1 Introduction

We consider the Hull-White model with constant variables. In this model the interest rate r_t satisfies the following stochastic differential equation (SDE):

$$dr_t = \sigma dW_t + (\theta - \alpha r_t)dt \tag{1}$$

where σ, θ, and α are constants.
By solving this SDE, we will get,

$$r_t = e^{-\alpha t}\left(r_0 + \frac{\theta}{\alpha}\left(e^{\alpha t} - 1\right) + \sigma \int_0^t e^{\alpha s} dW_s\right) \tag{2}$$

The price of a discount bond at time t with expiration time T is,

$$P(t,T) = E\left(e^{-\int_t^T r_u du} \mid \mathcal{F}_t\right) = E\left(e^{-\int_t^T r_u du} \mid r_t = x\right) \tag{3}$$

First we compute, $\int_t^T r_u du$,
$\int_t^T r_u du = \frac{r_0}{\alpha}\left(e^{-\alpha t} - e^{-\alpha T}\right) + \frac{\theta}{\alpha}\left[(T-t) - \frac{1}{\alpha}\left(e^{-\alpha t} - e^{-\alpha T}\right)\right] + \frac{\sigma}{\alpha}\int_0^t \left(e^{-\alpha(t-s)} - e^{\alpha(T-s)}\right) dW_s$
$+\frac{\sigma}{\alpha}\int_t^T \left(1 - e^{\alpha(T-s)}\right) dW_s$

$$E\left(e^{-\int_t^T r_u du} \mid r_t = x\right) = e^{\frac{-1}{\alpha}\left(1-e^{-\alpha(T-t)}\right)x + \left(\frac{\theta}{\alpha^2} - \frac{3\sigma^2}{4\alpha^3}\right) + \frac{1}{\alpha}\left(\frac{\sigma^2}{2\alpha} - \theta\right)(T-t) + \frac{1}{\alpha^2}\left(\frac{\sigma^2}{\alpha} - \theta\right)e^{-\alpha(T-t)} - \frac{\sigma^2}{4\alpha^3}e^{-2\alpha(T-t)}} \tag{4}$$

Thus,

$$P(t,T) = e^{A(t,T) - B(t,T)x} \tag{5}$$

Where,

$$B(t,T) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) \tag{6}$$

$$A(t,T) = \left(\frac{\theta}{\alpha^2} - \frac{3\sigma^2}{4\alpha^3}\right) + \frac{1}{\alpha}\left(\frac{\sigma^2}{2\alpha} - \theta\right)(T-t) + \frac{1}{\alpha^2}\left(\frac{\sigma^2}{\alpha} - \theta\right)e^{-\alpha(T-t)} - \frac{\sigma^2}{4\alpha^3}e^{-2\alpha(T-t)} \tag{7}$$

2 Computation of V_s

Some of the computations will be done with the help of Maple software.

The price of a call option at time s, with strike price K, and time of maturity t, on underlying bond with time of maturity T, is,

$$V_s = B_s E\left(B_t^{-1}\left(P\left(t,T\right)-K\right)^{+} \mid \mathcal{F}_s\right) \tag{8}$$

Where, $0 \leq s \leq t \leq T$.

First we compute V_0 , and then using the Markov property we get V_s from our expression for V_0.

We have,

$$V_0 = E\left(B_t^{-1}\left(P\left(t,T\right)-K\right)^{+}\right) \tag{9}$$

Since $B_t = e^{\int_0^t r_u du}$, we first compute $\int_0^t r_u du$.

$$\int_0^t r_u du = \left(\frac{r_0}{\alpha}-\frac{\theta}{\alpha^2}\right)+\frac{\theta}{\alpha}t+\frac{1}{\alpha}\left(\frac{\theta}{\alpha}-r_0\right)e^{-\alpha t}+\frac{\sigma}{\alpha}W_t-\frac{\sigma}{\alpha}e^{-\alpha t}\int_0^t e^{\alpha u}dW_u \tag{10}$$

So,

$$B_t^{-1} = \exp\left(C(t)-\frac{\sigma}{\alpha}W_t+\frac{\sigma}{\alpha}e^{-\alpha t}\int_0^t e^{\alpha u}dW_u\right) \tag{11}$$

where,

$$C(t) = -\left(\frac{r_0}{\alpha}-\frac{\theta}{\alpha^2}\right)-\frac{\theta}{\alpha}t-\frac{1}{\alpha}\left(\frac{\theta}{\alpha}-r_0\right)e^{-\alpha t} \tag{12}$$

