Rhodes University

Department Of Mathematics

Generalisations of Filters and Uniform Spaces

Murugiah Muraleetharan

A thesis submitted in fulfilment of the requirements for the Degree of

Master of Science In Mathematics

Abstract

The notion of a filter $\mathbb{F} \in 2^{2^X}$ has been extended to that of a : prefilter $\mathcal{F} \in 2^{I^X}$, generalised filter $f \in I^{2^X}$ and fuzzy filter $\varphi \in I^{I^X}$. A uniformity is a filter with some other conditions and the notion of a uniformity $\mathbb{D} \in 2^{2^{X \times X}}$ has been extended to that of a : fuzzy uniformity $\mathcal{D} \in 2^{I^{X \times X}}$, generalised uniformity $d \in I^{2^{X \times X}}$ and super uniformity $\delta \in I^{I^{X \times X}}$. We establish categorical embeddings from the category of uniform spaces into the categories of fuzzy uniform spaces, generalised uniform spaces and super uniform spaces and also categorical embeddings into the category of super uniform spaces from the categories of fuzzy uniform spaces and generalised uniform spaces.

KEYWORDS: Prefiter, Generalised filter, Fuzzy filter, Fuzzy uniform space, Generalised uniform space, Super uniform space, Embedding functor and Functor isomorphim.

A. M. S. SUBJECT CLASSIFICATION: 03E72, 04A20, 18A22, 54A40, 54E15.

Contents

ACKNOWLEDGEMENTS

In presenting this thesis, the result of 18 months of study within the mathematics department of Rhodes University, I wish to express my warm thanks to those who have given me assistance and encouragement during my time here.

First of all, I am indebted to the Head of Department Professor Wesley Kotze. I am grateful to him for his guidance and for the financial assistance he has provided through the FRD without which I could not have completed the course.

I also wish to place on record the very helpful support of my supervisor Dr. Mike Burton. Such progress as I have made in the present enquiry owes much to the useful and regular discussions I have had with him.

I am also indebted to my colleague Mr Alan Roy, a graduate of the University of Aberdeen in Scotland whose friendship has meant much to me. The peaceful orderly environment of Rhodes University coupled with the friendly atmosphere of the department I have been privileged to enjoy.

Above all, I would like to thank my family and in particular my father who was an inspiring mathematics teacher. From a young age his help and encouragement have proved invaluable in the development of my career as a mathematician.

PREFACE

Since the notion of a fuzzy set was introduced by Zadeh [84], there have been attempts to extend useful mathematical notions to this wider setting, replacing sets by *fuzzy sets*. The study of uniform space notion is facilitated by the notion of a *filter*. In [52] Lowen introduced the notion of a *fuzzy uniform space*. This was accomplished with the aid of a filters on I^X which Lowen [48] called *prefilters*. The notions of prefilter and fuzzy uniform space are extentions of filter and uniform space. We prove that the category of uniform spaces is embedded into the category of fuzzy uniform spaces.

The notion of a fuzzy filter is introduced in [22] and then a new generalised form of uniformity, called *super uniformity*, is defined and studied. It is based on the concept of fuzzy filter. In [15] the notion of a generalised uniform space is introduced and studied using prefilters, which are not the most natural analogue of the filter notion in this situation. With the collaboration of Burton and Gutiérrez we introduced and studied the notion of a *generalised filter* in [16]. In Chapter 8, we rewrite the basic theory of generalised uniform spaces with the aid of generalised filters.

We can see that the notion of a generalised uniform space and the notion of a super uniform space are extentions of uniform space and the category of uniform spaces is embedded into the categories of generalised uniform spaces and super uniform spaces. We prove that the category of generalised uniform spaces and the category of fuzzy uniform spaces are isomorphic. We also show that the categories of fuzzy uniform spaces and generalised unifrom spaces are embedded into the category of super uniform spaces. We first establish consistent notation. We have also slightly changed a few definitions and strengthened some theorems. This lead us to establish a nice categorical connection between the categories of uniform spaces, fuzzy uniform spaces, generalised uniform spaces and super uniform spaces.

Chapters 1 and 2 introduce basic results in fuzzy sets and fuzzy topology. Chapter 1 gives the introduction to fuzzy sets and deals with crisp subsets associated with a fuzzy set and fuzzy sets induced by maps. The second chapter is concerned with fuzzy topology: the fuzzy closure operator is a very useful tool to define a topology. We also see in this chapter continuous functions between fuzzy uniform spaces. In chapters 3 and 4 we record the standard results on filters and uniform spaces, since they give and an idea for consequent results in later chapters.

Chapter 5 explores the fundamental ideas of prefiters. We get prefilters from filters and filters from prefilters. We see the images and preimages of filters under a map and study convergence in fuzzy topological space. In chapter 6 we defined fuzzy topology as a prefilter plus other conditions. We deal some basic results in fuzzy neighbourhood spaces which are essential to find a fuzzy closure operator and then obtain a fuzzy topology from it. We find fuzzy topologies from fuzzy uniform space directly and from fuzzy neighbourhood spaces. Next we deal with convergence in uniform topology and uniformly continuous functions between fuzzy uniform spaces. In the last section we deal with α-level uniformities which turned out to be very useful for subsequent results.

Chapter 7 is concened with generalised filters: definitions, generalised filters from prefilters and prefilters from generalised filters and prime generalised filters. In Chapter 8 we deal with generalised uniform spaces and uniformly continuous functions between generalised uniform spaces. We will see fuzzy filters in Chapter 10 with more emphasis given to a fuzzy filter with characteristic value 1. We also obtain fuzzy filters from generalised filters and generalised filters from fuzzy filters. In Chapter 10 we give definitions for α-uniformities and super uniformities. Then we see uniformly continuous functions between α-uniform spaces and super uniform spaces.

Chapter 11 is the centeral chapter which connects the categories of uniform spaces, fuzzy uniform spaces, generalised uniform spaces and super uniform spaces. Here we see that fuzzy uniform spaces, generalised uniform spaces and super uniform spaces are extentions of uniform space using categorical embeddings from the category of uniform spaces into these categories. We can also obtain categorical embeddings into the category of super uniform spaces from the categories of fuzzy uniform spaces and generalised uniform spaces. Also we show that the category of fuzzy uniform spaces and the category of generalised uniform spaces are isomorphic.

Chapter 1

Fuzzy Sets

1.1 Introduction

The role of set theory has been formulated in the development of modern mathematics. However usual observables in our daily lives and conversation as well as scientific experimentation constitute ill-defined sets. For example, the set of old people, the class of tall men, the class of large numbers, the set of low temperatures etc.

In order to try to develop a theory for such ill-defined sets L.A. Zadeh [84] defined the notion of a fuzzy set as follows.

1.1.1 Definition
Let X be a set. A *fuzzy set* on X is a map from X into $[0, 1]$. That is, if μ is a fuzzy set on X then $\mu \in I^X$. Where $I = [0, 1]$ and I^X denotes the collection of all maps from X into I.

Since the notion of a fuzzy set was introduced the basic theorems of set theory have been extended to produce a calculus of fuzzy sets.

Lattice-dependent subsets

The unit interval $I = [0, 1]$ can be replaced with a complete lattice L.

Let X be a set and L be a complete lattice. Then an L-*subset* of X is a map from X to L.

Order-structure of L-subsets

L^X is equipped with order-theoretic structure induced from L, and so is a complete lattice. For example,

$$\mu \leq \nu \Leftrightarrow \forall\, x \in X,\ \mu(x) \leq \nu(x);$$
$$(\bigvee_{j \in J} \mu_j)(x) \equiv \bigvee_{j \in J} \mu_j(x),\ x \in X;$$
$$(\bigwedge_{j \in J} \mu_j)(x) \equiv \bigwedge_{j \in J} \mu_j(x),\ x \in X.$$

L is *de Morgan* iff L admits an order-reversing involution

$$' : L \longrightarrow L\ (a'' = a, a \leq b \Rightarrow a' \geq b')$$

in which case L^X is also de Morgan.

Remark
I is a complete de Morgan lattice, and so is I^X.

If X is a set and $A \subseteq X$, then we define the *characteristic function* of A, denoted 1_A by

$$1_A \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Note that $1_A \in \{0,1\}^X = 2^X$ and there is a natural bijection between $\mathcal{P}(X)$ and 2^X. If A' denotes the complement of A, we see that:

$\forall\ x \in X,\ 1_\emptyset(x) = 0;$
$\forall\ x \in X,\ 1_X(x) = 1;$
$\forall\ x \in X,\ 1_{A'}(x) = 1 - 1_A(x);$
$\forall\ x \in X,\ 1_{A\cup B}(x) = 1_A(x) \vee 1_B(x);$
$\forall\ x \in X,\ 1_{A\cap B}(x) = 1_A(x) \wedge 1_B(x);$
$A \subseteq B \Rightarrow 1_A \leq 1_B.$

Algebra on I^X

I^X is equipped with order-theoretic structure induced from I.

The *empty fuzzy set* 0 is : $\forall\ x \in X,\ 0(x) = 0;$
The *whole fuzzy set* 1 is : $\forall\ x \in X,\ 1(x) = 1;$
$\mu = \nu \Leftrightarrow \forall\ x \in X,\ \mu(x) = \nu(x);$
$\mu \leq \nu \Leftrightarrow \forall\ x \in X,\ \mu(x) \leq \nu(x);$
$(\mu \vee \nu)(x) \equiv \mu(x) \vee \nu(x),\ x \in X;$
$(\mu \wedge \nu)(x) \equiv \mu(x) \wedge \nu(x),\ x \in X;$
$(\bigvee_{j\in J} \mu_j)(x) \equiv \bigvee_{j\in J} \mu_j(x),\ x \in X;$
$(\bigwedge_{j\in J} \mu_j)(x) \equiv \bigwedge_{j\in J} \mu_j(x),\ x \in X;$
$\mu'(x) \equiv \mu(x)' = 1 - \mu(x),\ x \in X.$

In the case where I is the closed unit interval, we remind ourselves of the following basic facts:

1.1.2 Theorem

1. If $f \in I^{X\times Y}$ then,

$$\sup_{(x,y)\in X\times Y} f(x,y) = \sup_{x\in X} \sup_{y\in Y} f(x,y);$$

$$\inf_{(x,y)\in X\times Y} f(x,y) = \inf_{x\in X} \inf_{y\in Y} f(x,y);$$

$$\sup_{x\in X} \inf_{y\in Y} f(x,y) \leq \inf_{y\in Y} \sup_{x\in X} f(x,y).$$

2. If $X, Y \subseteq I$ then,

$$\sup\ X \wedge \sup\ Y = \sup_{x\in X} \sup_{y\in Y}\ x \wedge y;$$

$$\inf\ X \vee \inf\ Y = \inf_{x\in X} \inf_{y\in Y}\ x \vee y.$$

3. If $\mu, \nu \in I^X$ then

$$\sup\ (\mu \wedge \nu) \leq \sup\ \mu \wedge \sup\ \nu.$$

4. If $\nu \in I^X$ and $A, B \subseteq X$ then

$$\sup_{x\in A}\ \nu(x) \wedge \sup_{y\in B}\ \nu(y) = \sup_{x\in A}\ \sup_{y\in B}\ (\nu(x) \wedge \nu(y)).$$

1.2 Crisp Subsets of X Associated With a Fuzzy Set

If $\mu \in I^X$ and $\alpha \in I$ we define,

$$\mu^\alpha \stackrel{\text{def}}{=} \{x \in X : \mu(x) > \alpha\};$$

$$\mu_\alpha \stackrel{\text{def}}{=} \{x \in X : \mu(x) \geq \alpha\}.$$

These are the so-called α-*level*(or cut), strong and weak respectively. If a theory is to be fuzzified (for example group theory), a very useful type of theorem to have avilable is one which relates a property (such as normality) to its fuzzy analogue. Very often the theorem takes the form

$$\mu \text{ is fuzzy} - P \Leftrightarrow \forall\ \alpha \in I,\ \mu^\alpha \text{ is } P.$$

Or some variation of this. These α-levels theorems are extremly useful.

