# MOCK QUALIFYING EXAMINATION, ALGEBRA, PART A, 2019

September $n^2$, 2019

Solve any four questions; indicate which ones are supposed to be graded. You must show all work and justify all statements either by referring to an appropriate theorem or by providing a full solution.

**1.** Let $G$ be a group, and let $A$ be an abelian group. Let $\varphi\colon G \to \mathrm{Aut}(A)$ be a group homomorphism. Let $A \times_\varphi G$ denote the set $A \times G$ with the binary operation

$$(a, g)(a', g') = (a + \varphi(g)(a'), gg')\,.$$

(a) Prove that $A \times_\varphi G$ is a group.
(b) Find a map $\varphi\colon \boldsymbol{Z}_2 \to \mathrm{Aut}(\boldsymbol{Z}_m)$ such that the dihedral group $D_m$ is isomorphic to $\boldsymbol{Z}_m \times_\varphi \boldsymbol{Z}_2$. *Do not forget to prove the isomorphism!*

**2.** Let $G$ be a finite group, and let $\mathrm{Z}(G)$ denote the *center* of $G$.
(a) Prove that if $G/\mathrm{Z}(G)$ is cyclic, then $G$ is abelian.
(b) Prove that if $\mathrm{Aut}(G)$ is cyclic, then $G$ is abelian.
(c) Prove that if $\mathrm{Aut}(G)$ is nontrivial and cyclic, then $|\mathrm{Aut}(G)|$ must be even.
(d) Prove that there is no group with infinite cyclic automorphism group.

**3.**
(a) Prove that any subgroup of index 2 must be normal.
(b) How many index 2 subgroups are there of a free group on two generators? Write down these subgroups in terms of their generators.

**4.** An element $e$ in a ring $R$ is said to be idempotent if $e^2 = e$ . The center $\mathrm{Z}(R)$ of a ring $R$ is the set of all elements $x \in R$ such that $xr = rx$ for all $r \in R$. An element of $\mathrm{Z}(R)$ is called central. Two central idempotents $f$ and $g$ are called orthogonal if $fg = 0$. Suppose that $R$ is a unital ring.
(a) If $e$ is a central idempotent, then so is $1_R - e$, and $e$ and $1_R - e$ are orthogonal.
(b) $eR$ and $(1_R - e)R$ are ideals and $R = eR \times (1_R - e)R$.
(c) If $R_1, \ldots, R_n$ are rings with identity then the following statements are equivalent.
  (i) $R \cong R_1 \times \cdots \times R_n$
  (ii) $R$ contains a set of orthogonal central idempotents $e_1, \ldots, e_n$ such that $e_1 + \cdots + e_n = 1_R$ and $e_i R \cong R_i$, $1 \leq i \leq n$.
  (iii) $R = I_1 \times \cdots \times I_n$ where $I_k$ is an ideal of $R$ and $R_k \cong I_k$.

**5.**
(a) Give an example of a category in which a morphism between two objects is epic if and only if it is surjective.
(b) Give an example of a category $\mathcal{C}$ and of an epic morphism between two objects in $\mathcal{C}$ which is not surjective.

Mock Algebra Qualifying Examination, Fall 2019, Part b

Attempt as many questions as you like. A perfect score is 50.

Assume that all rings have identity.

1. (5 points) Let $V$ be a vector space over a field $K$ of dimension $r$. Let $f \in \mathrm{Hom}_K(V, K)$. Prove that if $f$ is non-zero, then it is surjective and determine the dimension of the kernel of $f$.

2. (7 points) (a) Suppose that $R$ and $S$ are commutative rings and that $M$ is a $(R, S)$-bimodule. This means that $M$ is a left $R$-module and a right $S$-module and the actions are compatible, i.e. $r(ms) = (rm)s$, for all $r \in R$, $s \in S$, and $m \in M$. Let $N$ be a left $S$-module. How does one define a left $R$-module structure on $M \otimes_S N$? What must you check to see that the action is well-defined? If we assume now in addition that $N$ is a $(S, R)$-bimodule that can you say about $M \otimes_S N$?