$P\left(t,T\right)-K > 0$ if and only if $r_t < \frac{A(t,T)-\log K}{B(t,T)}$ if and only if $\int_0^t e^{\alpha u}dW_u < D$, where

$$D(t,T) = \frac{e^{\alpha t}}{\sigma}\left(\frac{A(t,T)-\log K}{B(t,T)}\right)-\frac{1}{\sigma}\left(r_0+\frac{\theta}{\alpha}\left(e^{\alpha t}-1\right)\right) \tag{13}$$

Now we will compute $B_t^{-1}P\left(t,T\right)$ and $B_t^{-1}K$.

$$B_t^{-1}K = K\exp\left(C(t) - \frac{\sigma}{\alpha}W_t + \frac{\sigma}{\alpha}e^{-\alpha t}\int_0^t e^{\alpha u}dW_u\right) \tag{14}$$

and

$$B_t^{-1}P\left(t,T\right) = \exp\left(G(t,T) - \frac{\sigma}{\alpha}W_t + \sigma e^{-\alpha t}\left(\frac{1}{\alpha} - B(t,T)\right)\int_0^t e^{\alpha u}dW_u\right) \tag{15}$$

where $G(t,T)$,

$$G(t,T) = C(t) + A(t,T) - B(t,T)e^{-\alpha t}\left(r_0 + \frac{\theta}{\alpha}(e^{-\alpha t} - 1)\right) \tag{16}$$

$\int_0^t e^{\alpha u}dW_u \overset{d}{=} N\left(0, \int_0^t e^{2\alpha u}du\right)$ and $W_t \overset{d}{=} N\left(0,t\right)$

Let $X = \int_0^t e^{\alpha u}dW_u$ and $Y = W_t - \frac{E(XW_t)}{E(X^2)}X$.

Then we easily see that $E\left(X\right) = E\left(Y\right) = 0$ and $Cov(X,Y) = E\left(XY\right) = 0$. Since (X,Y) is a Gaussian vector, we have that X and Y are independent Gaussian variables.

$Var(X) = \left(\sigma_X\right)^2 = \frac{e^{2\alpha t}-1}{2\alpha}$ and $Var(Y) = \left(\sigma_Y\right)^2 = t + \left(\frac{e^{\alpha t}-1}{\alpha}\right)^2\left(1 - 2\frac{2\alpha}{e^{2\alpha t}-1}\right)$

Substituting X and Y in equations (14) and (15), we get,

$$B_t^{-1}P\left(t,T\right) = \exp\left(G(t,T) - \frac{\sigma}{\alpha}Y + \beta(t,T)X\right) \tag{17}$$

where

$$\beta(t,T) = \frac{\sigma}{\alpha}\frac{E\left(XW_t\right)}{E\left(X^2\right)} + \sigma e^{-\alpha t}\left(\frac{1}{\alpha} - B(t,T)\right) \tag{18}$$

$$B_t^{-1}K = K\exp\left(C(t) - \frac{\sigma}{\alpha}Y + \gamma(t)X\right) \tag{19}$$

where

$$\gamma(t) = \frac{\sigma}{\alpha}\frac{E\left(XW_t\right)}{E\left(X^2\right)} + \frac{\sigma}{\alpha}e^{-\alpha t} \tag{20}$$

Now we get that $E\left(B_t^{-1}\left(P\left(t,T\right) - K\right)^+\right) = I_1 - I_2$ where,

$$I_1=\int_{-\infty}^{\infty}\int_{-\infty}^{D}\exp\left(G(t,T)-\frac{\sigma}{\alpha}y+\beta(t,T)x\right)\frac{\exp\left(-\frac{x^2}{2\sigma_X^2}\right)}{\sqrt{2\pi}\sigma_X}\frac{\exp\left(-\frac{y^2}{2\sigma_Y^2}\right)}{\sqrt{2\pi}\sigma_Y}dxdy$$

$$I_2=\int_{-\infty}^{\infty}\int_{-\infty}^{D}K\exp\left(C(t)-\frac{\sigma}{\alpha}y+\gamma(t)x\right)\frac{\exp\left(-\frac{x^2}{2\sigma_X^2}\right)}{\sqrt{2\pi}\sigma_X}\frac{\exp\left(-\frac{y^2}{2\sigma_Y^2}\right)}{\sqrt{2\pi}\sigma_Y}dxdy$$