1.2.1 Lemma

If $\mu, \nu \in I^X$ then,

$$\begin{aligned} \mu = \nu \quad &\Leftrightarrow \forall\ \alpha \in I,\ \mu^\alpha = \nu^\alpha \\ &\Leftrightarrow \forall\ \alpha \in (0,1),\ \mu^\alpha = \nu^\alpha. \end{aligned}$$

Proof.

($\Leftarrow$)

Let $x \in X$ and $\mu(x) = \alpha$. If $\nu(x) > \alpha$ then $x \in \nu^\alpha$ and so $x \in \mu^\alpha$. Therefore $\mu(x) > \alpha$. This is a contradiction. Hence $\nu(x) \leq \alpha$. If $\nu(x) < \alpha$ then $\exists \beta$ such that $\nu(x) < \beta < \alpha$. Therefore $x \in \mu^\beta$ and so $x \in \nu^\beta$ and hence $\nu(x) > \beta$. This is a contradiction. Hence $\nu(x) = \alpha$. Since x is arbitrary. Therefore $\forall\ x \in X,\ \nu(x) = \mu(x)$. That is $\mu = \nu$.

Assume $\forall\ \alpha \in (0,1),\ \mu^\alpha = \nu^\alpha$. Now we have to show $\forall\ \alpha \in [0,1],\ \mu^\alpha = \nu^\alpha$. That is $\mu^1 = \nu^1$ and $\mu^0 = \nu^0$. But clearly $\mu^1 = \emptyset = \nu^1$ Let $x \in \mu^0$ then $\mu(x) > 0$. So $\exists \alpha > 0$ such that $\mu(x) > \alpha > 0$. Therefore $x \in \mu^\alpha = \nu^\alpha \Rightarrow \nu(x) > \alpha > 0 \Rightarrow x \in \nu^0$ and hence $\mu^0 \subseteq \nu^0$. Similarly $\nu^0 \subseteq \mu^0$. Therefore $\mu^0 = \nu^o$.

□

1.2.2 Lemma

If $\mu \in I^X$ then,

$$\mu = \sup_{\alpha \in (0,1)} \alpha 1_{\mu^\alpha}.$$

Proof.

If $x \in X$ then

$$\alpha 1_{\mu^\alpha}(x) = \begin{cases} \alpha & \text{if } \mu(x) > \alpha \\ 0 & \text{if } \mu(x) \leq \alpha. \end{cases}$$

Let $x \in X$ and $\mu(x) = \beta$. Then

$$\alpha 1_{\mu^\alpha}(x) = \begin{cases} \alpha & \text{if } \beta > \alpha \\ 0 & \text{if } \beta \leq \alpha. \end{cases}$$

Therefore

$$\sup_{\alpha \in (0,1)} \alpha 1_{\mu^\alpha}(x) = \beta.$$

□

1.2.3 Lemma

If $\mu, \nu, \mu(j), \nu(j) \in I^X,\ j \in J$ then

1. $(\mu \wedge \nu)^\alpha = \mu^\alpha \cap \nu^\alpha$;
2. $(\mu \vee \nu)^\alpha = \mu^\alpha \cup \mu^\alpha$;
3. $(\mu \wedge \nu)_\alpha = \mu_\alpha \cap \nu_\alpha$;
4. $(\mu \vee \nu)_\alpha = \mu_\alpha \cup \nu_\alpha$;
5. $\bigcup\limits_{j\in J} \mu(j)_\alpha \subseteq (\bigvee\limits_{j\in J} \mu(j))_\alpha$;
6. $(\bigwedge\limits_{j\in J} \mu(j))_\alpha = \bigcap\limits_{j\in J} \mu(j)_\alpha$;
7. $\bigcup\limits_{j\in J} \mu(j)^\alpha = (\bigvee\limits_{j\in J} \mu(j))^\alpha$;
8. $(\bigwedge\limits_{j\in J} \mu(j))^\alpha \subseteq \bigcap\limits_{j\in J} \mu^\alpha$;
9. $\bigcap\limits_{k\in K} \mu_{\alpha_k} = \mu_\alpha$ where $\alpha = \sup\limits_{k\in K} \alpha_k$;
10. $\bigcup\limits_{k\in K} \mu^{\alpha_k} = \mu^\alpha$ where $\alpha = \inf\limits_{k\in K} \alpha_k$;
11. $(\mu')_\alpha = (\mu^{1-\alpha})'$;
12. $(\mu')^\alpha = (\mu_{1-\alpha})'$;
13. $\mu_\alpha = \bigcap\limits_{\beta<\alpha} \mu^\beta$;
14. $\mu^\alpha = \bigcup\limits_{\beta>\alpha} \mu_\beta$.

PROOF.

(1)

$$\begin{aligned} x \in (\mu \wedge \nu)^\alpha &\iff (\mu \wedge \nu)(x) > \alpha \\ &\iff \mu(x) > \alpha \text{ and } \nu(x) > \alpha \\ &\iff x \in \mu^\alpha \cap \nu^\alpha. \end{aligned}$$

(2)

$$\begin{aligned} x \in (\mu \vee \nu)^\alpha &\iff (\mu \vee \nu)(x) > \alpha \\ &\iff \mu(x) > \alpha \text{ or } \nu(x) > \alpha \\ &\iff x \in \mu^\alpha \cup \nu^\alpha. \end{aligned}$$

(3)

$$\begin{aligned} x \in (\mu \wedge \nu)_\alpha &\iff (\mu \wedge \nu)(x) \geq \alpha \\ &\iff \mu(x) \geq \alpha \text{ and } \nu(x) \geq \alpha \\ &\iff x \in \mu_\alpha \cap \nu_\alpha. \end{aligned}$$

(4)

$$\begin{aligned} x \in (\mu \vee \nu)_\alpha &\iff (\mu \vee \nu)(x) \geq \alpha \\ &\iff \mu(x) \geq \alpha \text{ or } \nu(x) \geq \alpha \\ &\iff x \in \mu_\alpha \cup \nu_\alpha. \end{aligned}$$

(5)

$$\begin{aligned} x \in \underset{j\in J}{\cup} \mu(j)_\alpha &\iff \exists j_0 \in J \text{ such that } x \in \mu(j_0)_\alpha \\ &\iff \mu(j_0)(x) \geq \alpha \\ &\Longrightarrow (\underset{j\in J}{\vee} \mu(j))(x) \geq \alpha \\ &\iff x \in (\underset{j\in J}{\vee} \mu(j))_\alpha. \end{aligned}$$

(6)

$$\begin{aligned} x \in (\bigwedge_{j\in J}\mu(j))_\alpha &\iff \forall\ j \in J,\ \mu(j)(x) \geq \alpha \\ &\iff \forall\ j \in J,\ x \in \mu(j)_\alpha \\ &\iff x \in \bigcap_{j\in J}\mu(j)_\alpha. \end{aligned}$$

(7)

$$\begin{aligned} x \in \bigcup_{j\in J}\mu(j)^\alpha &\iff \exists j_0 \in J \text{ such that } \mu(j_0)(x) > \alpha \\ &\iff (\bigvee_{j\in J}\mu(j))(x) > \alpha \\ &\iff x \in (\bigvee_{j\in J}\mu(j))^\alpha. \end{aligned}$$

(8)

$$\begin{aligned} x \in (\bigwedge_{j\in J}\mu(j))^\alpha &\iff \bigwedge_{j\in J}\mu(j)(x) > \alpha \\ &\implies \forall\ j \in J,\ \mu(j)(x) > \alpha \\ &\iff x \in \bigcap_{j\in J}\mu(j)^\alpha. \end{aligned}$$

(9)

$$\begin{aligned} x \in \bigcap_{k\in K}\mu_{\alpha_k} &\iff \forall\ k \in K,\ \mu(x) \geq \alpha_k \\ &\iff \mu(x) \geq \sup_{k\in K}\alpha_k \\ &\iff x \in \mu_{\sup\limits_{k\in K}\alpha_k}. \end{aligned}$$

(10)

$$\begin{aligned} x \in \bigcup_{k\in K}\mu^{\alpha_k} &\iff \exists k_0 \in K \text{ such that } \mu(x) > \alpha_{k_0} \\ &\iff \mu(x) > \inf_{k\in K}\alpha_k \\ &\iff x \in \mu^{\inf\limits_{k\in K}\alpha_k}. \end{aligned}$$

(11)

$$\begin{aligned} x \in (\mu')_\alpha &\iff \mu'(x) \geq \alpha \iff 1-\mu(x) \geq \alpha \\ &\iff \mu(x) \leq \alpha \iff \mu(x) \not> 1-\alpha \\ &\iff x \in (\mu^{1-\alpha})'. \end{aligned}$$

(12)

$$\begin{aligned} x \in (\mu')^\alpha &\iff \mu'(x) > \alpha \iff \mu(x) < 1-\alpha \\ &\iff \mu(x) \not\geq 1-\alpha \iff x \in (\mu_{1-\alpha})'. \end{aligned}$$

(13)

$$\begin{aligned} x \in \bigcap_{\beta<\alpha}\mu^\beta &\iff \forall\ \beta < \alpha,\ \mu(x) > \beta \\ &\iff \mu(x) \geq \sup_{\beta<\alpha}\beta = \alpha \\ &\iff x \in \mu_\alpha. \end{aligned}$$

