac{3\pi}{2}.$$

Problem 1. What is the period, amplitude, maximum value, and minimum value of the function $f(\theta) = 5\cos\left(\pi\left(\theta - \frac{\pi}{3}\right)\right) - 2$.

Problem 2. What is the principle period of the function f, g, and h, where

$$f(\theta) = 2\sin(\theta - 1), \quad g(\theta) = \tan(3\pi\theta), \quad \text{and} \quad h(\theta) = \cos(7\theta)?$$

Problem 3. What is the sinusoidal function whose graph is given below?



Problem 4. Graph the function f given by $f(\theta) = 3\sin\left(2\left(\theta - \frac{\pi}{4}\right)\right) + 1$. What is its principle period?

Example 5. Show that $\cot(\theta) + \tan(\theta) = \sec(\theta)\csc(\theta)$.

Solution

We have that

$$\begin{aligned}
\cot(\theta) + \tan(\theta) &= \frac{\cos(\theta)}{\sin(\theta)} + \frac{\sin(\theta)}{\cos(\theta)} \\
&= \frac{\cos(\theta)}{\sin(\theta)} \cdot \frac{\cos(\theta)}{\cos(\theta)} + \frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{\sin(\theta)}{\sin(\theta)} \\
&= \frac{\cos^2(\theta) + \sin^2(\theta)}{\cos(\theta)\sin(\theta)} \\
&= \frac{1}{\cos(\theta)\sin(\theta)} \\
&= \frac{1}{\cos(\theta)}\frac{1}{\sin(\theta)} = \sec(\theta)\csc(\theta).
\end{aligned}$$

Problem 5. Show that

$$\cot(2x) = \frac{1 - \tan^2(x)}{2\tan(x)}.$$

Example 6. A point on the unit circle that has angle measure equal to θ has coordinates $\left(\frac{1}{3}, \frac{\sqrt{8}}{3}\right)$. Calculate (a) $\sec(\theta)$, (b) $\csc(\theta)$, and (c) $\cot(\theta)$.

Solution

$$\sec(\theta) = \frac{1}{\cos(\theta)} = 3$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{3}{\sqrt{8}}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{\sqrt{8}}$$

Example 7. Calculate $\sec(150°)$.

SOLUTION

$$\sec(150^\circ) = \frac{1}{\cos(150^\circ)} = \frac{2}{\sqrt{3}}.$$

Problem 6. Calculate the secant, cosecant, and cotangent function at the following angles:

(a) -30°,

(b) 120°,

(c) 150°.

Problem 7. What is the domain of the function tan and the function csc? Write the function $\tan(\theta)$ in terms of the function $\csc(\theta)$ where both are defined in quadrants I, II, III, and IV. It will help to recall the Pythagorean Identities.

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IV.2.5 Inverse Trigonometric Functions

Example 1. Calculate $\cos^{-1}(\cos(210°))$.

Solution

$150°$

Example 2. Calculate $\cos\left(\tan^{-1}(3x^2)\right)$.

Solution

First denote by θ the angle $\tan^{-1}(3x^2)$. Note that θ is an angle in quadrant I or IV because the range of the inverse tangent function is $(-\pi/2, \pi/2)$. Since $3x^2$ is positive or zero, the angle θ is the angle associated to a point p on the unit circle in quadrant I. The point $(1, 3x^2)$ lies on the ray from the origin to p and the vector V given by

$$V = \langle 1, 3x^2 \rangle$$

points in the direction of this ray. The length of V is

$$||V|| = ||\langle 1, 3x^2 \rangle|| = \sqrt{1 + 9x^4}$$

and so

$$\hat{V} + (0,0) = \left\langle \frac{1}{\sqrt{1+9x^4}}, \frac{3x^2}{\sqrt{1+9x^4}} \right\rangle = \left(\frac{1}{\sqrt{1+9x^4}}, \frac{3x^2}{\sqrt{1+9x^4}} \right).$$

Therefore, p projects onto the unit circle as the point

$$\left(\frac{1}{\sqrt{1+9x^4}}, \frac{3x^2}{\sqrt{1+9x^4}} \right)$$

and so

$$\cos(\theta) = \frac{1}{\sqrt{1+9x^4}} \quad \text{and} \quad \sin(\theta) = \frac{3x^2}{\sqrt{1+9x^4}}.$$

This problem may confuse you because there is an x involved that is being used as a variable but it is not the x-coordinate of a point. If the slope of the line were not $3x^2$ but just given as m, the problem might look easier. In this case, every point of the form (x, mx) lies on the line. Here, x now represents an x coordinate. Therefore, the point $(1, m)$ lies on the line. Project this point onto the unit circle to get the point $(1/\sqrt{1+m^2}, m/\sqrt{1+m^2})$. The x-coordinate of this point is the cosine of $\tan^{-1}(m)$ and the y-coordinate is the sine of this angle.

Example 3. Calculate $\sin^{-1}\left(\cos\left(\frac{8\pi}{7}\right)\right)$.

Solution

Again, it is best to draw pictures to see how this works. The answer is

$$\frac{\pi}{7} - \frac{\pi}{2}.$$

Example 4. Calculate $\tan^{-1}\left(\tan\left(\frac{9\pi}{11}\right)\right)$.

Solution

Here is a way of solving the problem without drawing a picture. Note that the tangent function is π-periodic. Therefore,

$$\tan\left(-{}^{2\pi}/_{11})\right) = \tan\left(\pi - {}^{2\pi}/_{11})\right) = \tan\left({}^{9\pi}/_{11})\right).$$

Therefore,

$$\tan\left(\tan\left({}^{9\pi}/_{11})\right)\right) = \tan\left(-{}^{2\pi}/_{11})\right) = -{}^{2\pi}/_{11}$$

since $-{}^{2\pi}/_{11}$ is in the range of the inverse tangent.

Example 5. Calculate $\cos\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$.

Solution

Let the angle θ be equal to $\sin^{-1}\left(\frac{1}{3}\right)$. The point p on the unit circle corresponding to angle θ lies in the first quadrant and has y-coordinate equal to ${}^1/_3$. The x-coordinate is therefore equal to x with

$$x = \sqrt{1 - ({}^1/_3)^2} = \frac{\sqrt{8}}{3}.$$

Therefore,

$$\cos\left(\sin^{-1}\left(\frac{1}{3}\right)\right) = \frac{\sqrt{8}}{3}.$$

Example 6. Calculate $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$.

SOLUTION

$$-\frac{\pi}{3}$$

Problem 1. Calculate the following:

(a) $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$,

(b) $\cos^{-1}\left(\cos\left(\frac{8\pi}{9}\right)\right)$,

(c) $\tan^{-1}\left(\tan\left(\frac{7\pi}{5}\right)\right)$,

(d) $\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$,

(e) $\cos^{-1}\left(\sin\left(-\frac{\pi}{5}\right)\right)$,

(f) $\sin\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$,

(g) $\cos\left(\sin^{-1}\left(\frac{1}{5}\right)\right)$,

(h) $\tan\left(\sin^{-1}\left(\frac{1}{7}\right)\right)$.

Problem 2. Calculate $\cos^{-1}(\cos(\theta))$ where θ is in Quadrant IV.

Problem 3. Calculate $\sin^{-1}(\cos(\theta))$ where θ is in Quadrant II.

Problem 4. The line L intersects the point $(2, 5)$ and makes a $60°$ angle with the x-axis. What is the equation for the line L?

Example 7. Let L be the line containing points $(2, 5)$ and $(6, 7)$. What is the angle that the line makes with the x-axis?

Solution

The slope of the line is

$$\frac{7-5}{6-2} = \frac{1}{2}.$$

The angle is therefore

$$\theta = \tan^{-1}\left(\frac{1}{2}\right).$$

Problem 5. Calculate $\cos\left(\tan^{-1}(3a^2)\right)$.

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IV.2.6 Equations Involving Trigonometric Functions

Example 1. Find all solutions of the equation $\cos^2(\theta) + 5\cos(\theta) + 6 = 0$.

Solution

$$\cos^2(\theta) + 5\cos(\theta) + 6 = (\cos(\theta) + 3)(\cos(\theta) + 2) = 0$$

so

$$\cos(\theta) = -3 \quad \text{or} \quad \cos(\theta) = -2.$$

Therefore, there are no solutions.

Example 2. Find all solutions of the equation $\sin^2(5\theta) - \sin(5\theta) - 1 = 0$.

Solution

Take

$$y = \sin(5\theta).$$

So,

$$y^2 - y - 1 = 0.$$

We therefore have that

$$y = \frac{1 \pm \sqrt{5}}{2}.$$

Since we must have that

$$|y| \leq 1,$$

we must have that

$$y = \frac{1 - \sqrt{5}}{2},$$

which is a negative number. Therefore,

$$\sin(5\theta) = \frac{1 - \sqrt{5}}{2}$$

and so there are two principle solutions.

