Putnam Practice Set #1

1. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,
$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$
2. Prove the following equality for all natural numbers n:
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$
3. In a room there are 10 people, each of which has age between 1 and 60 (ages are only integers). Prove that among them there are 2 groups of people, with no common person, the sum of whose ages is the same.
4. Define the Fibonacci sequence as $F_1 = 1, F_2 = 1$, and in general, $F_{n+2} = F_{n+1} + F_n$ for $n1$. (So, e.g., $F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21$, and so on.) (a) Prove that every third Fibonacci number is even, and the rest are odd. (b) More generally, prove that F_k divides F_{nk} for any n and k positive integers.
5. A battle ship is travelling on the number line. It starts at an unknown integer and moves at an unknown constant integer speed (integers per second.) You can fire a cannon once every second at an integer, destroying the ship if it is there. Come up with an algorithm for firing that is guaranteed to destroy the ship in a finite amount of time.
6. Take five arbitrary points on the surface of a sphere. Prove that there is a closed hemisphere (including the boundary) which contains at least four points.