(14)

$$\begin{aligned} x \in \bigcup_{\beta>\alpha}\mu_\beta &\iff \exists\beta_0 > \alpha \text{ such that } \mu(x) \geq \beta_0 \\ &\iff \mu(x) > \alpha \\ &\iff x \in \mu^\alpha. \end{aligned}$$

□

1.3 Fuzzy Sets Induced by Maps

For a function

$$f : X \longrightarrow Y$$

there corresponds a function

$$f^{\rightarrow} : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$$

where $f^{\rightarrow}(A) = \{f(x) : x \in A\}$ is called the *direct image* of $A \subseteq X$; and a function

$$f^{\leftarrow} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$$

where $f^{\leftarrow}(B) = \{x \in X : f(x) \in B\}$ is called the *preimage* of $B \subseteq Y$.
We define the analogues of these as follows.
If X and Y are sets, $f \in Y^X, \mu \in I^X$ and $\nu \in I^Y$ we define the *direct image* of μ, denoted by $f[\mu]$ and the *preimage* of ν denoted by $f^{-1}[\nu]$as follows:
For $y \in Y$,

$$f[\mu](y) \stackrel{\text{def}}{=} \sup_{f(x)=y} \mu(x)$$

with the convention that $\sup\ \emptyset = 0$ and

$$f^{-1}[\nu] \stackrel{\text{def}}{=} \nu \circ f.$$

It is straightforward to check these definitions reduce to usual ones in the case where $\mu = 1_A$ and $\nu = 1_B$ with $A \subseteq X$ and $B \subseteq Y$.

1.3.1 Theorem
Let X, Y, Z be sets and let $f \in Y^X, g \in Z^Y, \mu \in I^X, \nu \in I^Y$ and $\lambda \in I^Z$.
Let $(\mu_j : j \in J) \in (I^X)^J$ and $(\nu_j : j \in J) \in (I^Y)^J$. Then

1. $(g \circ f)[\mu] = g[f[\mu]]$;
2. $(g \circ f)^{-1}[\lambda] = f^{-1}[g^{-1}[\lambda]]$;
3. $f^{-1}[\bigvee_{j\in J} \nu_j] = \bigvee_{j\in J} f^{-1}[\nu_j]$;
4. $f^{-1}[\bigwedge_{j\in J} \nu_j] = \bigwedge_{j\in J} f^{-1}[\nu_j]$;
5. $f^{-1}[\nu'] = (f^{-1}[\nu])'$;
6. $\nu_1 \leq \nu_2 \Rightarrow f^{-1}[\nu_1] \leq f^{-1}[\nu_2]$;
7. $f[\bigvee_{j\in J} \mu_j] = \bigvee_{j\in J} f[\mu_j]$;
8. $f[\bigwedge_{j\in J} \mu_j] \leq \bigwedge_{j\in J} f[\mu_j]$;
9. $f[\mu]' \leq f[\mu']$, provided f is surjective;
10. $\mu_1 \leq \mu_2 \Rightarrow f[\mu_1] \leq f[\mu_2]$;
11. $f[f^{-1}[\nu]] \leq \nu$, with equality if f is surjective;
12. $\mu \leq f^{-1}[f[\mu]]$, with equality if f is injective;
13. $f[f^{-1}[\nu] \wedge \mu] \leq \nu \wedge f[\mu]$, with equality if f is injective.

PROOF.

(1)
$$\begin{aligned} g[f[\mu]](z) &= \sup_{g(y)=z} f[\mu](y) = \sup_{g(y)=z} \sup_{f(x)=y} \mu(x) \\ &= \sup_{g\circ f(x)=z} \mu(x) = (g\circ f)[\nu](z). \end{aligned}$$

(2)
$$\begin{aligned} (g\circ f)^{-1}[\lambda](x) &= \lambda((g\circ f)(x)) = \lambda(g(f(x))) = g^{-1}[\lambda](f(x)) \\ &= f^{-1}[g^{-1}[\lambda]](x). \end{aligned}$$

(3)
$$\begin{aligned} f^{-1}[\bigvee_{j\in J}\nu_j](x) &= (\bigvee_{j\in J}\nu_j)(f(x)) = \bigvee_{j\in J}\nu_j(f(x)) \\ &= \bigvee_{j\in J}(f^{-1}[\nu_j](x)) = (\bigvee_{j\in J} f^{-1}[\nu_j])(x). \end{aligned}$$

(4)
$$\begin{aligned} f^{-1}[\bigwedge_{j\in J}\nu_j](x) &= (\bigwedge_{j\in J}\nu_j)(f(x)) = \bigwedge_{j\in J}\nu_j(f(x)) \\ &= \bigwedge_{j\in J}(f^{-1}[\nu_j](x)) = (\bigwedge_{j\in J} f^{-1}[\nu_j])(x). \end{aligned}$$

(5)
$$\begin{aligned} f^{-1}[\nu'](x) &= \nu'(f(x)) = \nu(f(x))' = (f^{-1}[\nu](x))' \\ &= (f^{-1}[\nu])'(x). \end{aligned}$$

(6)
$$f^{-1}[\nu_1](x) = \nu_1(f(x)) \le \nu_2(f(x)) = f^{-1}[\nu_2](x).$$

(7)
$$\begin{aligned} f[\bigvee_{j\in J}\mu_j](y) &= \sup_{f(x)=y}(\bigvee_{j\in J}\mu_j)(x) = \bigvee_{f(x)=y}\ \bigvee_{j\in J}\mu_j(x) \\ &= \bigvee_{j\in J}\ \bigvee_{f(x)=y}\mu_j(x) = \bigvee_{j\in J}(f[\mu_j](y)) \\ &= (\bigvee_{j\in J} f[\mu_j])(y). \end{aligned}$$

(8)
$$\begin{aligned} f[\bigwedge_{j\in J}\mu_j](y) &= \sup_{f(x)=y}(\bigwedge_{jinJ}\mu_j)(x) = \bigvee_{f(x)=y}\ \bigwedge_{j\in J}\mu_j(x) \\ &\le \bigwedge_{j\in J}\ \bigvee_{f(x)=y}\mu_j(x) \\ &= \bigwedge_{j\in J}(f[\mu_j](y)) \\ &= (\bigwedge_{j\in J} f[\mu_j])(y). \end{aligned}$$

(9)
$$\begin{aligned} f[\mu]'(y) &= f[\mu](y)' = 1 - f[\mu](y) \\ &= 1 - \sup_{f(x)=y}\mu(x) \\ &\le \sup_{f(x)=y} 1-\mu(x) \text{ [since f is surjective]} \\ &= \sup_{f(x)=y}\mu'(x) = f[\mu'](y). \end{aligned}$$

(10)
$$f[\mu_1](y) = \sup_{f(x)=y}\mu_1(x) \le \sup_{f(x)=y}\mu_2(x) = f[\mu_2](y).$$

(11)
$$f[f^{-1}[\nu]](y) = \sup_{f(x)=y} f^{-1}[\nu](x) = \sup_{f(x)=y}\nu(f(x)) \le \nu(y).$$

If f is surjective then $f^{\leftarrow}(\{y\}) \neq \emptyset$ and so

$$f[f^{-1}[\nu]](y) = \nu(y).$$

(12)
$$f^{-1}[f[\mu]](x) = f[\mu](f(x)) = \sup_{f(z)=f(x)} \mu(z) \geq \mu(x).$$

If f is injective then $f(x) = f(z) \Rightarrow z = x$ and therefore
$$f^{-1}[f[\mu]](x) = \mu(x).$$

(13)
$$\begin{aligned} f[f^{-1}[\nu] \wedge \mu](y) &= \sup_{f(x)=y} (f^{-1}[\nu] \wedge \mu)(x) \\ &= \bigvee_{f(x)=y} (\nu(f(x)) \wedge \mu(x)) \\ &\leq \nu(y) \wedge (\bigvee_{f(x)=y} \mu(x)) \\ &= \nu(y) \wedge f[\mu](y) \\ &= (\nu \wedge f[\mu])(y). \end{aligned}$$

If f is injective then $f^{\leftarrow}(\{y\})$ is a singlton set and therefore
$$f[f^{-1}[\nu] \wedge \mu] = \nu \wedge f[\mu].$$

□

More information regarding fuzzy set can be found in [23, **?**, 25, 26, 32, 46, 49, 59, 60, 71, 85].

Chapter 2

Fuzzy Topology

2.1 Definitions and Fundamental Properties

In [19], Chang introduced the notion of a fuzzy topology as follows.

2.1.1 Definition
A *fuzzy topology* on X is a subset $\mathcal{T}$ of I^X satisfying

1. $0, 1 \in \mathcal{T}$;
2. $\mu, \nu \in \mathcal{T} \Rightarrow \mu \wedge \nu \in \mathcal{T}$;
3. $\forall j \in J,\ \mu_j \in \mathcal{T} \Rightarrow \bigvee_{j \in J} \mu_j \in \mathcal{T}$.

$(X, \mathcal{T})$ is called a *fuzzy topological space* (f.t.s) and the members of $\mathcal{T}$ the *fuzzy open sets* of X.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on X then if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we say $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$ or $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$.

In [44] Lowen defines a subset $\mathcal{T} \subseteq I^X$ to be a fuzzy topology on X if (1),(2),(3) hold as well as:
(4) $\forall\ \alpha \in I,\ \alpha 1_X \in \mathcal{T}$.

2.1.2 Examples

1. The discrete fuzzy topology on X: $\mathcal{T} = I^X$.
2. The indiscrete fuzzy topology on X: $\mathcal{T} = \{0, 1\}$.
3. Any topology τ on X generates a fuzzy topology on X — identify with the open sets their characteristic functions.
4. Given a topological space (X, τ), the family $\mathcal{T}_n$ of all lower semi continuous functions on X into $[0, 1]$ is a fuzzy topology on X which contains τ. We call $\mathcal{T}_n$ the *natural fuzzy topology* on a topological space (X, τ).

2.1.3 Definition
A fuzzy set μ in a f.t.s $(X, \mathcal{T})$ is *$\mathcal{T}$-closed (fuzzy closed)* iff $\mu' = 1 - \mu \in \mathcal{T}$.