Solution

The two principle solutions are

$$5\theta = A = \sin^{-1}\left(\frac{1-\sqrt{5}}{2}\right),$$

which is an angle in $(-\pi/2, 0)$ and

$$5\theta = B = \pi - \sin^{-1}\left(\frac{1-\sqrt{5}}{2}\right),$$

which is an angle in $(\pi, 3\pi/2)$. Therefore, for each integer k the angles

$$5\theta = A + 2\pi k \quad \text{and} \quad 5\theta = B + 2\pi k$$

are the solutions to the equation. We therefore have that

$$\theta = \frac{A}{5} + \frac{2\pi k}{5} \quad \text{and} \quad \theta = \frac{B}{5} + \frac{2\pi k}{5},$$

where k varies in $\mathbb{Z}$.

Example 3. Find all solutions of the equation $2\sin^2(5\theta) - \sin(5\theta) - 1 = 0$.

Solution

Take

$$y = 2\sin^2(5\theta) - \sin(5\theta) - 1.$$

So,

$$2y^2 - y - 1 = 0.$$

We therefore have that

$$y = \frac{1 \pm 3}{4} = -\frac{1}{2} \quad \text{or} \quad 1.$$

Solution

Consider the case when
$$y = -\frac{1}{2}.$$
In this case, we have two principle solutions
$$5\theta = \frac{7\pi}{6} \quad \text{or} \quad 5\theta = \frac{11\pi}{6}.$$
Consider the case when
$$y = 1.$$
In this case, we have one principle solution
$$5\theta = \frac{\pi}{2}.$$
We therefore have that for each integer k,
$$\frac{7\pi}{30} + \frac{2\pi k}{5};\ \frac{11\pi}{30} + \frac{2\pi k}{5};\ \text{and}\ \frac{\pi}{10} + \frac{2\pi k}{5}$$
are solutions.

Example 4. Find all solutions of the equation $\cos^2(\theta) - \frac{3}{4} = 0$.

Solution

So,
$$\cos(\theta) = \frac{\sqrt{3}}{2} \quad \text{or} \quad -\frac{\sqrt{3}}{2}.$$
The principle solutions to the first equation are
$$\theta = \frac{\pi}{6} \quad \text{and} \quad \theta = \frac{11\pi}{6}.$$

Solution

The principle solutions to the second equation are

$$\theta = \frac{5\pi}{6} \quad \text{and} \quad \theta = \frac{7\pi}{6}.$$

We therefore have that for each integer k,

$$\frac{\pi}{6} + 2\pi k;\ \frac{5\pi}{6} + 2\pi k;\ \frac{7\pi}{6} + 2\pi k \text{ and } \frac{11\pi}{6} + 2\pi k$$

are solutions.

Problem 1. Find all solutions of the equation

$$\cos^2(x) - \frac{1}{4} = 0.$$

Problem 2. Find all solutions of the equation

$$\cos^2(x) + 5\cos(x) + 6 = 0.$$

Problem 3. Find all solutions to the equation

$$\sin^2(\theta) - \sin(\theta) - 2 = 0.$$

Problem 4. Find all solutions to the equation

$$2\cos^2(\theta) + \sin(\theta) = 1.$$

Problem 5. Find all solutions to the equation

$$2\cos(3\theta) - 1 = 0.$$

Problem 6. How many times does the line given by $y = 1$ intersect the graph of cos on the interval $\left[-\frac{9\pi}{2}, \frac{7\pi}{2}\right]$?

Problem 7. How many times does the line given by

$$y = \frac{1}{2}$$

intersect the graph of sin on the interval $[-4\pi, 4\pi]$?

Problem 8. How many times does the line given by

$$y = 1$$

intersect the graph of tan on the interval $[-2\pi, 5\pi]$?

Problem 9. Find the **number of solutions** of

$$\sin^2(\theta) = 1$$

where θ is in the interval $[-A, A]$. Note that the number might be different for different values of A, but might also be the same if two values of A are close enough. It will be helpful to graph the function. Now, repeat this question but use the equation

$$\sin^2(\theta) = \frac{1}{2}.$$

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IV.2.7 Survey Problems with Implicit Information

Example 1. A tower is 500 feet away. Your eye level is 5 feet high. Your line of sight to the top of the tower makes an angle of $60°$ with the ground. How tall is the tower?

Solution

The tower has height h above eye level, where

$$\frac{h}{500} = \tan(60°) = \sqrt{3}.$$

Therefore,

$$h = 500\sqrt{3}.$$

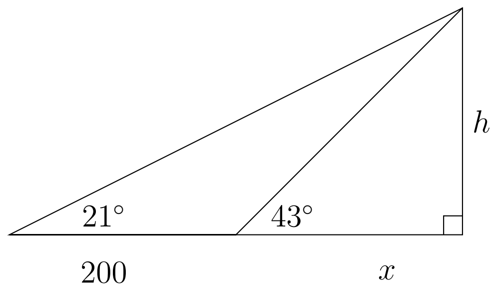
Since eye level is five feet, the height is

$$5 + 500\sqrt{3}.$$

Example 2. There is a building in front of you. The angle of elevation from your position to the top of the building is $21°$. You walk 200 feet towards the building and measure the angle of elevation to now be $43°$. How tall is the building?

Solution

Consider the figure below.



We have that

$$\tan(21^\circ) = \frac{h}{200+x} \quad \text{and} \quad \tan(43^\circ) = \frac{h}{x}.$$

Therefore,

$$200\tan(21^\circ) + x\tan(21^\circ) = (200+x)\tan(21^\circ) = h = x\tan(43^\circ),$$

and so

$$x(\tan(43^\circ) - \tan(21^\circ)) = 200\tan(21^\circ).$$

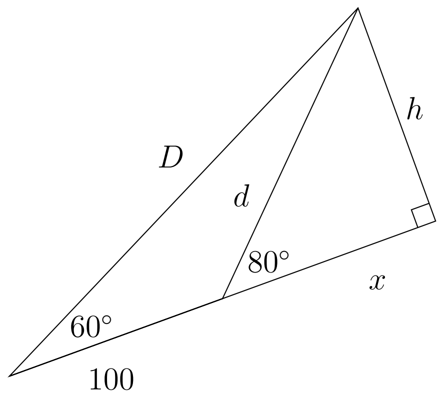
Therefore,

$$x = \frac{200\tan(21^\circ)}{\tan(43^\circ) - \tan(21^\circ)} \quad \text{and so} \quad h = \frac{200\tan(21^\circ)\tan(43^\circ)}{\tan(43^\circ) - \tan(21^\circ)}.$$

Example 3. You see a tree in the park. You walk 100 feet along a line so that the angle between the ray from your original position to your final position and the ray from your original position to the tree is 60 degrees. After walking 100 feet, you find that the angle between the ray pointing from your final position to the tree and the ray from your final position to your original position is 80 degrees. How far are you now from the tree and how far were you originally from the tree?

Solution

Consider the figure below.



We have that

$$\tan(60^\circ) = \frac{h}{100+x} \quad \text{and} \quad \tan(80^\circ) = \frac{h}{x}.$$

Therefore,

$$100\tan(60^\circ) + x\tan(60^\circ) = (100+x)\tan(60^\circ) = h = x\tan(80^\circ),$$

and so

$$x(\tan(80^\circ) - \tan(60^\circ)) = 100\tan(60^\circ).$$

Therefore,

$$x = \frac{100\tan(60^\circ)}{\tan(80^\circ) - \tan(60^\circ)} \quad \text{and so} \quad h = \frac{100\tan(60^\circ)\tan(80^\circ)}{\tan(80^\circ) - \tan(60^\circ)}.$$

Therefore,

$$D = \sqrt{(100+x)^2 + h^2} \quad \text{and} \quad d = \sqrt{x^2 + h^2}.$$

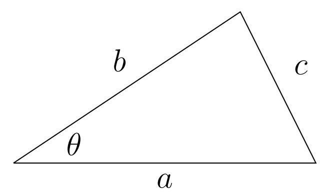
Problem 1. You are lying on the ground looking up at the top of a tower. Your line of sight makes an angle of 20° with the ground. The top of the tower is 1000 feet away. How tall is the tower?

Problem 2. A tower is 500 feet away. Your eye level is 5 feet high. Your line of sight to the top of the tower makes an angle of 60° with the ground. How tall is the tower?

Problem 3. You photograph a building that is 100 feet tall. The camera can only be propped up 20 feet high. The camera sensor is parallel to the building, this is to say that the camera is not tilted up or down. The lens on the camera has a 60° field of view, so it can see 30° above and below the axis of symmetry of the lens. How far away must you stand to photograph the entire building?

Problem 4. There is a building in front of you. The angle of elevation from your position to the top of the building is 21°. You walk 200 feet towards the building and measure the angle of elevation to now be 43°. How tall is the building?

Problem 5. You look up to the top of a tower. Your line of sight makes an angle of 35 degrees with the ground. You walk 100 away from the building and look up again to the top of the tower. Your line of sight is now makes an angle of 24° degrees with respect to the ground. What is the height of the tower?



Problem 6. Consider the figure to the right.

