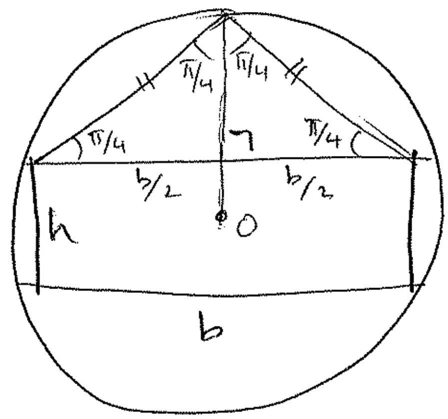
1



$$bh = \frac{1}{2} b\left(\frac{b}{2}\right)$$

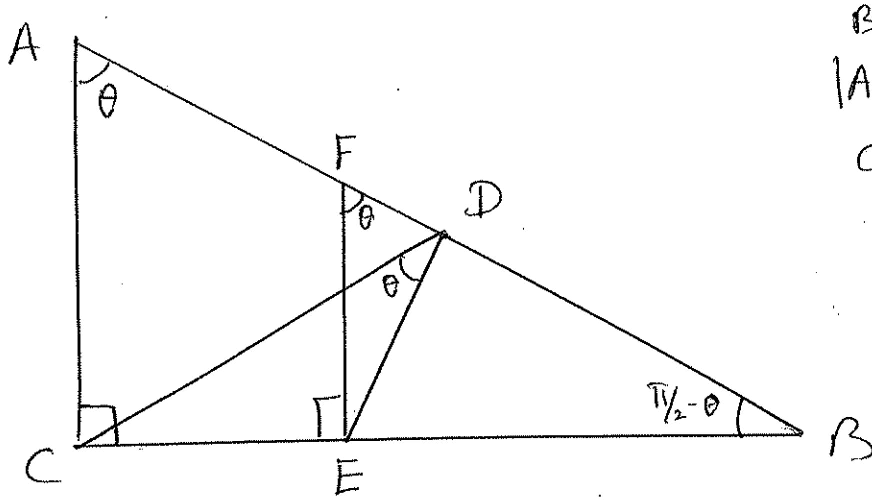
$$\Rightarrow h = \frac{b}{4}$$

$$1 = \frac{h}{2} + \frac{b}{2} = \frac{5}{8} b$$

$$\Rightarrow b = \frac{8}{5}$$

$$\Rightarrow h = \frac{2}{5}.$$

2



$$B\hat{A}C = \theta$$

$$|AC| = |AD| = 1$$

$$C\hat{D}E = \theta$$

$$A\hat{B}C = \pi/2 - \theta$$

$$E\hat{F}B = \theta$$

$$AB = AC \sec\theta = \sec\theta$$

$$BD = \sec\theta - 1$$

$$AC = AD \Rightarrow A\hat{C}D = A\hat{D}C = \pi/2 - \theta/2$$

$$\Rightarrow E\hat{D}B = \pi/2 - \theta/2$$

$$\Rightarrow D\hat{E}B = \frac{3\theta}{2}.$$

In $\triangle BDE$, $\dfrac{BD}{\sin(\frac{3\theta}{2})} = \dfrac{BE}{\sin(\pi/2 - \theta/2)}$

$$\Rightarrow BE = BD \cdot \frac{\cos(\theta/2)}{\sin(3\theta/2)}.$$

$$BE = (\sec\theta - 1)\frac{\cos(\theta/2)}{\sin(\frac{3\theta}{2})}.$$

$$\frac{EF}{BE} = \tan(\pi/2 - \theta)$$

$$\Rightarrow EF = BE\cot\theta = (\sec\theta - 1)\frac{\cos(\theta/2)}{\sin(\frac{3\theta}{2})}\cdot\frac{\cos\theta}{\sin\theta}.$$

$$= \frac{1-\cos\theta}{\cancel{\cos\theta}}\cdot\frac{\cos(\theta/2)}{\sin(\frac{3\theta}{2})}\cdot\frac{\cancel{\cos\theta}}{\sin\theta}$$

$$= 2\sin^2\left(\frac{\theta}{2}\right)\cdot\frac{\cos(\theta/2)}{\sin(\frac{3\theta}{2})}\cdot\frac{1}{2\sin(\frac{\theta}{2})\cdot\cos(\theta/2)}$$

