Putnam Practice Set #7 - Solutions

1. Find the number $R(n)$ of regions in which the plane can be divided by n straight lines.

   A formula that fits the first few cases is $R(n) = (n^2 + n + 2)/2$. We will prove by induction that it works for all $n \geq 1$. For $n = 1$ we have $R(1) = 2 = (1^2 + 1 + 2)/2$, which is correct. Next assume that the property is true for some positive integer n, i.e.: $R(n) = (n^2 + n + 2)/2$. We must prove that it is also true for $n + 1$, i.e., $R(n + 1) = (n^2 + 3n + 4)/2$.

   Solution: So lets look at what happens when we introduce the $(n + 1)st$ straight line. In general this line will intersect the other n lines in n different intersection points, and it will be divided into $n + 1$ segments by those intersection points. Each of those $n + 1$ segments divides a previous region into two regions, so the number of regions increases by $n + 1$. Hence: $R(n + 1) = R(n) + n + 1 = (n^2 + n + 2)/2 + n + 1 = (n^2 + 3n + 4)/2$. We are done by induction.

2. Define a set to be **selfish** if it has its own cardinality (number of elements) as an element of itself. Find, with proof the number of subsets of $\{1, 2, \ldots, n\}$ that are minimal selfish sets; that is, subsets that are selfish and do not properly contain any other selfish set.

   Solution: Let f_n denote the number of minimal selsh subsets of $\{1, 2, ..., n\}$. For $n = 1$ we have that the only selsh set of $\{1\}$ is $\{1\}$, and it is minimal. For $n = 2$ we have two selsh sets, namely $\{1\}$ and $\{1, 2\}$, but only $\{1\}$ is minimal. So $f_1 = 1$ and $f_2 = 1$. For n ¿ 2 the number of minimal selsh subsets of $\{1, 2, ..., n\}$ not containing n is equal to f_{n-1} . On the other hand, for any minimal selsh set containing n, by removing n from the set and subtracting 1 from each remaining element we obtain a minimal selsh subset of $\{1, 2, ..., n\}$. (Note since $\{1\}$ is selfish, 1 cannot be in the set.) Conversely, any minimal selsh subset of $\{1, 2, ..., n - 2\}$ gives raise to a minimal selsh subset of $\{1, 2, ..., n\}$ containing n by the inverse procedure. Hence the number of minimal selsh subsets of $\{1, 2, ..., n\}$ containing n is f_{n-2} . Thus $f_n = f_{n-1} + f_{n-2}$, which together with $f_1 = f_2 = 1$ implies that $f_n = F_n$ (nth Fibonacci number.)

3. Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cdots}}}}.$$

   Solution: We will prove that the answer is $(3 + 5)/2$.
   The value of the infinite continued fraction is the limit L of the sequence dened recursively $x_0 = 2207$, $x_{n+1} = 2207 - 1/x_n$, which exists because the sequence is decreasing (induction). Taking limits in both sides we get that $L = 2007 - 1/L$. Since $x_n > 1$ for all n (also proved by induction), we have that $L1$. If we call the answer r we have $r^8 = L$, so $r^8 + 1/r^8 = 2207$. Then

$$(r^4 + 1/r4)^2 = r^8 + 2 + 1/r^8 = 2 + 2207 = 2209,$$

   hence

$$r^4 + 1/r^4 = \sqrt{2209} = 47.$$

   Analogously,

$$(r^2 + 1/r^2)^2 = r^4 + 2 + 1/r^4 = 2 + 47 = 49,$$

   so

$$r^2 + 1/r^2 = \sqrt{49} = 7.$$

And

$$(r + 1/r)^2 = r^2 + 2 + 1/r^2 = 2 + 7 = 9,$$

so

$$r + 1/r = \sqrt{9} = 3.$$

From here we get $r^2 - 3r + 1 = 0$, hence $r = (3 \pm 5)/2$, but $r = L^{1/8} \geq 1$, so $r = (3 + 5)/2$.

4. Prove that if we paint every point of the plane in one of three colors, there will be two points one inch apart with the same color. Is this result necessarily true if we replace three by nine?

   Solution: We can prove the rst part by way of contradiction. Assume that we have colored the points of the plane with three colors such that any two points distance 1 at have different colors. Consider any two points A and B at distance $\sqrt{3}$. The circles of radius 1 and center A and B meet at two points P and Q, forming equilateral triangles APQ and BPQ. Since the vertices of each triangle must have different colors that forces A and B to have the same color. So any two points at distance $\sqrt{3}$ have the same color. Next consider a triangle DCE with $CD = CE = \sqrt{3}$ and $DE = 1$. The points D and E must have the same color as C, but since they are at distance 1 they should have different colors, so we get a contradiction.

   For the second part, if we replace three by nine then we can color the plane with nine different colors so that any two points at distance 1 have different colors: we can arrange them periodically in a grid of squares of size $2/3 \times 2/3$ as shown. If two points P and Q have the same color then either they belong to the same square and $PQ < (2/3)\sqrt{2} < 1$, or they belong to different squares and $PQ \geq 4/3 > 1$.

5. Call a set of positive integers **conspiratorial** if no three of them are pairwise relatively prime. What is the largest number of elements in any conspiratorial subset of integers 1 through 16?

   Solution: A conspiratorial subset of $S = \{1, 2, ..., 16\}$ has at most two elements from $T = \{1, 2, 3, 5, 7, 11, 13\}$, so it has at most $2+16-7 = 11$ numbers. On the other hand all elements of $S-T = \{4, 6, 8, 9, 10, 12, 14, 15, 16\}$ are multiple of either 2 or 3, so adding 2 and 3 we obtain the following 11-element conspiratorial subset: $\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}$.

   Hence the answer is 11.

6. Which is larger, 2010^{2010} or 2011^{2009}?

   Solutions: 2010^{2010} is larger.

   Let $f(x) = (2010 + x)^{2010-x}$. Then setting $y = f(x)$ we get

   $$\ln y = (2010 - x)\ln(2010 + x).$$

   So

   $$\frac{1}{y}y' = -\ln(2010 + x) + \frac{2010 - x}{2010 + x}.$$

   Hence

   $$f'(x) = (2010 + x)^{2010-x}\left(\frac{2010 - x}{2010 + x} - \ln(2010 + x)\right) < 0$$

   for $x \in [0, 1]$. (Since the first term is positive and the second term is negative. To see the second term is negative, note that

   $$\frac{2010 - x}{2010 + x} \leq 1 = \ln e \leq \ln(2010 + x)$$

   for all x in $[0, 1]$.) Hence f is decreasing on this whole interval. Therefore

   $$2010^{2010} = f(0) > f(1) = 2011^{2009}.$$