The lengths of the sides are a, b, and c as shown. Calculate c^2 in terms of a, b and θ. You will get no credit for memorizing the law of cosines, you must derive the formula.

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IV.2.8 Modeling the Superposition of Waves

This section is not included in the course.

Problem 1. Two waves on the string of a musical instrument superposition to give you a third wave. Suppose time is given in seconds. One wave is given by the equation $y = \sin(300t)$ and the other is given by $y = \sin(320t)$. The superposition of the two waves makes a sound (a hum) of a certain frequency, the amplitude of which varies periodically, making beats. The frequency of a wave is the number of cycles of the wave occurring each second, so it is the reciprocal of the period. What is the frequency of the hum and what is the frequency of the beats that you hear? Assume that time is given in seconds.

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IV.3.1 Properties of Exponential Functions

Example 1. Simplify the expression $\left(8zx^2y(2x^3y^{-3})^{-2}+z^{-1}\right)z^2$.

SOLUTION

$$2z^3x^{-4}y^7+z$$

Problem 1. Simplify the expression $\left(\dfrac{x^3y^4z^{-4}}{w^{-2}z^3xy^2}\right)^3$.

Example 2. Solve for x given by the equation $4^{2x}\cdot\left(\frac{1}{2}\right)^{x+2}=8\cdot(2^x)^{-2}$.

SOLUTION

Simplify the left hand side of the equation to obtain

$$4^{2x}\cdot\left(\frac{1}{2}\right)^{x+2}=(2^2)^{2x}\cdot\left(2^{-1}\right)^{x+2}=2^{4x}\cdot 2^{-x-2}=2^{3x-2}.$$

Simplify the left hand side of the equation to obtain

$$8\cdot(2^x)^{-2}=2^3\cdot 2^{-2x}=2^{3-2x}.$$

We therefore have that

$$3x-2=3-2x,$$

therefore,

$$5x=5$$

and so

$$x=1.$$

Problem 2. Solve for x in the equation

$$25^{x-2}=125^{3x+1}.$$

Example 3. Sketch the graph given by $y=3\left(2^{x-1}\right)+2$.

SOLUTION

Graph this using transformations and check your answer with a graphing program.

Problem 3. Sketch the graph of the function f where

$$f(x) = 2^{x-1} - 4.$$

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IV.3.2 The Logarithm and Its Properties

Example 1. Calculate $\log_3\left(3^4\right)$.

Solution

$$4$$

Example 2. Calculate $\log_5\left(\frac{1}{25}\right)$.

Solution

$$-2$$

Example 3. Calculate $\log_4(100) - 2\log_4(5)$.

Solution

$$\log_4(100) - 2\log_4(5) = \log_4(100) - \log_4(25) = \log_4\left(\frac{100}{25}\right) = \log_4(4) = 1.$$

Example 4. Calculate $\frac{\log_5(16)}{\log_5(4)}$.

Solution

$$\frac{\log_5(16)}{\log_5(4)} = \frac{\log_5\left(4^2\right)}{\log_5(4)} = \frac{2\log_5(4)}{\log_5(4)} = 2.$$

Example 5. Suppose that $\log_2(A) = e$, $\log_2(B) = \pi$, and $\log_2(C) = \sqrt{2}$. Calculate $\log_2\left(\frac{A^3C}{B^2}\right)$.

Solution

$$\begin{aligned}\log_2\left(\frac{A^3C}{B^2}\right) &= \log_2\left(A^3\right) + \log_2(C) - \log_2\left(B^2\right)\\ &= 3\log_2(A) + \log_2(C) - 2\log_2(B) = 3e + \pi - 2\sqrt{2}.\end{aligned}$$

Example 6. Calculate $\log_{\frac{1}{3}}(9)$.

Solution

$$\log_{\frac{1}{3}}(9) = \log_{\frac{1}{3}}\left(\left(\frac{1}{3}\right)^{-2}\right) = -2.$$

Problem 1. Compute the following.

(a) $\log_5\left(\frac{1}{25}\right) =$

(b) $\log_3\left(3^4\right) =$

(c) $\log_4(100) - 2\log_4(5) =$

(d) $\dfrac{\log_5(16)}{\log_5(4)} =$

(e) $\log_3(5) - \log_9(75)$.

Example 7. Solve the equation $\log_6(x+5) + \log_6 x = 2$.

Solution

$$\log_6(x+5) + \log_6 x = \log_6(x^2+5x) = 2 \quad \text{and so} \quad x^2+5x = 36.$$

We therefore have that

$$x^2 + 5x - 36 = (x+9)(x-4) = 0.$$

Since x must be positive, the only solution is

$$x = 4.$$

Problem 2. Solve for x where

$$\log_2(3x-4) = 3.$$

Problem 3. Solve for x where

$$2^x - 2^{-x} = 1.$$

Problem 4. Solve for x where

$$\log_{10}(x) + \log_{10}(x+3) = 1.$$

Problem 5. Find all solutions to the equation

$$\log_3(x-2) + \log_3(x-3) = \log_3(2x+24).$$

Problem 6. Solve for x given by the equation $\log_6(x+5) + \log_6 x = 2$.

Problem 7. Sketch the graph given by $y = 3\left(2^{x-1}\right) + 2$.

Problem 8. Sketch the graph given by $y = 2\log_4(x-1) + 3$.

Problem 9. Sketch the graph of the function f where $f(x) = \log_3(x-9) + 1$.

Example 8. Sketch the graph given by $y = \log_{1/2}(x)$.

Solution

Since

$$\log_2(1/2)\log_{1/2}(x) = \log_2(x) \quad \text{we have that} \quad \log_{1/2}(x) = \frac{\log_2(x)}{\log_2(1/2)} = -\log_2(x).$$

So, the graph given by

$$y = \log_{1/2}(x)$$

is the same graph as that give by

$$y = -\log_2(x).$$

Problem 10. Sketch the graph of the function f where $f(x) = \log_{1/3}(x+4) - 1$.

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IV.3.3 Symmetric Change

1 Does the data in the following table follow a linear or exponential growth model? Assuming that $A(t)$ is either a linear or exponential function, find a formula for $A(t)$ and calculate $A(5)$.

t	A(t)
0	4
1	7
2	10
3	13

2 Does the data in the following table follow a linear or exponential growth model? Assuming that $A(t)$ is either a linear or exponential function, find a formula for $A(t)$ and calculate $A(3)$.

t	A(t)
-4	2
-1	6
2	18
5	54

3 Does the data in the following table follow a linear or exponential growth model? Assuming that $A(t)$ is either a linear or exponential function, find a formula for $A(t)$ and calculate $A(4)$.

t	A(t)
-3	40
-1	20
1	10
3	5

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IV.3.4 Tuning a Guitar

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IV.3.5 The Natural Exponential and Logarithm

Example 1. Suppose your calculator only has a button for the natural logarithm and so you wish to write all logarithms in terms of the natural logarithm. Calculate $\log_3(7)$ in terms of the natural logarithm, $\ln(x)$.

Solution

$$\log_e(3)\log_3(7) = \log_e(7) \quad \text{so} \quad \log_3(7) = \frac{\ln(7)}{\ln(3)}.$$

Example 2. Solve the equation

$$\ln\left(3^{5x+1}\right) = 1.$$

Solution

Simplify the left hand side of the equation to obtain

$$\ln\left(3^{5x+1}\right) = (5x+1)\ln(3) = 1.$$

Therefore,

$$5x + 1 = \frac{1}{\ln(3)}$$

and so

$$x = \frac{1}{5}\left(\frac{1}{\ln(3)} - 1\right).$$

Example 3. Solve the equation

$$\ln(x^2 - 6) = \ln(x).$$

Solution

Since the natural logarithm is one to one, we have that

$$x^2 - 6 = x.$$

Therefore,

$$x^2 - x - 6 = (x-3)(x+2) = 0.$$

So we have that

$$x = 3 \quad \text{or} \quad x = -2.$$

However, -2 is not a permissible solution, so therefore

$$x = 3.$$

Example 4. Solve the equation

$$\ln(x^2 - 6) = \ln(x).$$

Solution

Since the natural logarithm is one to one, we have that

$$x^2 - 6 = x.$$

Therefore,

$$x^2 - x - 6 = (x-3)(x+2) = 0.$$

So we have that

$$x = 3 \quad \text{or} \quad x = -2.$$

However, -2 is not a permissible solution, so therefore

$$x = 3.$$

Example 5. Solve the equation

$$\ln\left(3^{5x+1}\right) = 1.$$

Solution

Simplify the left hand side of the equation to obtain

$$\ln\left(3^{5x+1}\right) = (5x+1)\ln(3) = 1.$$

Therefore,

$$5x+1 = \frac{1}{\ln(3)}$$

and so

$$x = \frac{1}{5}\left(\frac{1}{\ln(3)} - 1\right).$$

Example 6. Apply the laws for manipulating logarithms to simplify the expression $\ln\left(\frac{x\sqrt{y}}{z^5}\right)$.

