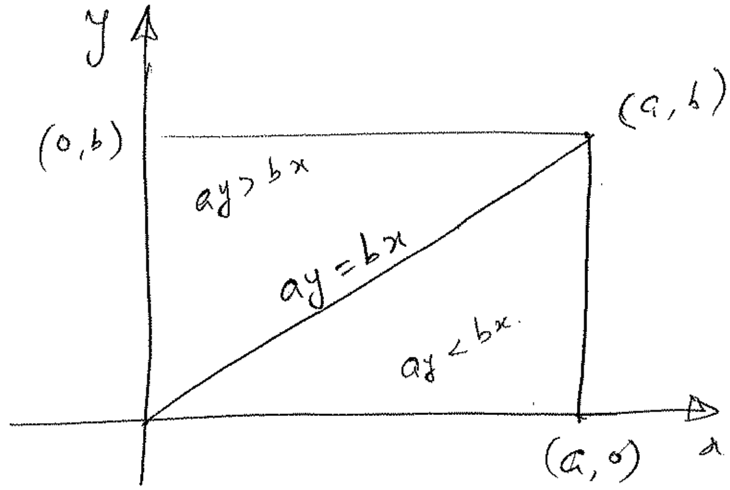
#6

Q1



$$\int_0^a \int_0^b e^{\max\{b^2x^2,\, a^2y^2\}}\, dy\, dx$$

$$= \int_0^a \int_0^{\frac{b}{a}x} e^{b^2x^2}\, dy\, dx + \int_0^b \int_0^{\frac{a}{b}y} e^{a^2y^2}\, dx\, dy$$

$$= \int_0^a \frac{b}{a} x\, e^{b^2x^2}\, dx + \int_0^b \frac{a}{b} y\, e^{a^2y^2}\, dy$$

$$= \frac{1}{2ab}\left[e^{b^2x^2}\right]_0^a + \frac{1}{2ab}\left[e^{a^2y^2}\right]_0^b = \frac{1}{2ab}\left(e^{a^2b^2} - 1\right) + \frac{1}{2ab}\left(e^{a^2b^2} - 1\right)$$

$$= \frac{1}{ab}\left(e^{a^2b^2} - 1\right).$$

Q2

$f : \mathbb{Z} \longrightarrow \mathbb{Z}$

$f(n) =$ [illegible] $- n$

(a) $f(f(n)) = n$, for all $n \in \mathbb{Z}$.

(b) $f(f(n+2)+2) = n$, for all $n \in \mathbb{Z}$.

(c) $f(0) = 1$

Sol$^{\underline{n}}$:

- $f(n) = 1 - n$ satisfies (a) – (c).

  (a) $f(f(n)) = f(1-n) = 1-(1-n) = n$.

  (b) $f(f(n+2)+2) = f(1-(n+2)+2) = f(1-n) = n$.

  (c) $f(0) = 1 - 0 = 1$.

- Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfies (a) – (c): Then

  $f(0) = 1$, $\quad f(f(0)) = 0$ [by (a)]

  $\Rightarrow f(1) = 0$

We use mathematical induction to show that

$f(n) = 1 - n$, for all $n \in \mathbb{Z}$.

from (b) $\quad f(f(n+2)+2) = n$, $\forall n$.

$\Rightarrow f(f(f(n+2)+2)) = f(n)$

$\Rightarrow f(n+2) + 2 = f(n)$ [by (a)]

$\Rightarrow f(n+2) = f(n) - 2$, for all $n \in \mathbb{Z}$.

$\Rightarrow$

We have $f(0)=1$, $f(1)=0$, and
$f(n+2) = f(n) - 2$, for all $n \in \mathbb{Z}$.

- Assume $f(k) = 1-k$ for all $k < n$ $(k \geq 0)$. Then

$$f(n) = f(n-2) - 2$$
$$= (1-(n-2)) - 2$$
$$= 1-n.$$

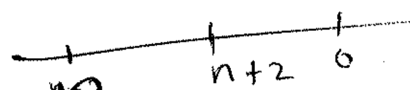
So by mathematical induction $f(n) = 1-n$ for all $n \in \mathbb{Z}_+ \cup \{0\}$.

Similarly using mathematical induction we can show that $f(n) = 1-n$, for all $n \in \mathbb{Z}_- = \{-1,-2,\dots\}$

So $f(n) = 1-n$ for all $n \in \mathbb{Z}$.

