Mock Algebra Qualifier, Part A
September 125, 2125

Do four out of the five problems.

1. Classify all groups of order 25.

2. (a) What is the definition of the *coproduct* of objects A and B in a category $\mathcal{C}$?
(b) How does the coproduct of $\mathbb{Z}_2$ and $\mathbb{Z}_5$ in the category of groups differ from their coproduct in the category of *abelian* groups?

2. Can a group of order 120 be simple?

2. Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.
(i) Determine the units in R.
(ii) Is the principal ideal (11) prime in R?

5. Let R be the ring of all rational numbers with denominators not divisible by 5. Prove that R is a local ring. If $\mathfrak{m}$ is the unique max ideal of R, can you describe the ring $R/\mathfrak{m}$?

Algebra B 2125. Qualifying Exam

Choose 5 questions out of 6.
All rings are assumed to be unital and all modules are assumed to be unitary unless specified otherwise.

1. For a ring R consider a short exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0\,.$$

Let I be an ideal of R. Prove that if $IM' = 0$, then $IM = 0 = IM''$. What can we say about M' if we're given that $IM = 0 = IM''$? Then suppose R is a PID. Recall that for a prime $\mathfrak{p} \in R$, $M(\mathfrak{p})$ denotes the submodule of all elements of M that have order some power of $\mathfrak{p}$. Show that the full subcategory of finitely generated R-modules whose objects satisfy $M = M(\mathfrak{p})$ is closed under taking extensions.

2. Describe the module $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{R}, \mathbb{Z})$. Is $\mathbb{R}$ a projective $\mathbb{Z}$-module?

3. Recall that an R-module M is divisible with respect to a subset $S \subset R$ if for every $m \in M$ and $s \in S$ there exists some $m' \in M$ such that $m = sm'$. We say that M is divisible if it is divisible with respect to $R \setminus \{0\}$.

(i) Describe $N \otimes_R M$ when N is a right torsion R-module and M is a divisible left R-module.

(ii) Let R be commutative and let S be a multiplicative set. Prove that if the natural homomorphism of R-modules $\varphi\colon M \to S^{-1}M$ where $m \mapsto sm/s$ is an isomorphism, then M is divisible with respect to S. Conversely, if M is divisible with respect to S, then φ is surjective.

4. Let V be a finite dimensional vector space over $\mathbb{C}$, and take φ in the endomorphism ring $\mathrm{End}_{\mathbb{C}}(V)$.

(i) Prove that φ defines a left $\mathbb{C}[x]$-module structure on V where, for $f \in \mathbb{C}[x]$ and $\boldsymbol{v} \in V$, $f(\varphi) \in \mathrm{End}_{\mathbb{C}}(V)$ and $f.\boldsymbol{v} := \big(f(\varphi)\big)(\boldsymbol{v})$.

(ii) We say a subspace $W \subset V$ is *φ-invariant* if $\varphi(W) \subset W$. Prove that W is φ-invariant if and only if W is a $\mathbb{C}[x]$-submodule of V under the action inducted by φ. Furthermore prove that $V_\varphi(\boldsymbol{v})$, the smallest φ-invariant subspace of V containing $\boldsymbol{v}$, is the cyclic submodule $\mathbb{C}[x]\boldsymbol{v}$.

5. Consider the matrices

$$M = \begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

What are the invariant factor and elementary divisor decompositions of the $\mathbb{Q}[x]$-module corresponding to M? (Hint: Recall that $\mathbb{Q}[x]$ is a PID and regard M as an element of $\mathrm{End}_{\mathbb{Q}}(\mathbb{Q}^4)$ and see the previous problem.) What are these decompositions if you consider the corresponding $\mathbb{C}[x]$-module instead? What is the Jordan canonical form of M considered as a matrix over $\mathbb{C}$? Determine, with proof, whether or not the matrices M and N are equivalent over $\mathbb{Q}$. Are M and N similar over $\mathbb{Q}$?

6. For a commutative unital integral ring R, let A be an $n \times n$ matrix over R. Prove that $\det(A) = 0$ if and only if the system of linear equations $A\boldsymbol{x} = 0$ has a nontrivial solution. Does either direction of this statement remain true if we drop the assumption that R is an integral domain?

Fields Qualifier 2125

1. Prove or disprove: the polynomial $x^{625} - x - 1$ is reducible over $\boldsymbol{F}_5$.

2. Construct a field with 121 elements and give the rules for addition and multiplication. If a is a generator, what is the multiplicative inverse of $1 + a$ in terms of your set of *minimal* generators.

3. Let $f = x^5 - 45x^3 + 35x^2 + 15$ and $g = x^{11} - 11$, both considered as polynomials in $\mathbb{Q}[x]$. Suppose $\alpha \in \mathbb{C}$ is a root of f. Prove or disprove: $\mathbb{Q}(\alpha)$ contains a root of g.

4. Prove that an algebraic extension $K \to F$ is normal over K if and only if every irreducible f in $K[x]$ factors in $F[x]$ as a product of irreducible factors all of which have the same degree.