Solution

$$\ln\left(\frac{x\sqrt{y}}{z^5}\right) = \ln(x) + \ln(\sqrt{y}) - \ln\left(z^5\right) = \ln(x) + \frac{1}{2}\ln(y) - 5\ln(z).$$

Problem 1. Find all solutions to the equation $x - xe^{5x+2} = 0$.

Problem 2. Find all solutions to the equation $5e^{1+3x} - 9e^{5-x} = 0$.

Problem 3. Find all solutions to the equation $4 + 2\ln\left(\frac{x}{5} + 2\right) = -3$.

Problem 4. Write the function f given by $f(x) = 3(5)^{2x-1}$ in terms of only the natural exponential function.

Problem 5. Suppose your calculator only has a button for the natural logarithm and so you wish to write all logarithms in terms of the natural logarithm. Calculate $\log_3(7)$ in terms of the natural logarithm, $\ln(x)$.

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IV.3.6 Exponential Growth and Decay

Example 1. A blob of a certain radioactive substance is decaying exponentially with respect to time. The mass is originally 8 grams. After 5 days, the mass is 4 grams. What is the half-life of the substance?

Solution

5 days

Example 2. A blob of a certain radioactive substance is decaying exponentially with respect to time. The mass is originally 8 grams. After 6 days, the mass is 1 gram. What is the half-life of the substance?

Solution

We will solve this problem in two ways. Notice that 1 gram is one eighth of the original amount. So, the substance has decayed for three half lives in six days. This means that the half life is two days. This is the quick way of seeing the answer.

Now we approach it more formulaically. The amount of the material in time t is give by

$$A(t) = 8b^t.$$

We have that

$$A(6) = 1 = 8b^6,$$

so

$$b = \left(\frac{1}{2}\right)^{\frac{1}{2}}.$$

Therefore,

$$A(t) = 8\left(\frac{1}{2}\right)^{\frac{t}{2}}.$$

If t is 2, then the factor by which we multiply 8 is $\frac{1}{2}$. The half life is therefore two hours.

Example 3. A blob of a certain radioactive substance is decaying exponentially with respect to time. The mass is originally 8 grams. After 5 days, the mass is 4 grams. What will the mass be in 15 days?

Solution

The mass decreases by a factor of $\frac{1}{2}$ in five days, and so it will decrease by a factor of $\frac{1}{2^2}$ in 10 days, and by a factor of $\frac{1}{2^3}$ in fifteen days. There will be only one gram of the substance remaining.

Example 4. A blob of bacteria undergoing exponential growth originally has a mass of 10 grams. After three days, the mass is 20 grams. What will the mass be after nine days?

Solution

After three more days, it will have a mass of 40 grams and three days later it will have a mass of 80 grams.

Example 5. A blob of a certain substance is increasing in size exponentially with respect to time. Originally, the mass is 10 grams. After five days, the mass is 17 grams. What will the mass of the blob be in 10 days?

Solution

Let $A(t)$ denote the amount of material in time t. Then

$$A(t) = 10b^t$$

and so

$$A(5) = 10b^5 = 17.$$

Therefore,

$$b = \left(\frac{17}{10}\right)^{\frac{1}{5}}$$

and so

$$A(t) = 10\left(\frac{17}{10}\right)^{\frac{t}{5}}.$$

We therefore have that

$$A(t) = 10\left(\frac{17}{10}\right)^{\frac{10}{5}} = 10\left(\frac{17}{10}\right)^{2}.$$

Example 6. A blob of a certain substance is growing in size exponentially with respect to time. Originally, the mass is 10 grams. After five days, the mass is 17 grams. What is the growth rate of the blob?

Solution

Let $A(t)$ denote the amount of material in time t. Then

$$A(t) = 10e^{kt}$$

and so

$$A(5) = 10e^{6k} = 17.$$

Therefore,

$$e^{6k} = \frac{17}{10}$$

and so

$$6k = \ln\left(\frac{17}{10}\right),$$

hence

$$k = \frac{1}{6}\ln\left(\frac{17}{10}\right).$$

The growth rate is k.

Example 7. A blob of a certain radioactive substance is decaying exponentially with respect to time. Originally, the mass is 10 grams. After two days, the mass is 8 grams. What is the decay rate of the blob? What fraction will have decayed in 7 days?

Solution

Let $A(t)$ denote the amount of material in time t. Then

$$A(t) = 10e^{-kt}$$

and so

$$A(5) = 10e^{-2k} = 8.$$

Therefore,

$$e^{-2k} = \frac{8}{10} = \frac{4}{5}$$

and so

$$-2k = \ln\left(\frac{4}{5}\right),$$

hence

$$k = -\frac{1}{2}\ln\left(\frac{4}{5}\right) = \frac{1}{2}\ln\left(\frac{5}{4}\right).$$

The decay rate is k.

Example 8. A blob of a certain radioactive substance is decaying exponentially with respect to time. After 10 days, the mass is 8 grams. After 12 days, the mass is 6 grams. What will the mass of the blob be in 20 days?

Solution

We will answer the problem in two ways, the first will be very quick and conceptual, the second will be more formulaic.

First way.

In two days, the mass decreases by a factor of $\frac{6}{8}$ which equals $\frac{3}{4}$. Therefore, each day, the mass will decrease by a factor of $\left(\frac{3}{4}\right)^{\frac{1}{2}}$. Ten days pass between ten days and 20 days, and so the mass decreases by a factor of $\left(\frac{3}{4}\right)^{5}$. We are therefore left with

$$8\left(\frac{3}{4}\right)^{5} \text{ grams.}$$

Second way.

Let $A(t)$ denote the amount of material in time t. Then

$$A(t) = A(0)e^{-kt}$$

and so

$$A(10) = A(0)e^{-10k} = 8 \quad \text{and} \quad A(12) = A(0)e^{-12k} = 6.$$

Therefore,

$$\frac{A(10)}{A(12)} = \frac{8}{6} = \frac{4}{3} = \frac{A(0)e^{-10k}}{A(0)e^{-12k}} = e^{2k}$$

and so

$$2k = \ln\left(\frac{4}{3}\right) \quad \text{that is} \quad k = \frac{1}{2}\ln\left(\frac{4}{3}\right).$$

We therefore have that

$$A(t) = A(0)e^{-\frac{t}{2}\ln\left(\frac{4}{3}\right)}$$

and since

$$A(10) = 8 = A(0)e^{-5\ln\left(\frac{4}{3}\right)},$$

we have that

$$A(0) = 8e^{5\ln\left(\frac{4}{3}\right)}.$$

Therefore,

$$A(t) = 8e^{5\ln\left(\frac{4}{3}\right)}e^{-\frac{t}{2}\ln\left(\frac{4}{3}\right)} = 8e^{\left(5-\frac{t}{2}\right)\ln\left(\frac{4}{3}\right)} = 8\left(\frac{4}{3}\right)^{\left(5-\frac{t}{2}\right)}.$$

So, if t is 20 then we have that

$$A(20) = 8\left(\frac{4}{3}\right)^{\left(5-\frac{20}{2}\right)} = 8\left(\frac{4}{3}\right)^{-5} = 8\left(\frac{3}{4}\right)^{5}.$$

Example 9. A blob of a certain radioactive substance is decaying exponentially with respect to time. After 10 days, the mass is 8 grams. After 12 days, the mass is 6 grams. What is the half-life of the radioactive substance?

Solution

We'll just do this one quickly but by being a bit more conceptual. The blob decays by a factor of $\frac{3}{4}$ in two days and so in will decay by a factor of $\left(\frac{3}{4}\right)^{\frac{1}{2}}$ per day. In t days, it will decrease by a factor of $\left(\frac{3}{4}\right)^{\frac{t}{2}}$. We are seeking a value T so that

$$\left(\frac{3}{4}\right)^{\frac{T}{2}} = \frac{1}{2}.$$

Take natural logarithms of each side to obtain the equality

$$\frac{T}{2}\ln\left(\frac{3}{4}\right) = \ln\left(\frac{1}{2}\right)$$

and so

$$T = 2\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{3}{4}\right)} = 2\frac{\ln\left(\frac{2}{1}\right)}{\ln\left(\frac{4}{3}\right)} = \frac{2\ln(2)}{\ln(4)-\ln(3)}.$$

The time T is the half life.

Problem 1. Suppose that f is given by

$$f(t) = 100(3)^t.$$

What is the exponential growth rate of f?

Problem 2. Suppose that A is defined by

$$A(t) = 1000e^{2t}.$$

When does A take the value 1500? How long does it take for the value of A to double?

Problem 3. A 3 gram blob of bacteria sits on a Petri dish. After three hours, there are 7 grams of bacteria. What is the growth rate of the bacteria and how many grams of bacteria will you have in 28 hours?

Problem 4. You have a sample of a certain radioactive substance. After two days, thirty percent of the sample has decayed. What is the half-life of the material and how much will have decayed after a week?