- $f(0)=1$, $f(-1) = +2$

Assume $f(k) = 1-k$ for all $0 \geq k > n$. Then



$\cancel{f(n) = f(n-2) - 2}$

$\cancel{f(n-2) =}$

$$f(n) = f(n+2) + 2$$
$$= (1-(n+2)) + 2$$
$$= 1-n.$$

(3) $f_1, f_2, \ldots, f_n : \mathbb{R}^n \longrightarrow \mathbb{R}$

with cont. second-order partial derivatives everywhere on $\mathbb{R}^n$.
and there are constants C_{ij} s.t

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = C_{ij} \qquad i, j = 1, 2, \ldots n.$$

Sol^n: Then $C_{ji} = -C_{ij}$ for all i, j $\quad (C^T = -C)$

Need to show that there are functions
$g : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $h_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ linear for all $i = 1, \ldots n$
such that

$$f_i + \frac{\partial g}{\partial x_i} = h_i \qquad i = 1, 2, \ldots n \quad \text{———} (*)$$

$$\Rightarrow \frac{\partial g}{\partial x_i} = h_i - f_i$$

$$\Rightarrow \frac{\partial^2 g}{\partial x_j \partial x_i} = \frac{\partial (h_i - f_i)}{\partial x_j}.$$

If g is C^2 then $\dfrac{\partial (h_i - f_i)}{\partial x_j} = \dfrac{\partial (h_j - f_j)}{\partial x_i}$

$$\Rightarrow \frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = C_{ij}$$

Let $h_i(x) = \frac{1}{2} \sum_{k=1}^{n} C_{ik} x_k$ then $\dfrac{\partial h_i}{\partial x_j} = \frac{1}{2} C_{ij}$,

and $\dfrac{\partial h_i}{\partial x_j} - \dfrac{\partial h_j}{\partial x_i} = \frac{1}{2} C_{ij} - \frac{1}{2} C_{ji} = C_{ij}$.

So $h_i(x) = \frac{1}{2} \sum_{k=1}^{n} C_{ik} x_k$, linear $i = 1, 2, \ldots n$ and g
with $\dfrac{\partial g}{\partial x_i} = h_i - f_i$ satisfy $(*)$.

4 $\vec{G}(x,y) = \left(\frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}, 0 \right)$.

Prove or disprove that there is a vector-valued function $\vec{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$ with the following properties:

(i) M, N, P have cont. partial derivatives for all $(x,y,z) \neq (0,0,0)$

(ii) $\text{Curl}\,\vec{F} = \vec{0}$ for all $(x,y,z) \neq (0,0,0)$

(iii) $\vec{F}(x,y,0) = \vec{G}(x,y)$

Sol^n: (iii) $\Rightarrow$ $\vec{F} = \vec{G}$ on xy-plane.

$M(x,y,0) = \frac{-y}{x^2+4y^2}$, $N(x,y,0) = \frac{x}{x^2+4y^2}$, $P(x,y,0)=0$

Stokes' Th^m

$\mathcal{S}$ - surface

s - arc length.

$$\iint_{\mathcal{S}} \text{Curl}\,\vec{F} \cdot d\vec{\mathcal{S}} = \oint_{\partial \mathcal{S}} \vec{F} \cdot d\vec{s}$$

(ii) $\|$ 0

$\|$ (iii)

$\oint_{\partial \mathcal{S}} \vec{G} \cdot d\vec{s}$ if $\partial \mathcal{S}$ on xy-plane

Take $\partial \mathcal{S}$ as $x^2+4y^2 = 4$, $z=0$, and $\mathcal{S}$ as any surface not containing $(0,0,0)$. Then

example: $x^2+4y^2+z^2 = 4$, $z \geq 0$.

$$\oint_{\partial \mathcal{S}} \vec{G} \cdot d\vec{s} = \int_0^{2\pi} \left(-\frac{\sin\theta}{4}, \frac{2\cos\theta}{4}, 0 \right) \cdot (-2\sin\theta, \cos\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi. \text{ Contradiction.}$$

∂S: $(2\cos\theta, \sin\theta, 0)$

$0 \leq \theta \leq 2\pi$

Hence, there is no such $\vec{F}$.

5 $\quad x, y \in (0,1)$

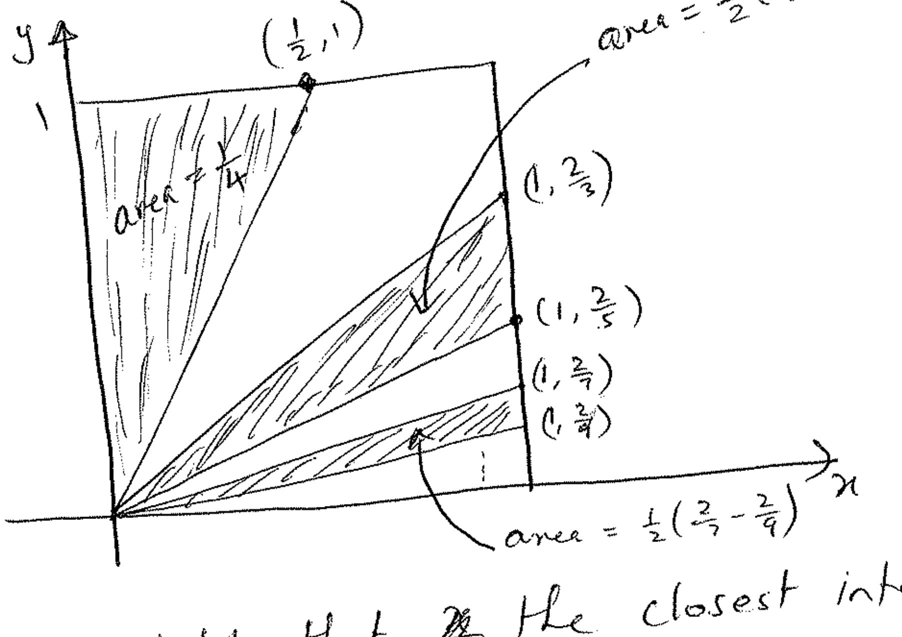
$0 < \frac{x}{y} < \frac{1}{2}, \quad \frac{3}{2} < \frac{x}{y} < \frac{5}{2}, \quad \frac{7}{2} < x < \frac{9}{2}, \ldots\ldots$

The probability that $\frac{x}{y}$ is exactly half an odd integer is 0, so ignore this possibility.

