

Putnam Practice Set #5

1. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?
2. Right triangle ABC has the right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so the $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F . Evaluate $\lim_{\theta \rightarrow 0} |EF|$. [Here $|XY|$ denotes the length of the line segment XY .]
3. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?
4. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC , and F the foot of the altitude from A . What is the length of BC ?
5. Let A, B, C denote distinct points with integer coordinates in \mathbf{R}^2 . Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of the segment XY and $[ABC]$ is the area of the triangle ABC .

6. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x -axis and let B be the area of the region lying to the right of the y -axis and to the left of s . Prove that $A + B$ depends only on the arc length, and not on the position, of s .