Problem 5. You ingest a certain dangerous bacteria at dinner by accident. When more than 100,000,000 of these organisms are in your gut, you will become violently ill. Their population doubles every 30 minutes. You ate dinner at 7pm and by 3am you are sick. At least how many of these bacteria did you ingest at dinner?

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IV.4.1 Intersections of Lines and Quadratic Curves

For the problems below, use the algebraic definition of tangency. While you may know how to solve these problems using Differential Calculus, we presently explore the algebraic notions.

Example 1. What is the equation of the line tangent to the graph of f at the point $(2, 7)$, where f is the function defined by the formula $f(x) = 3x^2 - 5$?

Solution

Suppose that the line L is tangent to the graph of f at $(2, 7)$. Then

$$L(x) = mx + b \quad \text{and} \quad L(2) = 2m + b = 7.$$

Solve for b to see that

$$b = 7 - 2m.$$

Therefore,

$$L(x) = mx + 7 - 2m.$$

Furthermore, the difference g equal to $f - L$ is given by

$$\begin{aligned} g(x) &= f(x) - L(x) \\ &= 3x^2 - 5 - (mx + 7 - 2m) \\ &= 3x^2 - mx - 12 + 2m. \end{aligned}$$

Since L and f intersect at exactly one point, m must be chosen so that $f(x) - L(x)$ has a unique solution. The quadratic formula gives the equality

$$x = \frac{m \pm \sqrt{m^2 - 12(2m - 12)}}{6}.$$

There is exactly one solution if and only if

$$m^2 - 12(2m - 12) = m^2 - 24m + 144 = (m - 12)^2 = 0,$$

hence

$$m = 12.$$

The slope of L is therefore 12 and so

$$L(x) = 12x - 17.$$

Problem 1. Let f be the function given by

$$f(x) = x^2 + 3.$$

Find the equation of the line tangent to the graph of f at the point $(1, 4)$.

Problem 2. Let f be the function given by

$$f(x) = x^2 - x + 2.$$

Find the equation of the line tangent to the graph of f at the point $(2, 4)$.

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IV.4.2 The Telescope

Example 1. A light ray traveling in the x–y plane along a line parallel to the y-axis from the positive y-direction to the surface of the parabolic mirror whose equation in the x–y plane is given by

$$y = \frac{1}{9}x^2$$

strikes the parabolic mirror at the point $(3, 1)$. Find an equation of the line on which the reflected light ray travels. Where does this line intersect the y-axis?

Solution

Denote by L the line tangent to the graph of the parabolic mirror at the point $(3, 1)$. The line L is given by

$$L(x) = mx + b \quad \text{and} \quad L(3) = 3m + b = 1$$

therefore,

$$b = 1 - 3m$$

and so

$$L(x) = mx + (1 - 3m).$$

Since

$$(f - L)(x) = \frac{1}{9}x^2 - mx + (3m - 1)$$

and the difference has exactly one zero,

$$m^2 - \frac{4}{9}(3m - 1) = m^2 - 2\frac{2}{3}m - \frac{4}{9} = \left(m - \frac{2}{3}\right)^2 = 0.$$

Therefore,

$$m = \frac{2}{3}.$$

If $L_\perp$ is a line perpendicular to L that intersects $(3, 1)$, then the equation for $L_\perp$ is

$$y = -\frac{3}{2}(x - 3) + 1 = -\frac{3}{2}x + \frac{11}{2}.$$

SOLUTION

If B is the light beam, then B intersects $(3,1)$ and $(3,2)$. We now calculate the reflection of $(3,2)$ across $L_\perp$. The line given by the equation

$$y = \frac{2}{3}(x-3) + 2 = \frac{2}{3}x$$

intersects $L_\perp$ when

$$\frac{2}{3}x = -\frac{3}{2}x + \frac{11}{2}.$$

Therefore,

$$\left(\frac{2}{3} + \frac{3}{2}\right)x = \frac{11}{2}$$

and so

$$x = \frac{11}{2} \cdot \frac{6}{13} = \frac{33}{13} \quad \text{and} \quad y = -\frac{3}{2} \cdot \frac{33}{13} + \frac{11}{2} = \frac{22}{13}.$$

Therefore, the reflection of $(3,2)$ is the point

$$\left(\left(\frac{33}{13}, \frac{22}{13}\right) - (3,2)\right) + \left(\frac{33}{13}, \frac{22}{13}\right) = \left(\frac{66}{13} - 3, \frac{44}{13} - 2\right) = \left(\frac{27}{13}, \frac{18}{13}\right).$$

The line intersecting

$$(3,1) \quad \text{and} \quad \left(\frac{27}{13}, \frac{18}{13}\right)$$

is the path of the reflected light ray, which has equation

$$y = \frac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}.$$

So the equation for the reflected light beam is

$$y = -\frac{5}{12}(x-3) + 1 = -\frac{5}{12}x + \frac{9}{4}.$$

In summary, the line on which the reflected light beam travels is given by

$$y = -\frac{5}{12}x + \frac{9}{4} \quad \text{and the beam intersects the } y\text{-axis at} \quad \left(0, \frac{9}{4}\right).$$

Problem 1. A light ray traveling in the x–y plane along a line parallel to the y-axis from the positive y-direction to the surface of the parabolic mirror whose equation in the x–y plane is given by

$$y = \frac{1}{16}x^2$$

strikes the parabolic mirror at the point $(8, 4)$. Find an equation of the line on which the reflected light ray travels. What is the y-intercept of this line?

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IV.4.3 Intersections of Lines and Higher Order Polynomial Curves

For the problems below, use the algebraic definition of tangency. While you may know how to solve these problems using Differential Calculus, we presently explore the algebraic notions.

Example 1. Let f be the function defined by the formula

$$f(x) = 3x^2 - 5.$$

What is the equation of the line tangent to the graph of f at the point $(2, 7)$?

Solution

Suppose that the line L is tangent to the graph of f at $(2, 7)$. Then

$$L(x) = mx + b \quad \text{and} \quad L(2) = 2m + b = 7.$$

Solve for b to see that

$$b = 7 - 2m.$$

Therefore,

$$L(x) = mx + 7 - 2m.$$

Furthermore, the difference g equal to $f - L$ is given by

$$\begin{aligned} g(x) &= f(x) - L(x) \\ &= 3x^2 - 5 - (mx + 7 - 2m) \\ &= 3x^2 - mx - 12 + 2m \\ &= a(x-2)^2 = ax^2 - 4ax + 4a. \end{aligned}$$

This gives us the inequality

$$3x^2 - mx - 12 + 2m = ax^2 - 4ax + 4a$$

and so

$$a = 3, \quad -4a = -12 = -m \implies m = 12.$$

The slope of L is therefore 12 and so

$$L(x) = 12x - 17.$$

Example 2. Let f be the function defined by the formula

$$f(x) = 2x^3 - x^2 + x + 5.$$

What is the equation of the line tangent to the graph of f at the point $(1, 7)$? Be sure to use the algebraic definition of tangency.

Solution

Suppose that the line L is tangent to the graph of f at $(1, 7)$. Then

$$L(x) = mx + b \quad \text{and} \quad L(1) = 1m + b = 7.$$

Solve for b to see that

$$b = 7 - m.$$

Therefore,

$$L(x) = mx + 7 - m.$$

Furthermore, the difference g equal to $f - L$ is given by

$$\begin{aligned} g(x) &= f(x) - L(x) \\ &= 2x^3 - x^2 + x + 5 - (mx + 7 - m) \\ &= 2x^3 - x^2 + (1 - m)x + (m - 2) \\ &= (ax + b)(x - 1)^2 \\ &= (ax + b)(x^2 - 2x + 1) = ax^3 + (b - 2a)x^2 + (a - 2b)x + 4b. \end{aligned}$$

This gives us the inequality

$$2x^3 - x^2 + (1 - m)x + (m - 2) = ax^3 + (b - 2a)x^2 + (a - 2b)x + 4b$$

and so

$$a = 2, \quad b - 2a = b - 4 = -1 \implies b = 3, \quad a - 2b = -4 = 1 - m \implies m = 5.$$

The slope of L is therefore 5 and so

$$L(x) = 5x + 2.$$

Problem 1. Let f be the function given by

$$f(x) = x^2 + 2x - 3.$$

Find the equation of the line tangent to the graph of f at the point $(2, 5)$.

Problem 2. Let f be the function given by

$$f(x) = x^3 - x + 2.$$

Find the equation of the line tangent to the graph of f at the point $(1, 2)$.

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IV.4.4 Intersections of Lines with Rational Curves

For the problems below, use the algebraic definition of tangency. While you may know how to solve these problems using Differential Calculus, we presently explore the algebraic notions.

Example 1. Let f be the function given by $f(x) = \dfrac{2x+1}{x-3}$. Find the equation of the line tangent to the graph of f at the point $(4, 9)$.