2.1.4 Definition
The *fuzzy interior* μ° of a fuzzy set μ is the join of all members of $\mathcal{T}$ contained in μ. That is,

$$\mu^\circ = \vee\{\nu \in I^X : \nu \in \mathcal{T},\ \nu \leq \mu\}.$$

This is the largest fuzzy open set contained in μ and

$$\mu \text{ is open iff } \mu = \mu^\circ.$$

2.1.5 Definition

The *fuzzy closure* $\bar{\mu}$ of a fuzzy set μ is the meet of all $\mathcal{T}$-closed sets which contain μ. That is,

$$\bar{\mu} = \wedge\{\nu \in I^X : \nu' \in \mathcal{T},\ \mu \le \nu\}.$$

Thus $\bar{\mu}$ is the smallest $\mathcal{T}$-closed set which contains μ and

$$\mu \text{ is closed iff } \mu = \bar{\mu}.$$

2.1.6 Proposition

(1) $\bar{\mu}' = (\mu')^\circ$ and $(\mu^\circ)' = \overline{(\mu')}$;
(2) $\bar{0} = 0,\ \mu \le \bar{\mu},\ \bar{\bar{\mu}} = \mu$ and $\overline{\mu \vee \nu} = \bar{\mu} \vee \bar{\nu}$.

PROOF

(1) a) Let $\mathcal{C}$ be the set of all $\mathcal{T}$- closed sets which contain μ. Then $\forall\ \nu \in \mathcal{C},\ \mu \le \nu$ and $\nu' \in \mathcal{T}$.
It is easy to show that $\{\nu' : \nu \in \mathcal{C}\}$ is the set of all $\mathcal{T}$-open sets contanied in μ'. Therefore

$$\bar{\mu} = \bigwedge_{\nu \in \mathcal{C}} \nu \text{ and } (\mu')^\circ = \bigvee_{\nu \in \mathcal{C}} \nu'.$$

So

$$(\bar{\mu})' = (\bigwedge_{\nu \in \mathcal{C}} \nu)' = \bigvee_{\nu \in \mathcal{C}} \nu'.$$

Hence $\bar{\mu}' = (\mu')^\circ$.

b) Let $\mathcal{C}$ be the set of all $\mathcal{T}$-open sets which are contained in μ. Then $\forall\, \nu \in \mathcal{C},\ \nu \in \mathcal{T}$ and $\nu \le \mu$. It is easy to show that $\{\nu' : \nu \in \mathcal{C}\}$ is the set of all $\mathcal{T}$-closed sets which contain μ'. Therefore

$$\mu^\circ = \bigvee_{\nu \in \mathcal{C}} \nu \text{ and } \bar{\mu'} = \bigwedge_{\nu \in \mathcal{C}} \nu'$$

and so

$$(\mu^\circ)' = (\bigvee_{\nu \in \mathcal{C}} \nu)' = \bigwedge_{\nu \in \mathcal{C}} \nu'.$$

Hence, $(\mu^\circ)' = \bar{\mu'}$.

(2) We have $0' = 1 \in \mathcal{T}$. So $\bar{0} = 0$.

We have $\bar{\mu} = \wedge\{\nu \in I^X : \nu' \in \mathcal{T},\ \nu \ge \mu\} \ge \mu$ and so $\bar{\mu}' = \vee\{\nu' \in I^X : \nu' \in \mathcal{T}\ \nu \ge \mu\} \in \mathcal{T}$. Therefore $\bar{\mu}$ is closed $\Rightarrow \bar{\mu} = \bar{\bar{\mu}}$.

Let $\mathcal{A} = \{\xi \in I^X : \xi' \in \mathcal{T},\ \mu \le \xi\}$ and $\mathcal{B} = \{\eta \in I^X : \eta' \in \mathcal{T},\ \nu \le \eta\}$. Then

$$\bar{\mu} \vee \bar{\nu} = (\bigwedge_{\xi \in \mathcal{A}} \xi) \vee (\bigwedge_{\eta \in \mathcal{B}} \eta) = \bigwedge_{\substack{\xi \in \mathcal{A} \\ \eta \in \mathcal{B}}} (\xi \vee \eta).$$

But $\mu \vee \nu \le \xi \vee \eta$ and $(\xi \vee \eta)' \in \mathcal{T}$.

If $\lambda \in I^X$ is such that $\mu \vee \nu \le \lambda$ and $\lambda' \in \mathcal{T}$ then $\mu \le \lambda,\ \lambda' \in \mathcal{T}$ and $\nu \le \lambda,\ \lambda' \in \mathcal{T}$. Therefore there exists $\xi \in \mathcal{A},\ \eta \in \mathcal{B}$ such that $\lambda = \xi$ and $\lambda = \eta$. Thus $\lambda = \xi \vee \eta$.

Therefore

$$\overline{\mu \vee \nu} = \bigwedge_{\substack{\xi \in \mathcal{A} \\ \eta \in \mathcal{B}}} (\xi \vee \eta).$$

Hence

$$\overline{\mu \vee \nu} = \bar{\mu} \vee \bar{\nu}.$$

$\square$

2.2 Fuzzy Closure Operator

In [53] the notion of a fuzzy closure operator is introduced. We adopt a slightly different definition of a fuzzy closure operator.

2.2.1 Definition
A *fuzzy closure operator* on X is a map $^{-}\colon I^X \longrightarrow I^X$ which fulfills the following properties

1. $0 = \bar{0}$;
2. $\forall\ \mu \in I^X,\ \mu \leq \bar{\mu}$;
3. $\forall\ \mu, \nu \in I^X,\ \overline{\mu \vee \nu} = \bar{\mu} \vee \bar{\nu}$;
4. $\forall\ \mu \in I^X,\ \bar{\bar{\mu}} = \bar{\mu}$.

This definition of fuzzy closure operator differs from the definition in [53], where it is required that
$\forall \alpha \in I,\ \alpha 1_X = \overline{\alpha 1_X}$.

It is shown that a fuzzy topology can be defined using a closure operator.

2.2.2 Theorem
If the map $^{-}\colon I^X \longrightarrow I^X$ is fuzzy closure operator then $\mathcal{T} = \{\mu' : \mu = \bar{\mu}\}$ is a fuzzy topology on X whose closure operation is just the operation $\mu \longrightarrow \bar{\mu}$.

Proof

We have $0 = \bar{0} \Rightarrow 0' = 1 \in \mathcal{T}$.
Since $1 \leq \bar{1}$. So $1 = \bar{1} \Rightarrow 1' = 0 \in \mathcal{T}$.
Let $\mu,\ \nu \in \mathcal{T}$. Then $\mu' = \bar{\mu'},\ \nu' = \bar{\nu'}$ and

$$\overline{(\mu \wedge \nu)'} = \overline{\mu' \vee \nu'} = \bar{\mu'} \vee \bar{\nu'} = \mu' \vee \nu' = (\mu \wedge \nu)'.$$

Therefore

$$((\mu \wedge \nu)')' = \mu \wedge \nu \in \mathcal{T}.$$

If $\mu \leq \nu$ then $\nu = \mu \vee \nu$ and hence

$$\bar{\nu} = \overline{\mu \vee \nu} = \bar{\mu} \vee \bar{\nu}$$

So $\bar{\mu} \leq \bar{\nu}$.
Let $\nu_j \in \mathcal{T}$ for each $j \in J$. Then

$$\bigwedge_{j \in J} \nu'_j \leq \nu'_j \Rightarrow \overline{\bigwedge_{j \in J} \nu'_j} \leq \bar{\nu'_j}.$$

Therefore

$$\overline{\bigwedge_{j \in J} \nu'_j} \leq \bigwedge_{j \in J} \bar{\nu'_j} = \bigwedge_{j \in J} \nu'_j$$

and hence

$$\overline{\bigwedge_{j \in J} \nu'_j} = \bigwedge_{j \in J} \nu'_j.$$

Hence

$$(\bigwedge_{j \in J} \nu'_j)' = \bigvee_{j \in J} \nu_j \in \mathcal{T}.$$

It remains to show that the resulting closure operation is just the operation $\mu \longrightarrow \bar{\mu}$.
Let $\mathcal{F} = \{\mu' : \mu \in \mathcal{T}\} = \{\mu : \mu = \bar{\mu}\}$.
We have to show that for each $\mu \in I^X,\ \bar{\mu}$ is the smallest element of $\mathcal{F}$ containing μ.
Now $\bar{\bar{\mu}} = \bar{\mu} \Rightarrow \bar{\mu} \in \mathcal{F}$ and we have $\mu \leq \bar{\mu}$.
If $\nu \in \mathcal{F}$ and $\mu \leq \nu$ then

$$\bar{\mu} \leq \bar{\nu} = \nu.$$

Thus $\bar{\mu}$ is the smallest element of $\mathcal{F}$ containing μ. □

2.3 Continuous Functions

In [19], the notion of a continuous function between fuzzy topological spaces is introduced and studied.

2.3.1 Definition
Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two fuzzy topological spaces. A function $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ is *continuous* iff $\forall\ \nu \in \mathcal{T}_2,\ f^{-1}[\nu] \in \mathcal{T}_1$.

2.3.2 Proposition
If $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ and $g : (Y, \mathcal{T}_2) \longrightarrow (Z, \mathcal{T}_3)$ are continuous functions then $g \circ f : (X, \mathcal{T}_1) \longrightarrow (Z, \mathcal{T}_3)$ is continuous.

Proof.

Let $\lambda \in \mathcal{T}_3$. Then $(g \circ f)^{-1}[\lambda] = f^{-1}[g^{-1}[\lambda]]$ and, since g is continuous, $g^{-1}[\lambda] \in \mathcal{T}_2$. Since f is continuous, $f^{-1}[g^{-1}[\lambda]] \in \mathcal{T}_1$. Therefore $(g \circ f)$ is continuous. □

2.3.3 Theorem
Let $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ be a function. Then the the following are equivalent

1. f is continuous,
2. For each $\mathcal{T}_2$-closed $\nu,\ f^{-1}[\nu]$ is $\mathcal{T}_1$-closed,
3. For each $\nu \in I^Y,\ \overline{f^{-1}[\nu]} \leq f^{-1}[\bar{\nu}]$,
4. For each $\mu \in I^X,\ f[\bar{\mu}] \leq \overline{f[\mu]}$.

Proof.

(1) ⇒ (2)
Let $\nu' \in \mathcal{T}_2$. Then $f^{-1}[\nu'] \in \mathcal{T}_1$ and $(f^{-1}[\nu])' = f^{-1}[\nu'] \in \mathcal{T}_1$.
Therefore $f^{-1}[\nu]$ is $\mathcal{T}_1 -$ closed .

(2) ⇒ (1)
Let $\nu \in \mathcal{T}_2$. Then ν' is $\mathcal{T}_2 -$ closed and so $f^{-1}[\nu'] = (f^{-1}[\nu])'$ is $\mathcal{T}_1 -$ closed . Thus $f^{-1}[\nu] \in \mathcal{T}_1$ and hence f is continuous.