(b) (3 points) Suppose now that $K$ is a field and let $V$, $W$ be vector space over $K$. Use (a) to show that $V \otimes_K W$ is also a vector space over $K$. What is the most natural way to find a basis for $V \otimes_K W$?

3. (5 points) (a) Let $V$, $W$ be vector spaces over a field $K$. How does one define a vector space structure on $\mathrm{Hom}_K(V, W)$? Suppose now that $W = K$. Given a basis for $V$, how would you produce a natural basis for $V^* = \mathrm{Hom}_K(V, K)$? More generally, if $\dim V = r$ and $\dim W = s$ and you are given bases for $V$ and $W$, find a natural basis for $\mathrm{Hom}_K(V, W)$.

(b) (10 points) Let $W$ be another vector space over $K$. Define the natural map of vector spaces $V^* \otimes W \to \mathrm{Hom}_K(V, W)$ and prove that it is an isomorphism of vector spaces.

4. (10 points) Let $R$ be the polynomial ring $\mathbf{C}[t]$ in one variable with coefficients in the complex numbers and let $I$ be the ideal generated by $t^2$ and let $M = R/I$. Prove that $M$ has a proper non-zero submodule and that $M$ cannot be written as a direct sum of proper non-zero submodules. Suppose now that we take $J$ to be the ideal generated by $t(t-1)$. Prove that the module $N = R/J$ is isomorphic to a direct sum of two proper non-zero submodules.

5. (5 points) Prove that an $n \times n$-matrix with entries in a field $K$ is invertible iff 0 is not an eigenvalue of the matrix.

6. (10 points) What is the companion matrix $A$ of the polynomial $q = x^2 - x + 2$? Prove that $q$ is the minimal polynomial of $A$.

7. (10 points) Suppose that $P_1$ and $P_2$ are $R$-modules. Prove that $P_1 \oplus P_2$ is projective iff $P_1$ and $P_2$ are projective.

8. (10 points) Let $0 \to L \to M \to N \to 0$ be a short exact sequence of $R$-modules such that we have a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, L) \longrightarrow \mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Hom}_R(N, N) \longrightarrow 0$$

Prove that the original short exact sequence is split.

# Mock Algebra Qualifier 2019 - Part C

**Do 4 out of the 5 problems.**

(1) Prove or disprove the following: If $K \to F$ is an extension (not necessarily Galois) with $[F : K] = 6$ and $\mathrm{Aut}_K(F)$ isomorphic to the Symmetric group $S_3$, then $F$ is the splitting field of an irreducible cubic in $K[x]$.

(2) Let $f = x^3 - x + 1 \in \boldsymbol{F}_3[x]$. Show that $f$ is irreducible over $\boldsymbol{F}_3$. Let $K$ be the splitting field of $f$ over $\boldsymbol{F}_3$. Compute the degree $[K : \boldsymbol{F}_3]$ and the number of elements of $K$.

(3) Let $K \subseteq F$ be a finite dimensional extension.
   (a) Define what it means for $F$ to be separable over $K$.
   (b) Prove from scratch that if $K$ is a finite field then $F$ is separable over $K$.
   (c) Prove that if $K$ is of characteristic zero then $F$ is separable over $K$.
   (d) Given an example of a non-separable finite dimensional extension.

(4) Let $F_{12}$ be a cyclotomic extension of $\mathbb{Q}$ of order 12. Determine $\mathrm{Aut}_{\mathbb{Q}}(F_{12})$ and all intermediate fields.

(5) Let $F = \mathbb{C}(t^4) \subset K = \mathbb{C}(t)$, where $t$ is a formal variable. Compute the Galois group $\mathrm{Aut}_F(K)$, and determine its subgroups and corresponding intermediate fields.