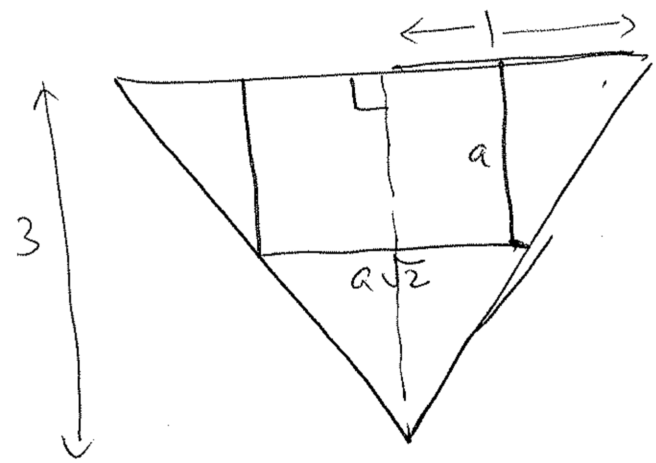
$$= \frac{\sin(\theta/2)}{\sin(3\theta/2)}.$$

$$\lim_{\theta\to 0} EF = \lim_{\theta\to 0}\frac{\sin(\theta/2)}{\sin(3\theta/2)}$$

$$= \lim_{\theta\to 0}\frac{\sin(\theta/2)}{\theta/2}\cdot\frac{3\theta/2}{\sin(3\theta/2)}\cdot\frac{1}{3}$$

$$= \frac{1}{3}.$$

3



Consider two opposite vertices of the cube that touch the cone. Then consider the vertical cross-section of the cone through those two vertices.

Let a be the side-length of the cube. Then length of the diagonal of a face is $a\sqrt{2}$.

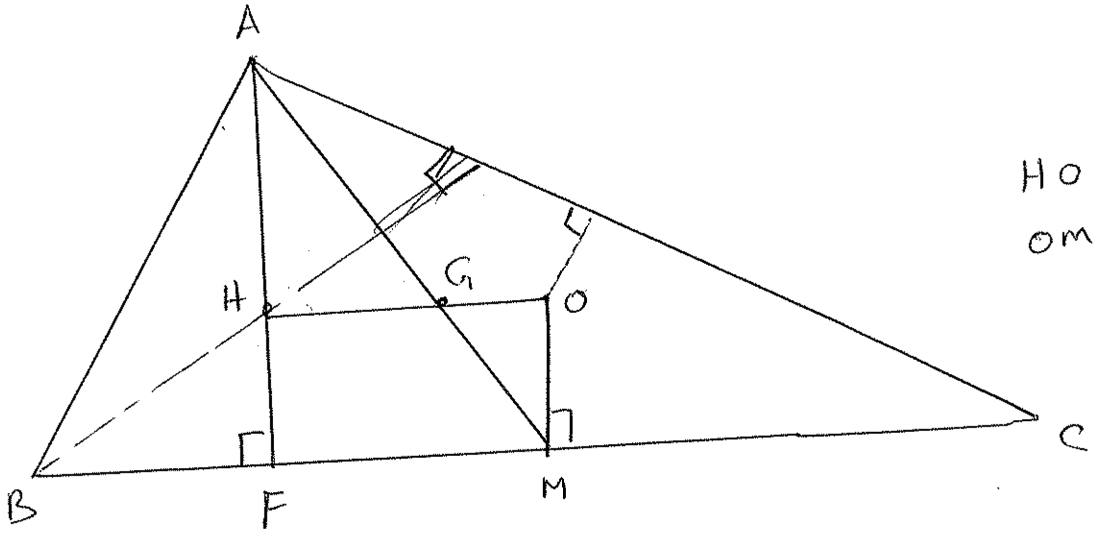
$$\frac{a\sqrt{2}}{2} = \frac{3-a}{3}$$

$$\Rightarrow 3\sqrt{2}\, a = 6 - 2a$$

$$\Rightarrow a = \frac{6}{3\sqrt{2}+2} = \frac{6(3\sqrt{2}-2)}{14}$$

$$= \frac{9\sqrt{2}-6}{7}.$$

4



$HO = 11$
$OM = 5$

H - the intersection of altitudes. (orthocenter)

O - the center of the circumscribed circle. (circumcenter)

G - Centroid.

HGO is a line (Euler line)

$\frac{AG}{GM} = \frac{2}{1}$. $\Rightarrow$ $AH = 10$.

$OA^2 = OH^2 + AH^2 = 11^2 + 10^2 = 221$ $(\because OA = OC)$

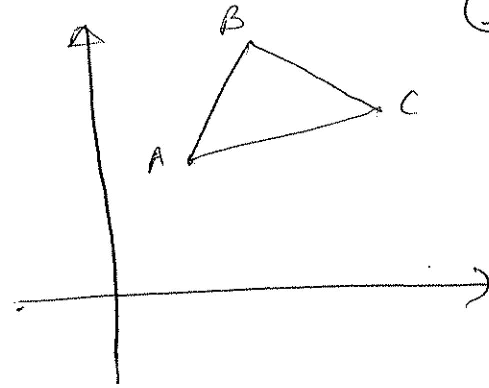
$MC^2 = OC^2 - OM^2 = 221 - 5^2$

$\Rightarrow$ $MC = 14$.

$BC = 28$.

5

(5) Coordinates of A, B, and C are integers in $\mathbb{R}^2$.



$$\text{area of } ABC = \frac{1}{2}\left|\det\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}\right|$$

$\Rightarrow$ $2[ABC]$ is a positive integer. $|AB|^2$ & $|BC|^2$ are also integers.