Solution

The line L, given by $L(x) = mx + b$, intersects f tangentially at $(4, 9)$ if and only if $(x-4)^2$ divides the numerator of the difference $(f - L)(x)$ when written in reduced form. Since L intersects $(4, 9)$, m and b are related by the equation

$$9 = 4m + b \quad \text{and so} \quad b = 9 - 4m.$$

Use this to obtain the equality

$$\begin{aligned}(f - L)(x) &= \frac{2x+1}{x-3} - (mx + 9 - 4m) \\ &= \frac{2x + 1 - (x-3)(mx + 9 - 4m)}{x-3} = \frac{-mx^2 + (-7 + 7m)x + (27 - 12m)}{x-3}.\end{aligned}$$

Since $(x-4)^2$ must divide the numerator, there is a constant a with

$$\begin{aligned}a(x-4)^2 &= ax^2 - 8ax + 16a \\ &= -mx^2 + (-7 + 7m)x + (27 - 12m).\end{aligned}$$

Therefore, a equals $-m$ and so

$$-8a = 8m = -7 + 7m, \quad \text{implying that} \quad m = -7$$

and so

$$L(x) = -7x + 37.$$

Problem 1. Let f be the function given by $f(x) = \dfrac{x}{x-2}$. Find the equation of the line tangent to the graph of f at the point $(4, 2)$.

Problem 2. Using the algebraic definition of tangency, find the equation of the line tangent to the graph of $\dfrac{2x}{x-1}$ at the point $(2, 3)$.

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IV.4.5 Symmetry of Tangency under Reflection

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Worksheets

See the ● Calendar page for a detailed timeline. Following the number of the worksheet, you will see the symbol ●. Clicking this symbol will return you to this page.

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Worksheet 1 ● Find The Error

1. Is the following statement true? If a, b, and c are real numbers and a and b are nonzero, then

$$\frac{c}{a} + \frac{c}{b} = \frac{c}{a+b}.$$

2. Is the following statement true? For all x in the real numbers, we have that

$$\sqrt{x^2+9} = x+3.$$

3. Is there anything wrong with the following statement? Since the absolute value of a number x is always positive, we have that

$$\begin{aligned} |x+y|^2 &= |(x+y)^2| \\ &= |x^2+2xy+y^2| = x^2+2|x||y|+y^2. \end{aligned}$$

4. Is the following statement true? If x is a real number, then

$$\sqrt{9-x}\sqrt{9-x} = 9-x.$$

Under what conditions will the statement be true?

5. Find the error in this argument. If $x = y$, then

$$x^2 = xy.$$

Subtracting y^2 from both sides, we obtain the inequality

$$x^2 - y^2 = xy - y^2.$$

We then write both sides as products, obtaining the equality

$$(x+y)(x-y) = y(x-y).$$

We therefore have that

$$x + y = y.$$

Since $x+y$ is equal to $2y$, we have that

$$2y = y, \quad \text{hence,} \quad 2 = 1.$$

6. Is the following statement true? Suppose that A, B, C, D and E are sets. If $A \cap B$ is a subset of E and if $C \cap D$ is a subset of E, then $(A \cup C) \cap (B \cup D)$ is a subset of E.

7. Construct a statement that appears to be true but is actually false. See if you can, as a group, come up with such a statement, show the statement is false, but convince a neighboring group that your statement is true.

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Worksheet 2 ● Working with Piecewise Functions

1. Piecewise defined functions f and g are said to be defined on commensurable domains if they are given explicitly by formulas on the same intervals. Suppose that

$$f(x) = \begin{cases} -2x+1 & \text{if } x < 0 \\ 3x+2 & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x+4 & \text{if } x < 2 \\ 2 & \text{if } x \geq 2. \end{cases}$$

Describe f and g as piecewise functions defined on commensurable domains. Use this to find all x with

$$f(x) = g(x).$$

2. Given the functions f and g above, use the description of f and g as functions defined on commensurable domains to find all x with

$$f(x) < g(x).$$

3. The piecewise defined function h is defined by

$$h(x) = \begin{cases} x+5 & \text{if } x < 2 \\ 10-x & \text{if } x \geq 2. \end{cases}$$

Find all x such that

$$h(x) < 0.$$

Find all x such that

$$h(x) \geq 0.$$

4. The function h is defined by a formula on the intervals $(-\infty, 2)$ and $[2, \infty)$. Intersect these intervals with the intervals you found in the previous problem.

5. For the function h above, write the function F given by

$$F(x) = |h(x)|$$

as a piecewise defined function.

6. Why might you need to write F as a piecewise defined function? Find all solutions to

$$F(x) = |2x-5|.$$

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Worksheet 3 ● Interpreting Graphical Information

The graph of the function f is given below (Figure 1). Refer to it for Problems 1–4.

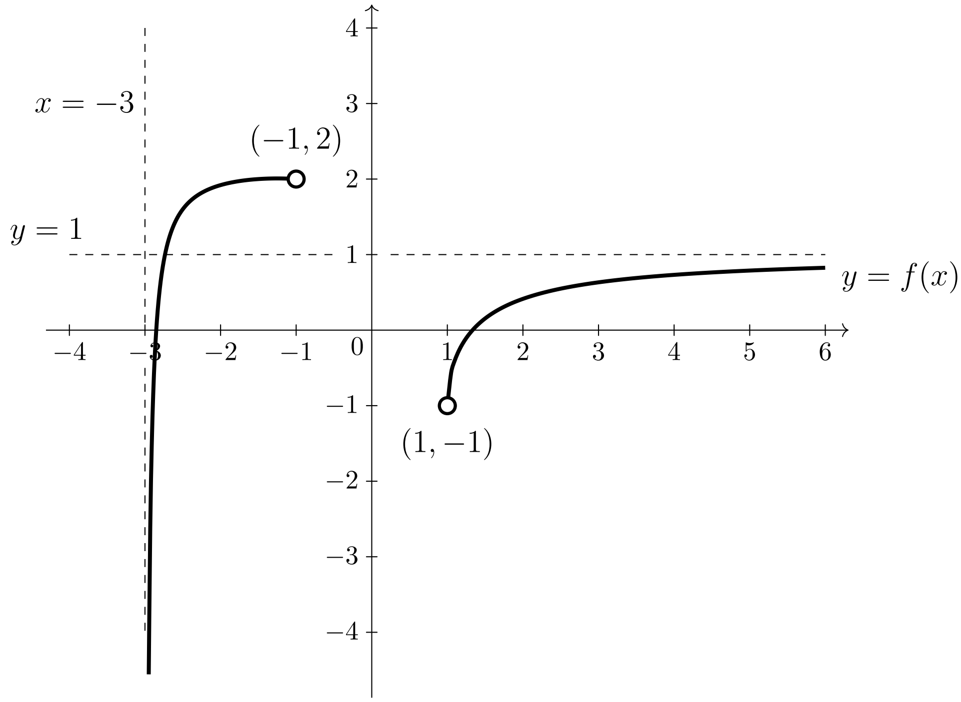

Figure 1

1. Find two points in the domain of f and two points not in the domain of f.
2. Find two points in the range of f and two points not in the range of f.
3. Graph on a real number line the domain of f.
4. Graph on a real number line the range of f.

The graph of the function a is given below (Figure 2). Refer to it for Problem 5.

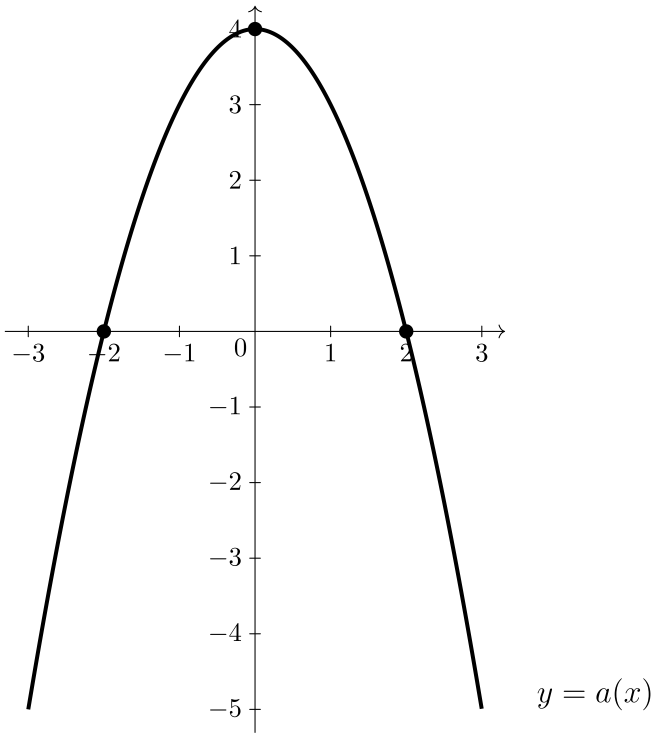

Figure 2

5. Let a be the function given by $a(x) = 4 - x^2$. The graph of a is given above (Figure 2). Find the coordinates for the three unlabeled points marked on the graph of a. Graph on a number line the set of all x with

(a) $a(x) > 0$,

(b) $a(x) \geq 0$,

(c) $a(x) < 0$,

(d) $a(x) \leq 0$.

