Research Statement

Murugiah Muraleetharan (Muralee)

My current research area is geometric evolution equations, in particular, evolution of curves by curvature, mean curvature flow, and Ricci flow. I have also worked in Topology (Uniform spaces) and Scientific computing (computational complexity in parallel programming) and have two refereed papers in topology and one unpublished preprint in computational complexity.

First I would like to explain about my very recent work in evolution of curves by curvature flow:

Let γ_0 be a given smooth embedded convex closed plane curve and $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^2$ be a one parameter smooth family of embedded curves satisfying $\gamma(., 0) = \gamma_0$. If k is the curvature and N is the inward unit normal then we say that γ evolves by the *curvature flow* (or curve shortening flow) if

$$\frac{\partial \gamma}{\partial t}(p,t) = k(p,t)N(p,t), \quad (p,t) \in S^1 \times [0, \omega) \tag{1}$$

The curvature vector $\mathbf{k} = kN$. If we let $s = s_t$ be the arclength on $\gamma_t = \gamma(., t)$, then $\mathbf{k} = (\partial^2/\partial s^2)\gamma$ and the equation (1) can be written in the following form $(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s_t^2})\gamma = 0$ making the quasilinear parabolic nature of the equation apparent. The existence, regularity, and long term behavior of solutions to this system have been extensively studied.

The first deep theorem about curvature flow was proved by Gage and Hamilton.

Theorem 1. *[9] Under the curvature flow, a convex curve remains convex and shrinks to a point. Furthermore it becomes asymptotically circular: If the evolving curve is dialated to keep the enclosed area constant, then the rescaled curve converges smoothly to a circle. i.e. convex curves shrink to round points.*

For higher dimensions, Huisken [14] proved that under mean curvature flow convex hypersurface in $\mathbb{R}^{n+1}$ contract smoothly to a single point in finite time and become spherical at the end of the contraction. The above result is not generally true for nonconvex embedded hypersurfaces. A barbell with a long, thin handle develops a singularity in the middle in short time. But under curvature flow for curves, Grayson [10] showed that the assumption of the convexity of the initial curve can be removed, and the result holds for arbitrary smooth embedded closed initial curves by showing embedded curves become convex without developing singularities.

Theorem 2. *[10] Under the curvature flow, embedded curves becomes convex and thus eventually shrink to round points.*

Later Grayson and Gage generalized the curvature flow to surfaces.

Theorem 3. *[11] A closed curve moving on a smooth compact Riemannian surface by curvature flow must either collapse to a point in a finite time or else converge to a simple closed geodesic as $t \to \infty$.*

Recently new direct proofs of Grayson's theorem [10] for curvature flow of embedded curves in planes have given by Hamilton [13] and Huisken [16]. Hamilton proved this using monotonicity of isoperimetric estimates, and Huisken proved it by obtaining a lower bound for the quotient of the extrinsic distance in the plane by the intrinsic distance along the curve.

We were able to extend the above techniques to the surface case and gave two different proofs of Grayson's theorem [11] for curvature flow of embedded curves in Riemannian surface, one using Hamilton's technique and the other one using Huisken's technique [17], [18], and [21].

We now state our main results: Let γ be a closed embedded curve evolving under the curvature flow in a compact surface M. If a singularity develops in finite time, then the curve shrinks to a point. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface. Using a local conformal diffeomorphism $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ between compact neighborhoods, we get a corresponding flow in the plane which satisfies the following equation:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2}\right) N' \tag{2}$$

where $\gamma'(p,t) = \phi(\gamma(p,t))$, k' is the curvature of γ' in U', N' is the unit normal vector, and J is smooth, bounded and bounded away from 0.

For a smooth embedded closed curve γ in $\mathbb{R}^2$, consider any curve Γ which divides the region enclosed by γ into two pieces with areas A_1 and A_2, where $A_1 + A_2 = A$ is the area enclosed by γ. Let L be the length of Γ. Define

$$G(\gamma, \Gamma) = L^2 \left(\frac{1}{A_1} + \frac{1}{A_2}\right), \quad \text{and} \quad \overline{G}(\gamma) = \inf_{\Gamma} G(\gamma, \Gamma).$$

First, we proved the following theorem.

