Mock Algebra Qualifier, Part A
September 00, 2018

Do four out of the six problems.

1. Let $G = \mathbb{Q}/\mathbb{Z}$, where $\mathbb{Q}$ and $\mathbb{Z}$ are considered as additive groups. Prove that for any positive integer n, G has a unique subgroup $G(n)$ of order n, and that $G(n)$ is cyclic.

2. (a) Show that a finite p-group has nontrivial center.
(b) Show if G is a finite group with order p^2, then G is abelian.

3. For groups $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$, provide a counterexample to each of the following statements
(i) $G_1 \simeq G_2$ and $N_1 \simeq N_2$ implies that $G_1/N_1 \simeq G_2/N_2$.
(ii) $G_1 \simeq G_2$ and $G_1/N_1 \simeq G_2/N_2$ implies that $N_1 \simeq N_2$.
(iii) $N_1 \simeq N_2$ and $G_1/N_1 \simeq G_2/N_2$ implies that $G_1 \simeq G_2$.

4. Let R be a UFD in which each nonzero prime ideal is maximal.
(a) Prove that if $a, b \in R$ and $(a, b) = 1$, then $ax + by = 1$ for some $x, y \in R$.
(b) Show that each ideal of R that is generated by two elements must be principal.

5. (a) Prove that $\mathbb{Q}[x]/(x^5 - 4x + 2)$ is a field.
(b) Prove that $\mathbb{Z}[x]/(x^5 - 4x + 2)$ is *not* a field.

6. For an integral domain R, and a,b in R with a being a unit, prove that the map $x \mapsto ax + b$ is an automorphism of $R[x]$ that restricts to the identity on R. Furthermore, prove that *every* automorphism of $R[x]$ that fixes R is of this type.

Algebra B 2018. Qualifying Exam

Choose 5 questions out of 6.
All rings are assumed to be unital and all modules are assumed to be unitary unless specified otherwise. Given a unital ring R, let $R^{\times}$ denote is group of units.

1. Let $\boldsymbol{k}[x]$ be the polynomial ring in one variable with coefficients in the field $\boldsymbol{k}$,
(a) Prove that $\boldsymbol{k}$ is a cyclic $\boldsymbol{k}[x]$-module, where the action is defined as $f.a = f(1)a$ for $a \in \boldsymbol{k}$ and $f \in \boldsymbol{k}[x]$. Why isn't $\boldsymbol{k}$ a free $\boldsymbol{k}[x]$-module?
(b) Since $\boldsymbol{k}$ is a cyclic $\boldsymbol{k}[x]$-module, there is a surjective map of modules $\boldsymbol{k}[x] \to \boldsymbol{k}$. What is the kernel of this map? Show that the corresponding short exact sequence doesn't split.
(c) Prove that $\boldsymbol{k}$ is not a projective $\boldsymbol{k}[x]$-module.

2. For a commutative unital integral ring R and unitary R-module M, recall that $\mathrm{Ann}_R(m) = \{r \in R \mid rm = 0\}$. Let $\tau(M) = \{m \in M \mid \mathrm{Ann}_R\, m \neq 0\}$. Prove that the assignment $M \mapsto \tau(M)$, $f \mapsto f|_{\tau(M)}$, for each R-module M and morphism of R-modules f defines a left-exact functor from the category of R-modules to the category of torsion R-modules. Why is it necessary to assume that R is an integral domain?

3. Let R be a unital ring and A and B be unitary R-modules.
(a) Sketch the construction of the abelian group $A \otimes_R B$.
(b) Prove that for $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \simeq \mathbb{Z}_{(m,n)}$, where (m,n) denotes the greatest common divisor of m and n.

4. Let R be the ring $\mathbb{C}[x]$, polynomials over the complex numbers.
(a) Prove that every cyclic R-module is isomorphic to $R/(f)$ for some $f \in \mathbb{C}[x]$.
(b) Prove a necessary and sufficient condition on f for the module $R/(f)$ to be a simple module
(c) How do the answers to the previous two parts change if we were working over $R = \mathbb{R}[x]$ instead?

5. Suppose that we have an exact sequence of vector spaces (which are modules over a division ring $\boldsymbol{k}$ and are inherently free)

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

such that $\dim_{\boldsymbol{k}} V_i$ is finite for each i. Prove that $\sum_{i=1}^{n} (-1)^i \dim_{\boldsymbol{k}} V_i = 0$.

6. Prove that if A and B are invertible matrices over a field $\boldsymbol{k}$, then $A + \lambda B$ is invertible for all but finitely many $\lambda \in \boldsymbol{k}$.

Fields Qualifier 2018

1. Determine the Galois group, three of its subgroups, and the corresponding intermediate fields of the splitting field of $f(x) = (x^3 - 2)(x^2 - 3)$ over $\boldsymbol{Q}$.

2. In the field $K(x)$, let $u = x^3/(x+1)$. Show that $K(x)$ is a simple extension of the field $K(u)$. What is $[K(x) : K(u)]$?

3. Given a tower of fields $F \to E \to K$, prove or disprove by providing a counterexample:

   (i) If K is normal over F, then K is normal over E.

   (ii) If K is normal over E and E is normal over F, then K is normal over F.

   (iii) If K is separable over F, then K is separable over E and E is separable over F.

4. For finite (nontrivial) extension of fields $F \to K$, prove that if K is algebraically closed, then $\mathrm{char} F = 0$, $[K : F] = 2$, and $K = F(\mathrm{i})$, where $\mathrm{i}^2 = -1$.