(e) Graph on a number line the range of a.

The graph of the function g is given below (Figure 3). Refer to it for Problem 6.

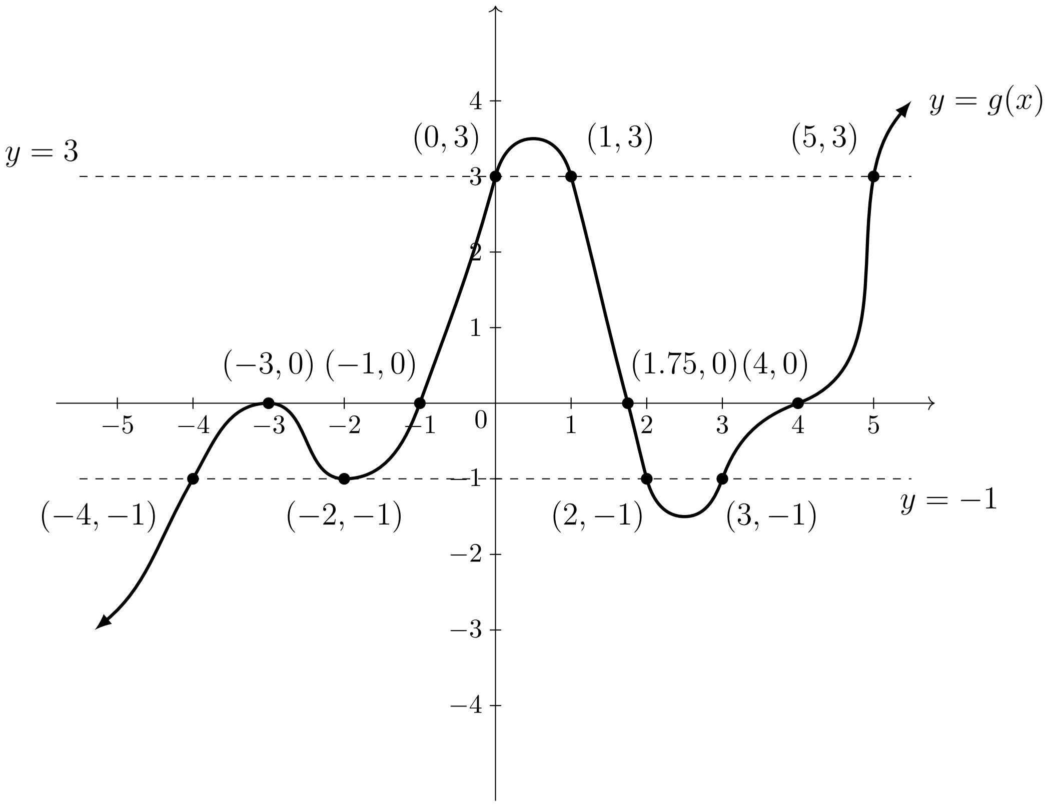

Figure 3

6. Graph on a number line the set of all x with

(a) $g(x) > 3$,

(b) $g(x) \geq -1$,

(c) $g(x) \leq 3$,

(d) $g(x) < -1$.

7. Let b be the function given by

$$b(x) = \frac{(x+3)(x-2)(x-4)^3}{(x-3)(x+5)^2}.$$

Use a computer or calculator to graph the function. Graph on a number line the set of all x with

(a) $b(x) > 0$,

(b) $b(x) \geq 0$,

(c) $b(x) < 0$,

(d) $b(x) \leq 0$.

If you did not have a way to graph the function, how else might you solve this problem?

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Worksheet 4 ● Perpendicular Lines and the Horizon

1. Draw in the coordinate plane the line that intersects the points $(1, 2)$ and $(5, 8)$. Find an equation for this line.

2. Find an equation for the line perpendicular to this line that intersects the point $(2, 7)$.

3. Draw a line in the plane. Call this line L. Draw a point, p, that is not on L. Let $L_\perp$ be the line perpendicular to L that intersects p. The line $L_\perp$ will intersect L at a point q. Use basic geometry to argue that q is the point on L that is closest to p.

4. Use the previous problem to find the point on the line given by

$$y = 2x + 1$$

that is closest to the point $(1, 4)$.

5. Suppose that p and q are two points on a circle of radius r with center O and that p, q, and O are not co-linear. Let m be the midpoint of $\overline{pq}$, the line segment with endpoints p and q. Show that $\overline{Om}$ and $\overline{pq}$ are perpendicular.

6. Suppose that a line L intersects a circle C centered at O in exactly one point, p. Use Problem 3 to show that $\overline{Op}$ is perpendicular to L.

7. Recall that the earth's radius is 4,000 miles. Suppose that you stand h feet above sea level and you look out across the ocean. What is the length of your line of sight? Make sure that you give the answer in miles. How far is the horizon away when you stand 6 feet above sea level? How about 9 feet? How about 100 feet?

8. In the previous problem, You calculated the distance to the horizon. This is not the distance one would have to travel on the earth to get to the horizon. Can you estimate this distance? This is to say, at most how far off is the length of the line of sight from length of the shortest arc that starts at your position and goes to where you see the horizon? Hint: You'll need to draw a picture and think geometrically, remembering a critical fact about triangles and arcs of circles.

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Worksheet 5 ● Units and Proportion

1. Note that one yard is three feet. Assume that a day is exactly 24 hours. Alice runs at a speed of 10 feet per second. What is this speed in units of yards and days?

2. Water leaks from a faucet at a rate of .2 cubic yards per week. How many cubic inches does it leak per minute?

3. A painter needs two cubic feet of paint to paint a wall with an area of 2000 square feet. The painter now needs to paint a wall with an area of 2000 square yards. How many cubic yards of paint does he need?

4. For the next two problems, remember that muscle strength and bone strength depend on the cross sectional area of the muscle and bone. Ignore complicating factors such as body mechanics. A giant stands 30 feet tall and has the exact same body proportions of a man who is five feet nine inches tall, weighs 170 pounds, and benchpresses 340 pounds. How much would you expect the giant to weigh? How much would you expect the giant to bench press? What is the man's benchpress to bodyweight ratio? What is the giant's benchpress to body weight ratio?

5. Suppose that a certain six foot tall man with a body weight of 180 pounds can squat 500 pounds. A mad scientist makes an exact copy of this man but scaled down to be one inch tall. How much does this copy weigh and how much can this copy squat? What is the squat weight to body weight ratio for the man and his miniature copy?

6. When you were a child, it was less painful for you to walk over small rocks at the beach or on a riverbank than it is today. Why is this the case?

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Worksheet 6 ● Vertices of Parabolas and Optimization

1. Find the minimum y value of a point on the graph given by the set of points satisfying

$$y = x^2 + x + 1.$$

2. Find the maximum y value of a point on the graph given by the set of points satisfying

$$y = -x^2 + 2x + 4.$$

3. You have 400 feet of fencing. You will fence in a rectangular region of your backyard. What should the side lengths of the rectangle be so that the area is maximal? What shape will the yard be?

4. You have 400 feet of fencing. You will fence in a rectangular region of your backyard with this fencing but can use an already existing stone wall as one side of the rectangular region. What should the side lengths of the rectangle be so that the area is maximal?

5. In the problem above, what should the side lengths of the rectangle be so that the area is minimal?

6. You have a fixed amount of fencing and you will only fence off square regions. Will you fence in more area by making a single square region or two square regions? Note that the two regions may have different side lengths. What is the smallest area that you can fence off if the two regions that you fence off are square?

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Worksheet 7 ● Linear Motion and Optimization

For the following problems, you will assume the following: Initially at time zero, Boat 1 is at $(2, 3)$ and Boat 2 is at $(9, 4)$. Boat 1 is traveling four miles per hour North (the positive y direction) and ten miles per hour East. Boat 2 is traveling two miles per hour North and one mile per hour East.

1. What are the vector equations for the positions of Boat 1 and Boat 2 as functions of time? How fast is each boat moving?

2. Independent of time, describe the paths of motion of Boat 1 and Boat 2. Do the lines that the two boats travel along intersect?

3. Will the boats crash into each other?

4. Can you change the speed of Boat 2 to ensure that the boats collide but maintain the same path of motion? If so how fast should Boat 2 go and what will be its new velocity vector? Hint: It may be helpful to consider the distance between the boats and scale the velocity vector of Boat 2 by a constant.

5. In the original setup, what is the distance between the boats as a function of time?

6. In the original setup, what is the closest the boats get to each other and at what time does this occur?

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Worksheet 8 ● Estimating Values of Trigonometric Functions

1. A point p on the unit circle is said to be a principle dyadic angle if the radian measure of p is zero or $\frac{2\pi}{2^n}$ for some natural number n. Draw a unit circle. Then mark and label the first seven non-zero principle dyadic angles on it, giving both degree and radian measures of the labeled angles.

2. Use the half-angle formula for the cosine function to calculate $\cos(90°)$, $\cos(45°)$, $\cos(22.5°)$, $\cos(11.25°)$, and $\cos(5.625°)$.