(2) ⇒ (4)
For $\mu \in I^X$,

$$\overline{f[\mu]} = \wedge\{\nu \in I^Y : \nu' \in \mathcal{T}_2,\ \nu \geq f\ [\mu]\}.$$

Therefore

$$f^{-1}[\overline{f[\mu]}] = \wedge\{f^{-1}[\nu] : \nu \in I^Y,\ \nu' \in \mathcal{T}_2,\ \nu \geq f[\mu]\} \text{ and}$$
$$\bar{\mu} = \wedge\{\lambda \in I^X : \lambda' \in \mathcal{T}_1,\ \lambda \geq \mu\}.$$

But

$$\nu' \in \mathcal{T}_2 \Rightarrow (f^{-1}[\nu])' \in \mathcal{T}_1$$

and

$$\nu \geq f[\mu] \Rightarrow \mu \leq f^{-1}[f[\mu]] \leq f^{-1}[\nu].$$

Thus

$$f^{-1}[\overline{f[\mu]} \geq \bar{\mu}$$

and so

$$\overline{f[\mu]} \geq f[f^{-1}[\overline{f[\mu]}]] \geq f[\bar{\mu}].$$

That is:

$$\overline{f[\mu]} \geq f[\bar{\mu}].$$

(4) $\Rightarrow$ (3)

For $\nu \in I^Y,\ f^{-1}[\nu] \in I^X$.

Therefore

$$f[\overline{f^{-1}[\nu]}] \leq \overline{f[f^{-1}[\nu]]} \leq \bar{\nu}$$

and so

$$f^{-1}[\bar{\nu}] \geq f^{-1}[f[\overline{f^{-1}[\nu]}]] \geq \overline{f^{-1}[\nu]}.$$

Thus

$$f^{-1}[\bar{\nu}] \geq \overline{f^{-1}[\nu]}.$$

(3) $\Rightarrow$ (2)

Let $\nu' \in \mathcal{T}_2$. Then $\nu \in I^Y$ and

$$\overline{f^{-1}[\nu]} \leq f^{-1}[\bar{\nu}] = f^{-1}[\nu].$$

Therefore

$$\overline{f^{-1}[\nu]} = f^{-1}[\nu]$$

and hence

$$f^{-1}[\nu] \text{ is } \mathcal{T}_1 - \text{ closed.}$$

□

More information regarding fuzzy topology can be found in [61, 65, 77, 79, 80].

Chapter 3

Filters

3.1 Introduction

The facts regarding fiters can be found in [8, 81] but, for convenience, we record the basics in this chapter. We will see the similar results for prefiters, generalised filters and fuzzy filters in the chapters 5,7 and 9 respectively.

3.1.1 Definitions

If X is a set, we call $\mathbb{F} \subseteq \mathcal{P}(X)$ a *filter* on X iff

1. $\mathbb{F} \neq \emptyset$ and $\emptyset \notin \mathbb{F}$;
2. $\forall\ F, G \in \mathbb{F},\ F \cap G \in \mathbb{F}$;
3. $\forall\ F \in \mathbb{F},\ F \subseteq G \Rightarrow G \in \mathbb{F}$.

If X is a set, we call $\mathbb{B} \subseteq \mathcal{P}(X)$ a *filter base* on X iff

1. $\mathbb{B} \neq \emptyset$ and $\emptyset \notin \mathbb{B}$;
2. $\forall\ F, G \in \mathbb{B},\ \exists B \in \mathbb{B} : B \subseteq F \cap G$.

For $\mathbb{S} \subseteq \mathcal{P}(X)$,

$$< \mathbb{S} > \stackrel{\text{def}}{=} \{Y \subseteq X : \exists S \in \mathbb{S} \text{ such that } S \subseteq Y\}.$$

If $A \subseteq X$ then $< \{A\} >$ is a filter on X.
If $\mathbb{F}$ and $\mathbb{G}$ are filters on X, we say that $\mathbb{G}$ is *finer* than $\mathbb{F}$ or $\mathbb{F}$ is *coarser* than $\mathbb{G}$, if $\mathbb{F} \subseteq \mathbb{G}$.
If a filter $\mathbb{F}$ is such that there exists a set A with $\mathbb{F} =< \{A\} >$, we call $\mathbb{F}$ a *principal* filter.
Note that

$$A \subseteq B \Rightarrow < \{B\} > \subseteq < \{A\} > .$$

We therefore expect principal filters which are generated by a singleton to be maximal. In other words $x \in X \Rightarrow < \{\{x\}\} >$ is maximal.

We say that a filter $\mathbb{F}$ is *fixed* iff $\cap\mathbb{F} \neq \emptyset$ and *free* if $\cap\mathbb{F} = \emptyset$.

We call $\mathbb{B} \subseteq \mathbb{F}$ to be a *base for* $\mathbb{F}$ iff $< \mathbb{B} > = \mathbb{F}$.

3.1.2 Proposition

1. If $\mathbb{B}$ is a filter base then $< \mathbb{B} >$ is a filter.
2. If $\mathbb{F}$ is a filter and $\mathbb{B} \subseteq \mathbb{F}$ satisfies $< \mathbb{B} > = \mathbb{F}$ then $\mathbb{B}$ is a filter base.

3.1.3 Examples

1. If (X, τ) is a topological space we define the *neighbourhood filter* $\mathbb{N}_x$ by

$$\mathbb{N}_x =< \{U : U \in \tau,\ x \in U\} > .$$

It is straightforward to check that $\mathbb{N}_x$ is a fixed filter.
2. Let (X, τ) be a topological space and let $\emptyset \neq A \subseteq X$. Define

$$\mathbb{F} \stackrel{\text{def}}{=} \{F \subseteq X : A \subseteq F^\circ\}.$$

Then $\mathbb{F}$ is a filter and $\mathbb{F} \subseteq < \{A\} >$.
3. On the real line $\mathbb{R}$ we have the free filter

$$< \{(a, \infty) : a \in \mathbb{R}\} > .$$

4. Let X be an infinite set. Define

$$\mathbb{F} \stackrel{\text{def}}{=} < \{A^c : A \text{ is finite }\} > .$$

It is easy to check that $\mathbb{F}$ is a filter.
5. On the natural numbers $\mathbb{N}$ we have the free filter

$$< \{(n, \infty) : n \in \mathbb{N}\} > .$$

6. On $\mathbb{R}^2$ we have the free filter

$$< \{B_r : 0 < r < \infty\} >$$

where

$$B_r \stackrel{\text{def}}{=} \{(x, y) : \sqrt{x^2 + y^2} < r\}.$$

3.2 Ultrafilters

3.2.1 Definition

Filters which are maximal (with respect to inclusion) are called *ultrafilters*. In other words, for a filter $\mathbb{F}$,

$$\mathbb{F} \text{ is ultra } \iff (\mathbb{F} \subseteq \mathbb{G},\ \mathbb{G} \text{ is a filter } \Rightarrow \mathbb{F} = \mathbb{G}).$$

3.2.2 Proposition

A principal filter $\mathbb{F} =< \{A\} >$ is ultra iff A is a singleton.

Proof.

Let $\mathbb{F} =< \{A\} >$ be an ultrafilter and let $x, y \in A$. Then $\mathbb{F} \subseteq < \{\{x\}\} >$ and $\mathbb{F} \subseteq < \{\{y\}\} >$. So, since $\mathbb{F}$ is maximal,

$$\mathbb{F} =< \{\{x\}\} >=< \{\{y\}\} > .$$

We therefore obtain

$$\{x\} \subseteq \{y\} \subseteq \{x\}$$

and from this we deduce that $x = y$.

On the other hand, let $\mathbb{F} =< \{x\} >$, $\mathbb{F} \subseteq \mathbb{G}$ and $G \in \mathbb{G}$. Then $\{x\} \in \mathbb{F} \subseteq \mathbb{G}$ and $G \in \mathbb{G}$ and so $\{x\} \cap G \neq \emptyset$. This means that $\{x\} \subseteq G$ and so $G \in \mathbb{F}$. □

Here is a surprising and useful characterisation of ultrafilters.

3.2.3 Theorem
Let $\mathbb{F}$ be a filter on a set X. Then

$$\mathbb{F} \text{ is ultra} \iff \forall A \subseteq X,\ A \in \mathbb{F} \text{ or } A^c \in \mathbb{F}.$$

Proof.

Let $\mathbb{F}$ be an ultrafilter, $A \subseteq X$ and $A \notin \mathbb{F}$. Define

$$\mathbb{G} = \{F \cap A^c : F \in \mathbb{F}\}.$$

Then $\mathbb{G}$ is a filter base since:
(i) $A \notin \mathbb{F} \Rightarrow \forall F \in \mathbb{F},\ F \not\subseteq A \Rightarrow \forall F \in \mathbb{F},\ F \cap A^c \neq \emptyset$.

(ii) If $F \cap A^c, G \cap A^c \in \mathbb{G}$ with $F, G \in \mathbb{F}$ then $F \cap G \in \mathbb{F}$ and so $F \cap G \not\subseteq A$. Thus $F \cap G \cap A^c \neq \emptyset$.
We have $\mathbb{F} \subseteq < \mathbb{G} >$ and hence $\mathbb{F} = < \mathbb{G} >$. Now $X \in \mathbb{F}$ and so $X \cap A^c = A^c \in \mathbb{F}$.

Conversely, let $\mathbb{F} \subseteq \mathbb{G}$ with $\mathbb{G}$ a filter and let $G \in \mathbb{G}$. If $G \notin \mathbb{F}$ then $G^c \in \mathbb{F}$ and hence $G^c \in \mathbb{G}$. But then $G \cap G^c = \emptyset \in \mathbb{G}$ and this contradiction establishes that $G \in \mathbb{F}$. Thus $\mathbb{F} = \mathbb{G}$ and so $\mathbb{F}$ is maximal. □

The following result will be appealed to many times and we note, in passing, that the Axiom of Choice is required.

3.2.4 Theorem
Every filter is contained in an ultrafilter.

Proof.

Let $\mathbb{F}$ be a filter on a set X and define

$$\mathcal{S} \stackrel{\text{def}}{=} \{\mathbb{G} : \mathbb{G} \text{ is a filter and } \mathbb{F} \subseteq \mathbb{G}\}.$$

Inclusion is a partial ordering on $\mathcal{S}$. Let $\mathcal{C}$ be a chain in $\mathcal{S}$. Then it is easy to check that $\mathbb{H} \stackrel{\text{def}}{=} \cup\mathcal{C}$ is a filter on X and $\mathbb{H}$ is an upper bound for $\mathcal{C}$. We appeal to Zorn's Lemma to deduce that $\mathcal{S}$ has a maximal element $\mathbb{G}$. Then $\mathbb{F} \subseteq \mathbb{G}$ and $\mathbb{G}$ is maximal. □

3.2.5 Definition
If $\mathbb{F}$ is a filter on a set X, we say that

$$\mathbb{F} \text{ is prime} \iff (F \cup G \in \mathbb{F} \Rightarrow F \in \mathbb{F} \text{ or } G \in \mathbb{F}).$$

3.2.6 Theorem
A filter is ultra iff it is prime.

Proof.