Now we can write $I_1$and I_2 as

$$I_1=e^{G(t,T)}\int_{-\infty}^{\infty}e^{-\frac{\sigma}{\alpha}y}\frac{\exp\left(-\frac{y^2}{2\sigma_Y^2}\right)}{\sqrt{2\pi}\sigma_Y}dy\int_{-\infty}^{D}e^{\beta(t,T)x}\frac{\exp\left(-\frac{x^2}{2\sigma_X^2}\right)}{\sqrt{2\pi}\sigma_X}dx$$

$$I_2=Ke^{C(t)}\int_{-\infty}^{\infty}e^{-\frac{\sigma}{\alpha}y}\frac{\exp\left(-\frac{y^2}{2\sigma_Y^2}\right)}{\sqrt{2\pi}\sigma_Y}dy\int_{-\infty}^{D}e^{\gamma(t)x}\frac{\exp\left(-\frac{x^2}{2\sigma_X^2}\right)}{\sqrt{2\pi}\sigma_X}dx$$

Suppose now that $X\sim N\left(0,\sigma^2\right)$, then

$E(1_{\{X\leq x\}}e^{aX}) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}e^{at}e^{-\frac{t^2}{2\sigma^2}}dt = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}e^{\frac{-1}{2}\left(\frac{t-a\sigma^2}{\sigma}\right)^2+\frac{a^2\sigma^2}{2}}dt$ making the substitution $u=\frac{t-a\sigma^2}{\sigma}$, we get,

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}e^{at}e^{-\frac{t^2}{2\sigma^2}}dt=e^{\frac{a^2\sigma^2}{2}}\Phi\left(\frac{x-a\sigma^2}{\sigma}\right)$$

where $\Phi\left(z\right)= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{\frac{-1}{2}s^2}ds$ =P$(N\left(0,1\right)\leq z)$

We use the above formula to compute the integral with respect to x and y.

$$I_1=e^{G(t,T)}\exp\left(\frac{\frac{\sigma^2}{\alpha^2}\sigma_Y^2}{2}\right)\exp\left(\frac{\left(\beta(t,T)\right)^2\sigma_X^2}{2}\right)\Phi\left(\frac{D-\beta(t,T)\sigma_X^2}{\sigma_X}\right)$$

$$I_2 = Ke^{C(t)} \exp\left(\frac{\frac{\sigma^2}{\alpha^2}\sigma_Y^2}{2}\right) \exp\left(\frac{(\gamma(t))^2\,\sigma_X^2}{2}\right) \Phi\left(\frac{D-\gamma(t)\sigma_X^2}{\sigma_X}\right)$$

After lengthy and tedious computation we can obtain that

$$I_1 = P(0,T)\Phi(\frac{\log(\frac{F_0}{K}) + \frac{1}{2}\Sigma_0^2}{\Sigma_0})$$

$$I_2 = P(0,t)K\Phi(\frac{\log(\frac{F_0}{K}) - \frac{1}{2}\Sigma_0^2}{\Sigma_0})$$

where $F_0 = \frac{P(0,T)}{P(0,t)}$ and $\Sigma_0^2 = \left(B(0,t)^2\right)\frac{\sigma^2\left(1-e^{-2\alpha t}\right)}{2\alpha}$.
Therefore by using the Markov property we get,

$$V_s = P(s,T)\Phi(\frac{\log(\frac{Fs}{K}) + \frac{1}{2}\Sigma_s^2}{\Sigma_s}) - P(s,t)\Phi(\frac{\log(\frac{Fs}{K}) - \frac{1}{2}\Sigma_s^2}{\Sigma_s}) \tag{21}$$

where $F_s = \frac{P(s,T)}{P(s,t)}$ and $\Sigma_s^2 = \left(B(s,t)^2\right)\frac{\sigma^2\left(1-e^{-2\alpha(t-s)}\right)}{2\alpha}$

3 Conclusion

We observe that the formula for V_s is similar to the usual Black-Scholles formula for a stock.