3. Mark off all angles in the first quadrant on a circle that are multiples of $5.625°$.

4. A dyadic angle is an angle that is a sum of principle dyadic angles, this is to say that its measure in radians is zero or $2\pi \times \frac{m}{2^n}$ for some natural numbers m and n. Show that the sum and average of two dyadic angles is again a dyadic angle.

5. By averaging dyadic angles, one can quickly approximate any angle by dyadic angles using a method very similar to the bisection method. This is to say that given an angle θ, one can find dyadic angles A and B such that θ lies between A and B and A and B are as close to each other as we like. Approximate $18°$ by averaging dyadic angles, then write the approximating dyadic angles as sums of principle dyadic angles.

6. Approximate $\cos(18°)$ and $\sin(18°)$. At worst, how far off from the exact answer can your approximation be? Neglect the error that you get in your calculation of the trigonometric functions on the approximating dyadic angles. Test the accuracy of your approximation with a calculator.

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Worksheet 9 ● Restriction of Domain

1. Graph the function $f(x) = (x+1)^2$. Then reflect the graph of f about the line $y = x$. (Comment: When you reflect the graph about the line $y = x$, you interchange the roles of the x and y coordinates.) Is the reflection of this graph the graph of a function?

2. Restrict the domain of f given by $f(x) = (x+1)^2$ in the simplest way possible so that f becomes invertible on that domain and the range of the inverse is an interval containing zero.

3. Consider the function, f, that projects the unit circle, $\mathcal{C}$, onto the x–axis. This is the function

$$f\colon \mathcal{C} \to [-1,1] \quad \text{defined by} \quad f((a,b)) = a.$$

Is this function invertible? If not, find the largest arc, $\mathcal{A}$, on the circle and containing $(1,0)$ where it is invertible. You should highlight this arc $\mathcal{A}$.

4. Consider the function, g, that projects the unit circle onto the y–axis. This is the function

$$g\colon \mathcal{C} \to [-1,1] \quad \text{defined by} \quad g((a,b)) = b.$$

Is this function invertible? If not, find the largest arc, $\mathcal{B}$, on the circle and containing $(0,1)$ where it is invertible. You should highlight this arc $\mathcal{B}$.

5. Denote respectively by f^{-1} and g^{-1} the inverse of the restriction of f on $\mathcal{A}$ and the inverse of the restriction of g on $\mathcal{B}$. Calculate

$$f^{-1}\left(\frac{1}{3}\right), \quad f^{-1}\left(-\frac{1}{3}\right), \quad g^{-1}\left(\frac{2}{3}\right), \quad \text{and} \quad g^{-1}\left(-\frac{2}{3}\right).$$

6. Calculate

(a) $f^{-1}\left(f\left(\left(2/5, \sqrt{21}/5\right)\right)\right)$
(b) $f^{-1}\left(f\left(\left(2/5, -\sqrt{21}/5\right)\right)\right)$
(c) $g^{-1}\left(g\left(\left(2/5, \sqrt{21}/5\right)\right)\right)$
(d) $g^{-1}\left(g\left(\left(-2/5, \sqrt{21}/5\right)\right)\right)$.
(e) $g^{-1}\left(f\left(\left(2/5, \sqrt{21}/5\right)\right)\right)$
(f) $g^{-1}\left(f\left(\left(2/5, -\sqrt{21}/5\right)\right)\right)$
(g) $f^{-1}\left(g\left(\left(2/5, \sqrt{21}/5\right)\right)\right)$
(h) $f^{-1}\left(g\left(\left(-2/5, \sqrt{21}/5\right)\right)\right)$.

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Worksheet 10 ● Unrestricted Growth

Continuously compounding interest and unrestricted population growth are both examples of quantities modeled by geometric growth. Suppose that you eat something contaminated with pathogenic e-coli. You ingest only 1,000 of these bacteria. Their population doubles every twenty minutes and undergoes unrestricted growth.

1. How many bacteria will there be in 20 minutes?

2. How many bacteria will there be in 40 minutes?

3. Set up an equation that describes the number of bacteria you will have at time t. What is the growth rate of this population?

4. How many bacteria will you have after 50 minutes?

5. You will get violently ill once you have 500,000,000 of these bacteria in you. How long will that take?

6. Suppose now that you have injested an unknown strain of bacteria. What you do know is that their population is undergoing unrestricted growth. If you have 1,000,000 bacteria at noon and 10,000,000 at 3:00pm, how many will you have at 4:00pm? What is the growth rate of this population and what is the doubling time of the population?

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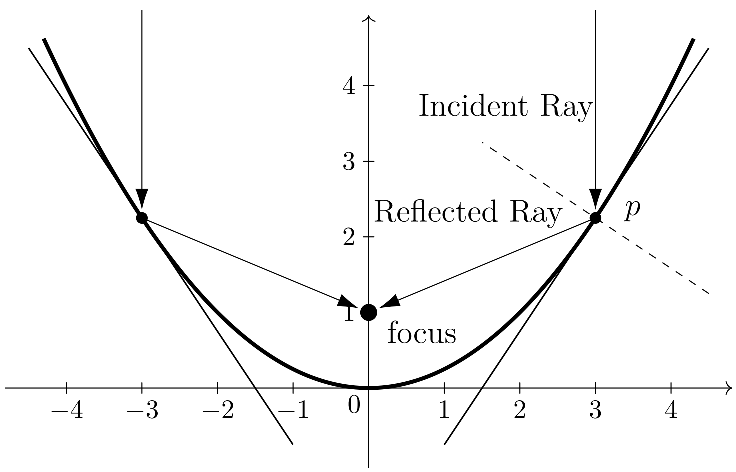
Worksheet 11 ● Tangency and Telescopes

Parabolic mirrors are important in telescope construction because all incoming parallel light rays to a reflective parabolic surface reflect off the surface onto the same point, the focus of the parabola. This enables light from a source to be gathered over a large region. In this worksheet we will seek to understand this reflection principle. We need one fact about reflections in two dimensions that we can regard as experimentally determined: When light falls onto a reflective curve, hitting a point p, it reflects in such a way that the incident ray and the reflected ray are reflections across the line intersecting p and perpendicular to the line tangent to the curve at p. The picture below illustrates this principle.



1. Graph the function f given by $f(x) = 3x^2$ and draw at least three incoming (incident) light rays parallel to the axis of symmetry of the graph of f, the line $x = 0$.

2. Where should the incident light ray that moves along the line $x = 0$ reflect?

3. Find the line, L, tangent to the graph of f at the point $(2, 12)$.

4. Find an equation for the line, $L_\perp$, perpendicular to L that intersects $(2, 12)$.

5. Find an equation for the path of motion of the reflection across $L_\perp$ of the incident light ray parallel to the y-axis and intersecting $(2, 12)$.

6. Redo your above calculations, where L is now tangent to the graph of f at the point $(a, f(a))$, $L_\perp$ is perpendicular to L and intersects $(a, f(a))$, and the incident light ray parallel to the y-axis intersects $(a, f(a))$. Show that all reflected light rays intersect the same point, the focus.

7. Let A be a positive real number. Redo the previous problem, but now where f is given by $f(x) = Ax^2$.

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Worksheet 12 ● Tangency and Inverse Functions

1. Let f be the function defined by $f(x) = x^2$ on the positive real numbers. Find the equation of the line tangent to the graph of f at the point $(3, 9)$.

2. Graph the reflection of the graph of f and the line tangent to the graph of f at the point $(3, 9)$ about the line $y = x$.

3. The reflection of the graph of f about $y = x$ is the graph of a function, g. What function is it? The reflection of the line above is also a line. What line is it and what is its slope? Where do the reflected line and reflected graph meet?

If f is a rational function and L_1 is the line tangent to the graph of f at the point $(x, f(x))$ then define by $f'(x)$ the slope of L. If f^{-1} is the inverse function of f, then a line L_2 is tangent to the graph of f^{-1} at a point (a, b) if and only if the reflection of L_2 about the line $y = x$ is tangent to the graph of f at the point (b, a). In this case, let $(f^{-1})'(a)$ be the slope of L_2.

4. Calculate $(f^{-1})'(4)$ if f is the function given by $f(x) = x^3 + 2x + 1$. Note that $f(1)$ is equal to 4.

5. Repeat the above procedure but with the function f given by

$$f(x) = \frac{3x + 1}{x + 1},$$

but with the selected point on the graph being $(1, 2)$. After doing this, find the inverse of f and calculate the equation of the line tangent to the graph of f^{-1} at $(2, 1)$. Compare the equations of the two lines that you obtain. Carefully graph the two functions and the two lines as well as the line given by the locus of points $y = x$ to see if this makes sense.

6. Suppose that n is a natural number. Suppose that $g'(x) = x^{\frac{1}{n}}$. Use the above procedure to calculate $g'(a)$ for any non-zero a if n is odd and for strictly positive a if n is even. Why do we not attempt to calculate $g'(0)$?