Let $\mathbb{F}$ be an ultrafilter, $F \cup G \in \mathbb{F}, F \notin \mathbb{F}$. Define

$$\mathbb{G} \stackrel{\text{def}}{=} \{H : H \cup F \in \mathbb{F}\}.$$

It is straighforward to check that $\mathbb{G}$ is a filter and $G \in \mathbb{G}$. Furthermore, if $K \in \mathbb{F}$ then $K \subseteq K \cup F \in \mathbb{F}$ and so $K \in \mathbb{G}$. Consequently we have $\mathbb{F} \subseteq \mathbb{G}$ and, since $\mathbb{F}$ is ultra, $\mathbb{F} = \mathbb{G}$. Thus $G \in \mathbb{F}$ since $G \in \mathbb{G}$.

To prove the converse, let $A \subseteq X$. Then $A \cup A^c = X \in \mathbb{F}$ and hence $A \in \mathbb{F}$ or $A^c \in \mathbb{F}$. Thus, $\mathbb{F}$ is ultra. □

3.2.7 Theorem

Let $\mathbb{F}$ be a filter on a set X. Then

$$\mathbb{F} = \cap\, \mathbb{P}(\mathbb{F}).$$

Where

$$\mathbb{P}(\mathbb{F}) \stackrel{\text{def}}{=} \{\mathbb{K} : \mathbb{K} \text{ is an ultrafilter and } \mathbb{F} \subseteq \mathbb{K}\}.$$

PROOF.

The inclusion $\mathbb{F} \subseteq \cap\mathbb{P}(\mathbb{F})$ is obvious so we show that $\cap\,\mathbb{P}(\mathbb{F}) \subseteq \mathbb{F}$. To this end let $K \in \cap\,\mathbb{P}(\mathbb{F})$. If $K \notin \mathbb{F}$ then for each $F \in \mathbb{F},\ F \not\subseteq K$. Thus we have

$$\forall F \in \mathbb{F},\ F \cap K^c \neq \emptyset.$$

Let

$$\mathbb{K} \stackrel{\text{def}}{=} < \{F \cap K^c : F \in \mathbb{F}\} > .$$

It is easy to see that $\mathbb{K}$ is a filter containing $\mathbb{F}$. Let $\mathbb{H}$ be an ultrafilter containing $\mathbb{K}$. We therefore have

$$\mathbb{F} \subseteq \mathbb{K} \subseteq \mathbb{H} \in \mathbb{P}(\mathbb{F}).$$

Now $K^c = X \cap K^c \in \mathbb{K}$ and so $K^c \in \mathbb{H}$. But we also have $K \in \cap\mathbb{P}(\mathbb{F}) \subseteq \mathbb{H}$ and so $K \cap K^c = \emptyset \in \mathbb{H}$. This contradiction establishes the result. □

3.2.8 Definitions

We say that a subset $\mathcal{S} \subseteq \mathcal{P}(X)$ has *the finite intersection property* (FIP) if every finite subcollection from $\mathcal{S}$ has nonempty intersection. We let

$$\wp_f(\mathcal{S}) \stackrel{\text{def}}{=} \{\mathcal{C} \subseteq \mathcal{S} : \mathcal{C} \text{ is finite }\}.$$

Then

$$\mathcal{S} \text{ has FIP} \iff \forall \mathcal{C} \in \wp_f(\mathcal{S}),\ \cap\mathcal{C} \neq \emptyset.$$

So a filter has FIP.

If $\mathcal{S}$ has FIP we can construct a filter containing $\mathcal{S}$ as follows. Let

$$\lceil \mathcal{S} \rceil \stackrel{\text{def}}{=} \{\cap\mathcal{C} : \mathcal{C} \in \wp_f(\mathcal{S})\}.$$

So $\lceil \mathcal{S} \rceil$ denotes the set of all intersections of finite subsets of $\mathcal{S}$. It is easy to check that

$$\mathcal{S} \text{ has FIP } \Rightarrow \lceil \mathcal{S} \rceil \text{ is a filter base}$$

and so

$$\mathcal{S} \text{ has FIP } \Rightarrow < \lceil \mathcal{S} \rceil > \text{ is a filter.}$$

Furthermore,

$$\mathcal{S} \subseteq < \lceil \mathcal{S} \rceil > .$$

Of course, a filter $\mathbb{F}$ is closed with respect to the formation of finite intersections and supersets and so

$$< \lceil \mathbb{F} \rceil > = \mathbb{F}.$$

If $\mathbb{F}$ and $\mathbb{G}$ are filters on a set X, we say that they are *compatible* , and write $\mathbb{F} \sim \mathbb{G}$, if every element of $\mathbb{F}$ meets every element of $\mathbb{G}$. In other words

$$\mathbb{F} \sim \mathbb{G} \iff \forall\ F \in \mathbb{F},\ \forall\ G \in \mathbb{G},\ F \cap G \neq \emptyset.$$

If $\mathbb{F} \sim \mathbb{G}$ we can construct a filter which contains them both.

3.2.9 Theorem
If $\mathbb{F} \sim \mathbb{G}$ then

$$[\mathbb{F}, \mathbb{G}] \stackrel{\text{def}}{=} < \{F \cap G : F \in \mathbb{F},\ G \in \mathbb{G}\} >$$

is a filter and $\mathbb{F}, \mathbb{G} \subseteq [\mathbb{F}, \mathbb{G}]$.
Furthermore $[\mathbb{F}, \mathbb{G}]$ is the smallest filter containing both $\mathbb{F}$ and $\mathbb{G}$.

The proof of this is straightforward. The fact that $[\mathbb{F}, \mathbb{G}]$ is the smallest filter containing both $\mathbb{F}$ and $\mathbb{G}$ tempts some authors to write $\mathbb{F} \vee \mathbb{G}$ for $[\mathbb{F}, \mathbb{G}]$ and this is fine as long as we realise that the set of filters on a set X is not a lattice. This is because two filters need to be compatible in order for the supremum to exist. If they are not compatible then $[\mathbb{F}, \mathbb{G}] = \mathcal{P}(X)$ which is not a filter. In particular, two different ultrafilters have no filter as a supremum.

In the case where $\mathbb{G} =< \{A\} >$ for some set $A \subseteq X$, we write

$$[\mathbb{F}, \mathbb{G}] = [\mathbb{F}, A]$$

and the compatibility requirement is that

$$\forall\ F \in \mathbb{F},\ F \cap A \neq \emptyset.$$

Thus we have

3.2.10 Corollary

$$\forall\ F \in \mathbb{F},\ F \cap A \neq \emptyset \Rightarrow [\mathbb{F}, A] \text{ is a filter and } \mathbb{F} \subseteq [\mathbb{F}, A].$$

Now here is a useful observation.

3.2.11 Corollary
If $\mathbb{F}$ is an ultrafilter on X and $A \subseteq X$ then

$$\forall F \in \mathbb{F},\ F \cap A \neq \emptyset \iff A \in \mathbb{F}.$$

Proof.

$$\begin{aligned}\forall F \in \mathbb{F},\ F \cap A \neq \emptyset \quad &\Rightarrow \mathbb{F} \subseteq [\mathbb{F}, A] \\ &\Rightarrow \mathbb{F} = [\mathbb{F}, A] \\ &\Rightarrow A \in [\mathbb{F}, A] = \mathbb{F}.\end{aligned}$$

The converse is obvious. □

3.3 Topological Notions in Terms of Filters

The fundamental topological notions of convergence, closure and continuity can be described using filters.

3.3.1 Definition
Let (X, τ) be a topological space and $\mathbb{F}$ a filter on X. A point $x \in X$. is said to be a *limit* point of $\mathbb{F}$, if $\mathbb{F}$ is finer than the neighbourhood filter $\mathbb{N}_x$.
$\mathbb{F}$ is also said to be *coverge* to x and we write $\mathbb{F} \longrightarrow x$.
In other words

$$\mathbb{F} \to x \iff \mathbb{N}_x \subseteq \mathbb{F}.$$

A filter base $\mathbb{B}$ *converges* to x iff $<\mathbb{B}>\longrightarrow x$
We define,

$$\lim \mathbb{F} \stackrel{\text{def}}{=} \{x \in X : \mathbb{F} \longrightarrow x\}.$$

3.3.2 Definition

Let (X, τ) be a topological space and a filter $\mathbb{F}$ on X has $x \in X$ as a *cluster* point if every member of $\mathbb{F}$ meets every member of $\mathbb{N}_x$. We write $\mathbb{F} \succ x$
In other words

$$\mathbb{F} \succ x \iff \forall F \in \mathbb{F},\ \forall V \in \mathbb{N}_x,\ F \cap V \neq \emptyset.$$

A filter base $\mathbb{B}$ *clusters* at x iff $<\mathbb{B}>\succ x$
We define,

$$\operatorname{adh} \mathbb{F} \stackrel{\text{def}}{=} \{x \in X : \mathbb{F} \succ x\}$$

3.3.3 Examples

(1) On the real line $\mathbb{R}$ with the usual topology the filter

$$\mathbb{F} =< \{(0, \frac{1}{n}) : n \in \mathbb{N}\} >$$

converges to 0.
To check this, let V be a neighbourhood of 0. Then there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq V$. Thus there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and so $(0, \frac{1}{n}) \subseteq (-\varepsilon, \varepsilon) \subseteq V$. We have shown that

$$\forall V \in \mathbb{N}_0,\ \exists F \in \mathbb{F},\ F \subseteq V$$

and this means that $\mathbb{N}_0 \subseteq \mathbb{F}$.

(2) On the real line $\mathbb{R}$ with the usual topology, let

$$\mathbb{F} =< \{(0, m(n)) : n \in \mathbb{N}\} >$$

where

$$m(n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Then $\mathbb{F} \succ 0$ since if $(-\varepsilon, \varepsilon) \subseteq V \in \mathbb{N}_0$ and $(0, m(n)) \subseteq F \in \mathbb{F}$ it is clear that $V \cap F \neq \emptyset$.

3.3.4 Theorem

$$\mathbb{F} \succ x \iff \exists \text{ a filter } \mathbb{G} \text{ with } \mathbb{F} \subseteq \mathbb{G} \text{ and } \mathbb{G} \longrightarrow x.$$

PROOF.