The closest integer to $\frac{x}{y}$ is even iff

$$0 < \frac{x}{y} < \frac{1}{2} \quad \text{or} \quad \frac{4n-1}{2} < \frac{x}{y} < \frac{4n+1}{2} \quad \text{for some}$$

integer $n \geq 1$.



$y = 2x$
$y = \frac{2}{3}x$
$y = \frac{2}{5}x$
$y = \frac{2}{7}x$
$y = \frac{2}{9}x.$

The probability that ~~x/y~~ the closest integer to $\frac{x}{y}$ is even $=$ areas of shaded regions

$$= \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{2}{4n-1} - \frac{2}{4n+1}\right) = \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{4n-1} - \frac{1}{4n+1}\right)$$

$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$$

$$= \frac{1}{4} + \left(1 - \frac{\pi}{4}\right) = \frac{5}{4} - \frac{\pi}{4} \qquad \text{from Leibniz's formula}$$

Leibniz's formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Proof:

$$\frac{\pi}{4} = \tan^{-1}(1)$$
$$= \int_0^1 \frac{1}{1+x^2}\,dx.$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

$$\Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n, \quad |-x^2| < 1 \Rightarrow x^2 < 1 \Rightarrow -1 < x < 1$$

$$= \underbrace{(1 - x^2)}_{>0} + \underbrace{(x^4 - x^6)}_{>0} + \underbrace{(x^8 - x^{10})}_{>0} + \cdots$$

$$f_1(x) = 1 - x^2, \quad f_2(x) = (1-x^2) + (x^4 - x^6), \quad f_3(x) = (1-x^2) + (x^4 - x^6) + (x^8 - x^{10})$$

$$\Rightarrow 0 \le f_1(x) \le f_2(x) \le f_3(x) \le \cdots \text{ and}$$
$$\lim_{n\to\infty} f_n(x) = \frac{1}{1+x^2}.$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx \overset{MCT}{=} \lim_{n\to\infty} \int_0^1 f_n(x)\,dx.$$

So $\int_0^1 \frac{1}{1+x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{10} + \cdots$

6 $\binom{n}{0}, \binom{n}{1}, \dots \binom{n}{n}$.

G_n the geometric mean of the above binomial coefficients

Show: $\lim_{n\to\infty} \sqrt[n]{G_n} = \sqrt{e}$.

Sol^n: $$\binom{n}{0}\cdot\binom{n}{1}\cdots\binom{n}{n} = \prod_{k=0}^{n}\binom{n}{k} = \prod_{k=0}^{n}\frac{n!}{k!(n-k)!}$$

$$= \frac{(n!)^{n+1}}{(1!\,2!\cdots n!)^2}$$

$$= n^{(n+1-2)}\cdot(n-1)^{n+1-4}\cdot(n-2)^{n+1-6}\cdots 2.1^{n+1-2(n-1)(n+1-2n)}$$

$$= \prod_{k=1}^{n}(n-k+1)^{n+1-2k}$$

$$\sum_{k=1}^{n}(n+1-2k) = 0 \implies (n+1)^{\sum_{k=1}^{n} n+1-2k} = 1.$$

So $$\binom{n}{0}\cdot\binom{n}{1}\cdots\binom{n}{n} = \prod_{k=1}^{n}\left(\frac{n-k+1}{n+1}\right)^{n+1-2k} = \prod_{k=1}^{n}\left(1-\frac{k}{n+1}\right)^{n+1-2k}$$

$$G_n = \sqrt[n+1]{\binom{n}{0}\cdots\binom{n}{n}} = \prod_{k=1}^{n}\left(1-\frac{k}{n+1}\right)^{1-\frac{2k}{n+1}}.$$

$$\implies \frac{1}{n}\ln G_n = \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{2k}{n+1}\right)\ln\left(1-\frac{k}{n+1}\right).$$

$$\lim_{n\to\infty} \frac{1}{n} \ln G_n = \int_0^1 (1-2x) \ln(1-x)\, dx$$ (by Riemann Sum)

By integration by parts, and

$$\int_0^1 (1-2x) \ln(1-x)\, dx = \lim_{a\to 1^-} \int_0^a (1-2x) \ln(1-x)\, dx$$

$$= \frac{1}{2}.$$

$$\Rightarrow \lim_{n\to\infty} \ln \sqrt[n]{G_n} = \frac{1}{2}.$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{G_n} = \sqrt{e}.$$