Theorem A. *If $\gamma'(\cdot, t)$ is evolving by the parabolic flow (2), and t_0 is close enough to the blow-up time $\omega < \infty$, then there is some $\varepsilon > 0$ such that $\overline{G}(\gamma'(\cdot, t)) > \epsilon$ for all $t \in [t_0, \omega)$.*

Define the extrinsic and intrinsic distance functions $d, l : \Gamma \times \Gamma \times [0,T] \to \mathbb{R}$ by

$d(p,q,t) = |\gamma(p,t) - \gamma(q,t)|_{\mathbb{R}^2}$ and $l(p,q,t) = \int_p^q ds_t = s_t(q) - s_t(p)$ where Γ is either S^1 or an interval.

Next, we proved the following two theorems.

Theorem B. *Let $\gamma : I \times [0,T] \to \mathbb{R}^2$ be a smooth embedded solution of the flow (2), where I is an interval such that l is smoothly defined on $I \times I$. Suppose $\frac{d}{l}$ attains a local minimum at (p_0, q_0) in the interior of $I \times I$ at time $t_0 \in [0,T]$. Then*

$$\frac{d}{dt}\left(\frac{d}{l}\right)(p_0, q_0, t_0) \geq 0,$$

with equality if and only if γ is a straight line.

We now define the smooth function $\psi : S^1 \times S^1 \times [0,T] \to \mathbb{R}$ by $\psi(p,q,t) := \frac{L(t)}{\pi} \sin\left(\frac{l(p,q,t)\pi}{L(t)}\right)$.

Theorem C. *Let $\gamma : S^1 \times [0,T] \to \mathbb{R}^2$ be a smooth embedded solution of the flow (2). Suppose $\frac{d}{\psi}$ attains a local minimum $(\frac{d}{\psi})(p_0, q_0, t_0) < 1$ at some point $(p_0, q_0) \in S^1 \times S^1$ at time $t_0 \in [0,T]$. Then*

$$\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \geq 0,$$

with equality if and only if $\frac{d}{\psi} \equiv 1$ or $\gamma(S^1, \cdot)$ is a circle.

Theorem D. *Let γ be a closed embedded curve evolving by curvature flow on a smooth compact Riemannian surface. If a singularity develops in finite time, then the curve converges to a round point in the C^∞ sense.*

We proved the theorem D first using theorem A, and then using theorems B and C. First we can find the dilation-invariant estimates for the derivatives of the curvature in term of the maximum curvature, and obtains a sequence of dilations of the solutions along a blow up sequence which converges to a smooth nontrivial family of convex curves γ_∞. This limit is a solution of the curvature flow, and is complete. If the singularities are of type I, then by Huisken's monotonicity principle [15], the limit is a homothetically shrinking soliton, and the only embedded one is the circle. Hence if the forming singularity is type-I, then the curve converges to a round point in the C^∞ sense. In type II singularity, the limit solution exists for all time with curvature satisfying $0 < k \leq 1$, and $k = 1$ at the origin at $t = 0$. The strong maximum principle applied to Harnack estimate [12] shows the limit is a translating soliton, the grim reaper. In the grim reaper, a horizontal line segment has length $L < \pi$, while if it is high enough, it encloses an arbitrarily large area A_1, while there is still an arbitrarily large area A_2 on the other side if we go out far enough. If the grim reaper is to be the limit, then the original curve comes arbitrarily close to it after translating, rotating, and dilating; all of which do not affect the constant $\overline{G}$. But then we must have $\overline{G} \to 0$, which is impossible.

