Putnam Practice Set #6

1. Evaluate
$$\int_0^a \int_0^b e^{\max\{b^2x^2,\ a^2y^2\}}\ dy\ dx$$
where a and b are positive integers.

2. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:
   (a) $f(f(n)) = n$, for all integers n;
   (b) $f(f(n+2)+2) = n$, for all integers n;
   (c) $f(0) = 1$.

3. Suppose $f_1(x), f_2(x), ..., f_n(x)$ are functions of n variables $x = (x_1, ..., x_n)$ with continuous second-order partial derivatives every where on $\mathbf{R}^n$. Suppose further that there are constants c_{ij} such that
$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$
for all i and j, $1 \le i \le n$, $1 \le j \le n$. Prove that there is a function $g(x)$ on $\mathbf{R}^n$ such that $f_i + \frac{\partial g}{\partial x_i}$ is linear for i, $1 \le i \le n$. (A linear function is one of the form $a_0 + a_1x_1 + a_2x_2 + ... + a_nx_n$.)

4. Let
$$\overrightarrow{G}(x,y) = \left(\frac{-y}{x^2+4y^2},\ \frac{x}{x^2+4y^2},\ 0\right)$$
Prove or disprove that there is a vector-valued function
$$\overrightarrow{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$$
with the following properties:
   (a) M, N, P have continuous partial derivatives for all $(x,y,z) \neq (0,0,0)$;
   (b) $Curl\,\overrightarrow{F} = \overrightarrow{0}$ for all $(x,y,z) \neq (0,0,0)$;
   (c) $\overrightarrow{F}(x,y,0) = \overrightarrow{G}(x,z)$.

5. Two real numbers x and y are chosen at random in the interval $(0,1)$ with respect to the uniform distribution. What is the probability that the closet integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

6. Denote by G_n the geometric mean of the binomial coefficients
$$\binom{n}{0}, \binom{n}{1}, ..., \binom{n}{n}.$$
Prove that
$$\lim_{n\to\infty} \sqrt[n]{G_n} = \sqrt{e}.$$