Mock Algebra Qualifier, Part A
August 16, 2048

Do four out of the five problems.

1. (a) Prove that any subgroup of index 2 must be normal.
(b) How many index 2 subgroups are there of the free group on two generators? Write down these subgroups in terms of their generators.

2. Let G be a finite group, and take $H < G$. Let P be a Sylow p-subgroup of G for some prime number p and suppose that $|H|$ is divisible by p. Prove that if P is normal in G, then $P \cap H$ is a Sylow p-subgroup of H.

3. Let R be a unique factorization domain, and let F be it's field of fractions. For a monic polynomial $f \in R[x]$, prove that if $r \in F$ is a root of f, the $r \in R$.

4. Let R be a commutative ring with 1_R. Recall that a multiplicative subset S of R is said to be *saturated* if for $x, y \in R$ we have that if $xy \in S$ then $x, y \in S$.
(a) Prove that if S is a saturated multiplicative subset of R, then $R \setminus S$ is a union of prime ideals of R.
(b) Prove that the set of zero-divisors of R is a union of prime ideals of R.

5. (a) For a commutative ring R with 1_R, let $\mathrm{J}(R)$ denote the intersection of all the max ideals of R. Prove that if $x \in \mathrm{J}(R)$ then $x + 1$ is invertible in R.
(b) For the ring R of rational numbers with odd denominator, prove that $\mathrm{J}(R)$ consists of all the rational numbers with odd denominator and even numerator.

Algebra B 2048. Qualifying Exam

Choose 5 questions out of 6.
All rings are assumed to be unital and all modules are assumed to be unitary unless specified otherwise. Given a unital ring R, let $R^\times$ denote is group of units.

1. For a unital ring R and unitary R-module A Write out the details of the isomorphisms

$$\mathrm{Hom}_R(R, A) \simeq A \simeq R \otimes_R A\,.$$

Supposing that F is a finite dimensional free R module, prove that $F \otimes_R A \simeq \mathrm{Hom}_R(F, A)$.

2. (a) For P_1 and P_2 projective modules over $\mathbb{Z}$, Prove that $P_1 \oplus P_2$ is a projective $\mathbb{Z}$-module.
(b) Considering the natural left-module structure of $\mathbb{Z}$ on itself by multiplication, prove that the submodule $2\mathbb{Z}$ is *not* an injective $\mathbb{Z}$ module.
(c) Prove that $\mathbb{Z}_2$ is *not* a projective $\mathbb{Z}$-module.

3. Let R be a commutative ring and let M and N be free R-modules of the same finite rank over R. Prove that if $\varphi \in \mathrm{Hom}_R(M, N)$ is surjective, then it must be an isomorphism. Why do we need to assume that R is commutative? Is this same statement true if we assume that φ is injective instead?

4. Give examples of the following:
(a) A submodule of a finitely generated module that is not finitely generated.
(b) A projective module that is not free.
(c) A free module that is not torsion-free.
(c) A torsion-free module that is not free.

5. For a field K, say that a $K[x]$-module M is nilpotent if for every non-unit $p \in K[x]$, we have $p^n M = 0$ for sufficiently large n. Prove that a finitely generated nilpotent indecomposable $K[x]$-module is isomorphic to $K[x]/(x^k)$ for some $k > 0$.

6. Let M and N be square matrices over a field. Recall that we say M and N are *equivalent* if there exist invertible matrices P and Q such that $M = QNP^{-1}$, and M and N are similar if there exists invertible P such that $M = PNP^{-1}$. Let $\mathrm{Mat}_3(\boldsymbol{F}_7)$ denote the ring of of 3×3 matrices over the field with seven elements. Under matrix equivalence, how many equivalence classes of matrices are there in $\mathrm{Mat}_3(\boldsymbol{F}_7)$? How many similarity classes of matrices are there in $\mathrm{Mat}_3(\boldsymbol{F}_7)$?

Fields Qualifier 2048

Do any 3 problems.

1. Suppose that F has characteristic p and that K is a finite extension of F. Prove that if $p \nmid [K : F]$ then K is a separable extension of F

2. Let K be a field with 9 elements. Prove from scratch that K has an extension of degree 2. and that any two such extensions are isomorphic over K.

3. Let $\zeta \in \mathbb{C}$ be a primitive nth-root of unity, and let $L = \mathbb{Q}(\zeta)$.

   (i) Show that $\mathbb{Q} \to L$ is a Galois extension.
   (ii) For any $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(L)$, show that $\sigma(\zeta) = \zeta^i$ for some integer i.
   (iii) Use the previous part to show that $\mathrm{Aut}_{\mathbb{Q}}(L)$ is abelian.

4. Compute the Galois group of $X^3 + 3$ over the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and over the field with seven elements $\boldsymbol{F}_7$.