$(\Rightarrow)$
If $\mathbb{F} \succ x$ let $\mathbb{G} = [\mathbb{F}, \mathbb{N}_x]$. Then $\mathbb{N}_x \subseteq \mathbb{G}$ and $\mathbb{F} \subseteq \mathbb{G}$ means that $\mathbb{G}$ is finer than $\mathbb{F}$ and $\mathbb{G} \longrightarrow x$.
$(\Leftarrow)$
Conversely, if $\mathbb{F} \subseteq \mathbb{G}$ and $\mathbb{G} \longrightarrow x$ then $\mathbb{N}_x \subseteq \mathbb{G}$. Let $V \in \mathbb{N}_x$ and $F \in \mathbb{F}$. Then $F \in \mathbb{G}$ and $V \in \mathbb{G}$ and so $F \cap V \neq \emptyset$. □

The closure can be described in terms of filters.

3.3.5 Theorem
Let (X, τ) be a topological space and $A \subseteq X$. Then

$$x \in \overline{A} \iff \exists\, \mathbb{F},\ \mathbb{F} \text{ a filter on } X\ , A \in \mathbb{F},\quad \mathbb{F} \longrightarrow x.$$

PROOF.

($\Rightarrow$)
If $x \in \bar{A}$ let $\mathbb{F} = [\mathbb{N}_x, A]$. Then $A \in \mathbb{F}$ and $\mathbb{F} \longrightarrow x$.
($\Leftarrow$)
Conversely, let $A \in \mathbb{F}$ and $\mathbb{F} \longrightarrow x$. Then $\mathbb{N}_x \subseteq \mathbb{F}$ and hence if $V \in \mathbb{N}_x$ we have $V \in \mathbb{F}$ and so $V \cap A \neq \emptyset$. Since V is arbitrary, $x \in \overline{A}$. □

Continuity can be described in terms of filters.
The following proposition is straightforward.

3.3.6 Proposition
Let $f : X \longrightarrow Y$ be a function and $\mathbb{F}$ is a filter on X then

$$f[\mathbb{F}] = \{f^{\rightarrow}(F) : F \in \mathbb{F}\}$$

is a filter base on Y.

3.3.7 Theorem
Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be a function. Then

$$f \text{ is continuous at } x_0 \iff \forall \mathbb{F},\ (\mathbb{F} \text{ is a filter and } \mathbb{F} \longrightarrow x_0 \Longrightarrow < f[\mathbb{F}] > \longrightarrow f(x_0)).$$

PROOF.

($\Rightarrow$)
Let f be continuous at x, $\mathbb{F} \longrightarrow x$ and $V \in \mathbb{N}_{f(x)}$. Then there exists $U \in \mathbb{N}_x$ such that $f[U] \subseteq V$. Thus $f[U] \in f[\mathbb{F}]$ and so $V \in < f[\mathbb{F}] >$. Since V is arbitrary we have shown that $\mathbb{N}_{f(x)} \subseteq < f[\mathbb{F}] >$.
($\Leftarrow$)
Conversely, let $x \in X$ and $V \in \mathbb{N}_{f(x)}$. Then, since $\mathbb{N}_x \rightarrow x$, we have $< f[\mathbb{N}_x] > \longrightarrow f(x)$. This means that $\mathbb{N}_{f(x)} \subseteq < f[\mathbb{N}_x] >$ and so $V \in < f[\mathbb{N}_x] >$. Therefore there exists $U \in \mathbb{N}_x$ such that $f[U] \subseteq V$. Since V and x are arbitrary, we have shown that f is continuous. □

Chapter 4

Uniform Spaces

We will see later fuzzy uniform spaces, generalised uniform spaces and super uniform spaces. In this chapter we include only some basic results regarding uniform spaces and we will see later similar results for the above mentioned uniform spaces. More literature regarding uniform spaces can be found in [8, 37, 81].

4.1 Introduction

If X is a set then we define the following notation.

1. $\Delta = \Delta(X) \stackrel{\text{def}}{=} \{(x,x) : x \in X\}$.
2. If $U, V \subseteq X \times X$ then

$$U \circ V \stackrel{\text{def}}{=} \{(x,y) \in X \times X : \exists z \in X \text{ such that } (x,z) \in V \text{ and } (z,y) \in U\}.$$

3. If $U \subseteq X \times X$ then $U_s \stackrel{\text{def}}{=} \{(x,y) : (y,x) \in U\}$.

Note that

1. $A \subseteq B \Rightarrow A_s \subseteq B_s$.
2. $A \subseteq E$ and $B \subseteq F \Rightarrow A \circ B \subseteq E \circ F$.
3. $U^n \stackrel{\text{def}}{=} \underbrace{U \circ U \circ \cdots \circ U}_{n \text{ factors}}$.

   If $\Delta \subseteq U$ then $U \subseteq U^2 \subseteq U^3 \subseteq \ldots \subseteq U^n$.

4.1.1 Definitions

If X is set then $\mathbb{D} \subseteq \mathcal{P}(X \times X)$ is called a *uniformity* on X iff

1. $\mathbb{D}$ is a filter;
2. $\forall\, U \in \mathbb{D},\ \Delta \subseteq U$;
3. $\forall\, U \in \mathbb{D},\ U_s \in \mathbb{D}$;
4. $\forall\, U \in \mathbb{D},\ \exists V \in \mathbb{D} : V \circ V \subseteq U$.

We call $(X, \mathbb{D})$ a *uniform space.*

If X is a set and $\mathbb{B} \subseteq \mathcal{P}(X \times X)$ is called a *uniform base* on X iff

1. $\mathbb{B}$ is a filter base;
2. $\forall\ B \in \mathbb{B},\ \Delta \subseteq B$;
3. $\forall\ B \in \mathbb{B},\ \exists D \in \mathbb{B} : D_s \subseteq B$;
4. $\forall\ B \in \mathbb{B},\ \exists D \in \mathbb{B} : D \circ D \subseteq B$.

We call $\mathbb{B} \subseteq \mathbb{D}$ a *base for* $\mathbb{D}$ if $< \mathbb{B} >= \mathbb{D}$. So $\mathbb{B}$ is a base for $\mathbb{D}$ iff $\forall\ D \in \mathbb{D},\ \exists B \in \mathbb{B} : B \subseteq D$.

4.1.2 Proposition

1. If $\mathbb{B}$ is a uniform base then $< \mathbb{B} >$ is uniformity.
2. If $\mathbb{D}$ is a uniformity and $\mathbb{B} \subseteq \mathbb{D}$ satisfies $< \mathbb{B} >= \mathbb{D}$ then $\mathbb{B}$ is a uniform base.

Proof.

(1) Let $\mathbb{B}$ be a uniform base then $\mathbb{B}$ is filter base. So,

$$\mathbb{B} \text{ is a filter base } \Rightarrow < \mathbb{B} > \text{ is a filter .}$$

$$\mathbb{D} =< \mathbb{B} >= \{D : \exists\ B \in \mathbb{B} \text{ such that } B \subseteq D\}$$

If $D \in \mathbb{D}$ then$\exists B \in \mathbb{B}$ such that $B \subseteq D$. Thus $\Delta \subseteq B \subseteq D$.
If $D \in \mathbb{D}$ then $\exists B \in \mathbb{B}$ such that $B \subseteq D$. So $\exists E \in \mathbb{B}$ such that $E_s \subseteq B$. Therefore $E \subseteq B_s \subseteq D_s$ and hence $D_s \in \mathbb{D}$.
If $D \in \mathbb{D}$ then $\exists B \in \mathbb{B}$ such that $B \subseteq D$. So, $\exists E \in \mathbb{B}$ such that $E \circ E \subseteq B \subseteq D$.

(2) We have $\mathbb{D}$ is filter, $\mathbb{B} \subseteq \mathbb{D}$ and $< \mathbb{B} >= \mathbb{D}$. Therefore, $\mathbb{B}$ is a filter base.

$$\mathbb{D} = \{D : \exists B \in \mathbb{B} \text{ such that } B \subseteq D\}$$

If $B \in \mathbb{B}$ then $B \in \mathbb{D}$. So $\Delta \subseteq B$.
If $B \in \mathbb{B}$ then $B \in \mathbb{D}$. So $\exists E \in \mathcal{D}$ such that $E \circ E \subseteq B$. Therefore $\exists F \in \mathbb{B}$ such that $F \subseteq E$ and hence

$$F \circ F \subseteq E \circ E \subseteq B$$

If $B \in \mathbb{B}$ then $B \in \mathbb{D}$. So $B_s \in \mathbb{D}$. Therefore $\exists E \in \mathbb{B}$ such that $E \subseteq B_s \Rightarrow E_s \subseteq B$. □

4.1.3 Examples

(1) Let (X, ρ) be a pseudometric space and $\varepsilon > 0$. Let

$$D^\rho_\varepsilon = \{(x, y) : \rho(x, y) < \varepsilon\};$$

$$\mathbb{B}_\rho = \{D^\rho_\varepsilon : \varepsilon > 0\};$$

$$\mathbb{D}_\rho =< \mathbb{B}_\rho > .$$

Then $\mathbb{D}_\rho$ is a uniformity on X. Note that $\mathbb{D}_\rho = \mathbb{D}_{2\rho}$ and so different metrices generate the same uniformity.

(2) Let $\mathbb{B} = \{\Delta\}$ and $\mathbb{D} =< \mathbb{B} >$. Then $\mathbb{B}$ is a uniform base and $\mathbb{D}$ is a uniformity called the *discrete uniformity.*

(3) If $\mathbb{D} = \{X \times X\}$ then $\mathbb{D}$ is called the trivial uniformity.

(4) For $r \in \mathbb{R}$, let $B_r = \Delta \cup \{(x, y) \in \mathbb{R} \times \mathbb{R} : x > r,\ y > r\}$ then $\mathbb{B} = \{B_r : r \in \mathbb{R}\}$ is a uniform base on $\mathbb{R}$.

We call $D \in \mathbb{D}$ *symmetric* if $D = D_s$.

4.1.4 Proposition
(1) A uniformity has a base of symmetric elements.
(2) $D \in \mathbb{D},\ n \in \mathbb{N} \Rightarrow \exists$ symmetric $E \in \mathbb{D} : E^n \subseteq D$.

Proof.

(1) Let $(X, \mathbb{D})$ be a uniform space and $\mathbb{B} = \{D \in \mathbb{D} : D = D_s\}$. Then $< \mathbb{B} >= \mathbb{D}$:
Let $D \in \mathbb{D}$ and $E = D \cap D_s$. Then $E \in \mathbb{B}$ and $E \subseteq D$.