Some results in Topology and Scientific Computing:

Out of my masters degree thesis, title - *Generalizations of Filters and Uniform Spaces* (General Topology), I have two papers [6] and [7]. In my thesis, I extended the notion of a filter $\mathbb{F} \in 2^{2^X}$ to that of a: prefilter $\mathcal{F} \in 2^{I^X}$, generalized filter $f \in I^{2^X}$, and fuzzy filter $\phi \in I^{I^X}$. A uniformity is a filter with some other conditions and I extended the notion of a uniformity $\mathbb{D} \in 2^{I^{X\times X}}$ to that of: fuzzy uniformity $\mathcal{D} \in 2^{I^{X\times X}}$, generalized uniformity $d \in I^{2^{X\times X}}$, and super uniformity $\delta \in I^{I^{X\times X}}$. I established categorical embeddings from the category of uniform spaces into the categories of fuzzy uniform spaces, generalized uniform spaces, and super uniform spaces and also categorical embeddings into the category of super uniform spaces from the categories of fuzzy uniform spaces and generalized uniform spaces.

In my masters degree in computer science, I focused on Computational Complexity, and Parallel and Scientific Computing. In my unpublished work [26], I implemented the parallel LU-decomposition method for six different data layouts - column block, row block, column cyclic, row cyclic, block grid and scatted grid. I used LogP model to analysis the running time of algorithms since it's not depend on the structure of the network and implemented the algorithms using MPI. The result we got both from the theoretical analysis approaches the implementation result as the matrix size increases. This shows that communication time dominates when the matrix size is small, and computation time dominates when the matrix size is large. By using the LogP model, we successfully predicted the running time of the algorithm. Hence, the data layout should be carefully chosen since it takes an important role in parallel implementation.

Current and Future work:

We have mentioned above how Hamilton and Huisken simplified Grayson's theorem using isoperimetric estimates to rule out certain type of behavior. They both have used previous results concerning the classification of singularities - was based on Huisken's monotonicity formula and Hamilton's Harnack estimates, in addition to their isoperimetric estimates to rule of the type II singularities. We would like to get stronger bound on curvature so that we could avoid using the heavy machineries and give a direct proof for evolving curves in the plane. In the surface case, in addition to using Huisken's monotonicity formula and Hamilton's Harnack estimates, we have been using Oaks' result – if the initial curve is embedded, and the singularity develops in finite time, then curves shrink to a point. The Oaks'result is based on a series of papers by Angenent on a more general theory of parabolic equations for curves on surfaces. We also expect to give a more direct proof for evolving curves in the surface.

We have been also working on curvature flow of curves in compact or complete Riemannian manifold (M^n, g). In compact manifold, the short time existence for the flow is known but for non-compact complete manifold we expect to prove the short time existence using Shi type estimates. These dilation invariant estimates can also give long time existence and play an important role in singularity analysis. It is also interesting to study the evolution of curves in a manifold when the underlying manifold evolves by Ricci flow. Perelman third paper deals with this problem and has some results.

Sobolev imbedding and heat kernel estimates to study Ricci flows, especially in the case of surgeries: The proof of the Poincaré conjecture contains two essential steps, one is the proof of local non-collapsing with or without surgeries and the other is the classification of backward limits of ancient κ solutions. Perelman used his W entropy to prove the local non-collapsing for smooth Ricci flows and then used his new analytical functional, reduced length and volume to proof non-collapsing with surgeries and also for the classification of the limits. But W entropy and it's monotonicity imply certain uniform Sobolev inequalities along Ricci flows, and Qi Zhang used this to prove local non-collapsing results with or without surgeries and the classification of the backward limits of ancient κ solutions. Thus, Zhang gave a simpler proof of Poincaré conjecture using W entropy, Sobolev imbedding, and related heat kernel estimates, and bypassed using the reduced distance and volume but still followed the framework by Perelman. The reduced distance and volume are still needed for the proof of the Geometrization conjecture, specifically for Perelman's no local collapsing theorem II with surgeries. I have very interested in the above recent work of Zhang and have been interested in to see whether it is possible to give a simpler proof of the Geometrization conjecture using W entropy and related log Sobolev inequalities and heat kernel estimates.

