Putnam Practice Set #3

1. Show that
$$\sum_{n \in A} \frac{1}{n} < \infty,$$
where A is the set of positive integers that do not contain a '9' in their decimal expansion.

2. Evaluate
$$\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2 + n + 1),$$
where $\operatorname{Arccot}(t)$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot \theta = t$.

3. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a non-zero function g defined on (a, b) such that the wrong product rule is true for x in (a, b).

4. Find all real-valued continuously differentiable functions f on the real line such that for all x,
$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

5. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that
$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$