We have $\quad (|AB| + |BC|)^2 < 8\cdot[ABC] + 1$. ——— (1)

By Arithmetic mean - Geometric mean Inequality

$$|AB|^2 + |BC|^2 \geq 2\,|AB|\cdot|BC| \quad \text{——— } (*)$$

$$[ABC] = \frac{1}{2}|AB|\cdot|BC| \operatorname{Sin}(A\hat{B}C)$$

$$\Rightarrow \quad 2[ABC] \leq |AB|\cdot|BC| \quad \text{——— } (**)$$

$$8[ABC]+1 > (|AB|+|BC|)^2 = |AB|^2 + |BC|^2 + 2\,|AB|\cdot|BC|$$
$$\overset{(**)}{\geq} |AB|^2 + |BC|^2 + 4\,[ABC]$$

$$|AB|^2 + |BC|^2 \overset{(*)}{\geq} 2|AB|\cdot|BC| \overset{(**)}{\geq} 4\,[ABC]$$

$$\Rightarrow |AB|^2 + |BC|^2 + 4[ABC] + 1 \geq 8\,[ABC] + 1.$$

So we have

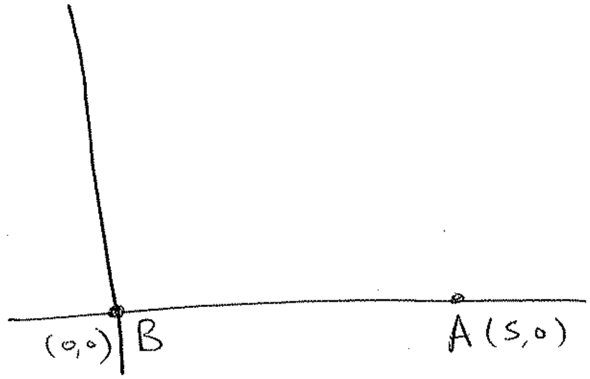
$$|AB|^2 + |BC|^2 + 4[ABC] < 8[ABC]+1 \leq |AB|^2 + |BC|^2 + 4[ABC] + 1$$

So we have equality in both $(*)$ & $(**)$

$\Rightarrow$ $|AB| = |BC|$ and $A\hat{B}C = \pi/2$

5

Sol^n 2 :



(s>0) $s \in \mathbb{Z}$.

$C \overset{?}{=} (0, s)$.

$x, y \in \mathbb{Z}$.

Let $C = (x, y+s)$.

Need to show $x = y = 0$.

$|AB| = s$, $BC^2 = x^2 + (y+s)^2$, $[ABC] = \frac{1}{2} s . (y+s)$

$$\left(|AB| + |BC|\right)^2 < 8\,[ABC] + 1$$

$$\Rightarrow \left(s + \sqrt{x^2 + (y+s)^2}\right)^2 < \cancel{8}4s(y+s) + 1.$$

$$s^2 + x^2 + (y+s)^2 + 2s\sqrt{x^2+(y+s)^2} < 4s(y+s) + 1$$

$$s^2 + 2ys + s^2 - 4sy - 4s^2 + 2s\sqrt{x^2+(y+s)^2} < 1 - x^2 - y^2$$

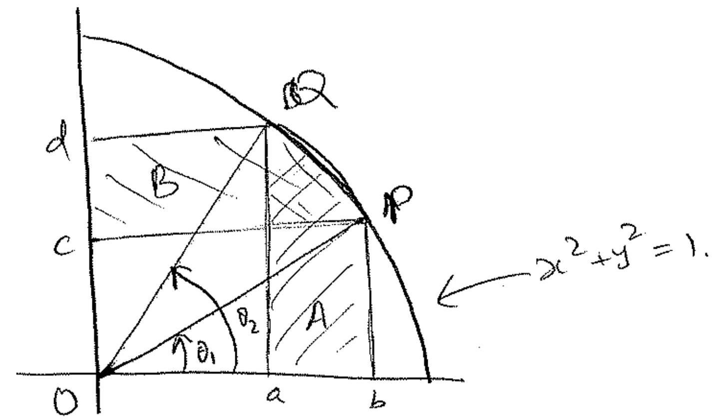
$$-2sy - 2s^2 + 2s\sqrt{x^2+(y+s)^2} < 1 - x^2 - y^2$$

$$2s\Big(\underbrace{\sqrt{x^2+(y+s)^2} - (y+s)}_{\geq 0}\Big) < 1 - x^2 - y^2$$

$$\Rightarrow 1 - x^2 - y^2 > 0 \Rightarrow x^2 + y^2 < 1$$

$$\Rightarrow x = y = 0$$

6



$$A = \int_a^b y\,dx\,, \qquad B = \int_c^d x\,dy$$

$$A + B = \int_{\theta_1}^{\theta_2} -y\,dx + x\,dy$$

$$x = \cos\theta\,, \quad y = \sin\theta$$

$$\Rightarrow \quad A + B = \int_{\theta_1}^{\theta_2} (\sin^2\theta + \cos^2\theta)\,d\theta = \theta_2 - \theta_1 = \text{arc length of } PQ.$$

OR $\quad A + B = 2$ area of OPQ. $= \theta_2 - \theta_1$