References

[1] Steven J. Altschuler, *Singularities of the curve shrinking flow for space curves*, J. Differential Geometry **34** (1991), 491–514.

[2] Steven J. Altschuler and Matthew A. Grayson, *Shortening space curves and flow through singularities*, J. Differential Geometry **35** (1992), 283–298.

[3] Sigurd Angenent, *Parabolic equations for curves on surfaces part i. curves with p-integrable curvature*, Annals of Mathematics **132** (1990), 451–483.

[4] ______, *On the formation of singularities in the curve shortening flow*, J. Differential Geometry **33** (1991), 601–633.

[5] ______, *Parabolic equations for curves on surfaces part ii. intersections, blow-up and generalized solutions*, Annals of Mathematics **133** (1991), 171–215.

[6] M. H. Burton, M. Muraleetharan, and J. Gutierrez Garcia, *Generalized filters 1*, Fuzzy Sets and Systems, North-Holland **106** (1999), 275–284.

[7] ______, *Generalized filters 2*, Fuzzy Sets and Systems, North-Holland **106** (1999), 393–400.

[8] Huai-Dong Cao and Xi-Peng Zhu, *A complete proof of the poincaré and geometrization conjectures: application of hamilton-perelman theory of ricci flow*, Asian J. Math. **10** (2006), no. 2, 165–492.

[9] M. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geometry **23** (1986), 69–96.

[10] Matthew A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geometry **26** (1987), 285–314.

[11] ______, *Shortening embedded curves*, Annals of Mathematics **129** (1989), 71–111.

[12] Richard S. Hamilton, *Harnack estimate for the mean curvature flow*, Journal of Differential Geometry (1995), no. 1, 215–226.

[13] ______, *Isoperimetric estimates for the curve shrinking flow in the plane*, Annals of Mathematics Studies **137** (1995), 201–222.

[14] Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geometry **20** (1984), 237–266.

[15] ______, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geometry **31** (1990), 285–299.

[16] ______, *A distance comparison principle for evolving curves*, Asian J. Math. **2** (1998), 127–133.

[17] David Johnson and M. Muraleetharan, *Singularity analysis of evolving curves using the distance comparison principle*, Submitted.

[18] ______, *Singularity formation of embedded curves evolving on surfaces by curvature flow*, Submitted.

[19] Bruce Kleiner and John Lott, *Notes on the perelman paper*, http://arXiv.org/math.DG/0605667v1 (2006).

[20] John Morgan and Gang Tian, *Ricci flow and the poincaré conjecture*, AMS and Clay Institute (2006).

[21] M. Muraleetharan, *Evolution of curves by curvature flow*, Ph.D. thesis, Lehigh University, May 2006.

[22] Jeffrey A. Oaks, *Singualrities and self intersections of curve evolving on surfaces*, Indiana University Mathematical Journal **43** (1994), no. 3, 959–981.

[23] Grisha Perelman, *Finite extinction time for the solutions to the ricci flow on certain three manifolds*, http://arXiv.org/math.DG/0307245.

[24] ______, *Ricci flow with surgery on three manifolds*, http://arXiv.org/math.DG/0303109.

[25] ______, *The entropy formula for the ricci flow and its geometric application*, http://arXiv.org/math.DG/0211159v1 (2002).

[26] E. E. Santos and M. Muraleetharan, *Analysis and implementation of parallel lu-decomposition with different data layouts*, unpublished (2000).

[27] Terence Tao, *Course blog at ucla*, terrytao.wordpress.com/2008.

[28] Qi S. Zhang, *Heat kernel bounds, ancient κ solutions and the poincaré conjecture*, arXiv:0812.2460 (2009).

[29] ______, *Strong non-collapsing and uniform sobolev inequalities for ricci flow with surgeries*, Pacific J. Math **239** (2009), no. 1, 179–200.