(2) Let $D \in \mathbb{D}$ and $n \in \mathbb{N}$. Then $\Delta \subseteq D$ and $\Delta^n = \Delta \subseteq D$. □

4.2 The Uniform Topology

Let $x \in X,\ A \subseteq X$ and $U \subseteq X \times X$.

$$U(x) \stackrel{\text{def}}{=} \{y \in X : (x, y) \in U\}$$

and

$$U(A) \stackrel{\text{def}}{=} \bigcup_{x \in A} U(x) = \{y \in X : \exists x \in A \text{ such that } (x, y) \in U\}.$$

4.2.1 Theorem
Let $(X, \mathbb{D})$ be a uniform space. Then
$\beta_x = \{U(x) : U \in \mathbb{D}\}$ is a neighbourhood base at x.

Proof.

(i) We have $x \in U(x)$ because $(x, x) \in \Delta \subseteq U$.
(ii) Let $U(x), V(x) \in \beta_x$. Then $U, V \in \mathbb{D}$ and so $U \cap V \in \mathbb{D}$. Therefore

$$\begin{aligned} U(x) \cap V(x) &= \{y \in X : (x, y) \in U \text{ and } (x, y) \in V\} \\ &= \{y \in X : (x, y) \in U \cap V\} \\ &= (U \cap V)(x) \in \beta_x. \end{aligned}$$

(iii) If $U(x) \in \beta_x$ then $U \in \mathbb{D}$.
We seek $V(x) \in \beta_x$ such that if $y \in V(x)$ then $\exists W(y) \in \beta_y$ with $W(y) \subseteq U(x)$. We have

$$U \in \mathbb{D} \Rightarrow \exists V \in \mathbb{D} \text{ such that } V \circ V \subseteq U.$$

Let $y \in V(x)$. Then $(x, y) \in V$. If $z \in V(y)$ then $(y, z) \in V$ and so $(x, z) \in V \circ V \subseteq U$. Therefore $z \in U(x)$. Hence $V(y) \subseteq U(x)$. That is $\exists V(y) \in \beta_y$ with $V(y) \subseteq U(x)$. □

4.2.2 Lemma
Let $(X, \mathbb{D})$ be a uniform space and $\mathbb{B}$ is a base for $\mathbb{D}$. Then
$\beta_x = \{B(x) : B \in \mathbb{B}\}$ is also a neighbourhood base at x.

Proof.

(i) We have $x \in B(x)$. because $(x, x) \in \Delta \subseteq B$.
(ii) Let $U(x), V(x) \in \beta_x$. Then $U, V \in \mathbb{B}$ and so $\exists B \in \mathbb{B}$ such that $B \subseteq U \cap V$. But $U(x) \cap V(x) = (U \cap V)(x)$ and therefore $B(x) \subseteq (U \cap V)(x)$.
(iii) Let $U(x) \in \beta_x$. Then $U \in \mathbb{B}$. Therefore $\exists V \in \mathbb{B}$ such that $V \circ V \subseteq U$. If $y \in V(x)$ then $(x, y) \in V$. If $z \in V(y)$ then $(y, z) \in V$ and so $(x, z) \in V \circ V \subseteq U$. Therefore $z \in U(x)$. Hence $V(y) \subseteq U(x)$. That is $\exists V(y) \in \beta_y$ with $V(y) \subseteq U(x)$.
□

Thus if $(X, \mathbb{D})$ a uniform space then

$$\beta_x = \{U(x) : U \in \mathbb{D}\}$$

is a neighbourhood base at x. Therefore

$$\mathbb{N}_x = \{V \subseteq X : B \subseteq V \text{ for some } B \in \beta_x\}$$

is a neighbourhood system at x, and hence

$$\tau_{\mathbb{D}} = \{U \subseteq X : \forall\, x \in X,\ \exists V \in \mathbb{N}_x \text{ such that } V \subseteq U\}$$

is a topology on X.
The same topology is produced if any base β is used in place of $\mathbb{D}$.
We call $\tau_{\mathbb{D}}$ the *uniform topology* generated by $\mathbb{D}$.

There is a simple expression for $\tau_{\mathbb{D}}$-closure:

4.2.3 Theorem
If $\mathbb{B}$ is a base for $\mathbb{D}$ then

$$\bar{A} = \bigcap_{U \in \mathbb{B}} U(A).$$

PROOF.

Let $x \in \bar{A}$ and $U \in \mathbb{B}$. Let $V \in \mathbb{B}$ be symmetric with $V \subseteq U$. Then $V(x)$ is a neighbourhood of x and so $V(x) \cap A \neq \emptyset$. Let $a \in V(x) \cap A$ then (x, a) and so $(a, x) \in V$. Therefore $x \in V(a) \subseteq V(A) \subseteq U(A)$. Thus $\forall\, U \in \mathbb{B},\ x \in U(A)$ and this means that

$$x \in \bigcap_{u \in \mathbb{B}} U(A).$$

Now let $x \in \bigcap_{U \in \mathbb{B}} U(A)$ and $V(x)$ be a basic neighbourhood of x with V symmetric. Then $x \in V(A) \Rightarrow \exists a \in A$ such that $(a, x) \in V$. Therefore $(x, a) \in V$ and so $a \in V(x)$ which means that $V(x) \cap A \neq \emptyset$. Hence $x \in \bar{A}$.

$\square$

4.3 Uniformly Continuous Functions

Let $f : X \longrightarrow Y$ be a mapping. Then

$$f \times f : X \times X \longrightarrow Y \times Y, \quad (x, y) \mapsto (f(x), f(y))$$

is a mapping.
For $D \subseteq X \times X$,

$$(f \times f)^{\rightarrow}(D) = \{(f(x), f(y)) : (x, y) \in D\}$$

and for $E \subseteq Y \times Y$

$$(f \times f)^{\leftarrow}(E) = \{(x, y) \in X \times X : (f(x), f(y)) \in E\}.$$

4.3.1 Definition
Let $(X, \mathbb{D})$ and $(Y, \mathbb{E})$ be uniform spaces and $f : X \longrightarrow Y$ a mapping. f is said to be *uniformly continuous* if $\forall\, E \in \mathbb{E},\ \exists D \in \mathbb{D} : (f \times f)^{\rightarrow}(D) \subseteq E$. That is,

$$\forall\, E \in \mathbb{E},\ (f \times f)^{\leftarrow}(E) \in \mathbb{D}.$$

For example:

If (X, ρ) and (Y, d) are metric spaces then $(X, \mathbb{D}_\rho)$ and $(Y, \mathbb{D}_d)$ are uniform spaces. Then

$$\begin{aligned}
&f : X \longrightarrow Y \text{ is uniformly continuous}\\
&\text{iff } \forall\ \varepsilon > 0,\ \exists\delta > 0 \text{ such that } \rho(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon\\
&\text{iff } \forall\ \varepsilon > 0,\ \exists\delta > 0 \text{ such that } (x,y) \in D^\rho_\delta \Rightarrow (f(x), f(y)) \in D^d_\varepsilon\\
&\text{iff } \forall\ V \in \mathbb{D}_d,\ \exists U \in \mathbb{D}_\rho \text{ such that } (f \times f)^\rightarrow(U) \subseteq V.
\end{aligned}$$

So uniform continuity in the uniform space context generalises uniform continuity in metric spaces.

4.3.2 Theorem
If $f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ is uniformly continuous function then $f : (X, \tau_{\mathbb{D}}) \longrightarrow (Y, \tau_E)$ is continuous.

Proof.

Let $V(f(x))$ be a $\tau_{\mathbb{E}}$ -neighbourhood of $f(x)$ with $V \in \mathbb{E}$. Since f is uniformly continuous function. Therefore $\exists U \in \mathbb{D}$ such that $(f \times f)^\rightarrow(U) \subseteq V$ and so $f^\rightarrow(U(x)) \subseteq V(f(x))$. Since $z \in f^\rightarrow(U(x)) \Rightarrow \exists y \in U(x)$ such that $f(y) = z$. So

$$y \in U(x) \Rightarrow (x,y) \in U \Rightarrow (f(x), f(y)) \in V \Rightarrow f(y) = z \in V(f(x)).$$

□

4.3.3 Lemma
If $f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ and $g : (Y, \mathbb{E}) \longrightarrow (Z, \mathbb{F})$ are two uniformly continuous mappings then
$g \circ f : (X, \mathbb{D}) \longrightarrow (Z, \mathbb{F})$ is uniformly continuous.

Proof.

Let $F \in \mathbb{F}$. Then, since g is uniformly continuous,

$$(g \times g)^\leftarrow(F) \in \mathbb{E}.$$

Since f is uniformly continuous,

$$(f \times f)^\leftarrow((g \times g)^\leftarrow(F)) \in \mathbb{D}.$$

That is

$$((g \circ g) \times (g \circ g))^\leftarrow(F) = (f \times f)^\leftarrow((g \times g)^\leftarrow(F)) \in \mathbb{D}.$$

Hence $(g \circ f)$ is uniformly continuous.

□

Chapter 5

Prefilters

5.1 Introduction

In [48], Lowen introduced the notion of a prefilter and since then it has been studied. The facts regarding prefiters can be found in [50, 52, 62, 10]. In this chapter we summarize these properties of prefilters.

First we state some notations here. I will denote the unit interval, I_0 and I_1 are the intervals $(0,1]$, $[0,1)$. For any $\mu \in I^X$ and $\varepsilon \in I$ we define

$$(\mu+\varepsilon)(x) \stackrel{\text{def}}{=} (\mu(x)+\varepsilon) \wedge 1 \text{ and } (\mu-\varepsilon)(x) \stackrel{\text{def}}{=} (\mu(x)-\varepsilon) \vee 0$$

$$\sup\ \mu \stackrel{\text{def}}{=} \sup_{x\in X} \mu(x) \text{ and } \inf\ \mu \stackrel{\text{def}}{=} \inf_{x\in X} \mu(x).$$

5.1.1 Definitions

If X is a set, $\emptyset \neq \mathcal{F} \subseteq I^X$ is called a *prefilter* (on X) iff

1. $0 \notin \mathcal{F}$;
2. $\forall\ \nu,\mu \in \mathcal{F},\ \nu \wedge \mu \in \mathcal{F}$;
3. $\forall\ \nu \in \mathcal{F},\ \nu \leq \mu \Rightarrow \mu \in \mathcal{F}$.

If X is a set, $\emptyset \notin \mathcal{B} \subseteq I^X$ is called a *prefilter base* (on X) iff

1. $0 \notin \mathcal{B}$;
2. $\forall\ \nu,\mu \in \mathcal{B},\ \exists\lambda \in \mathcal{B} : \lambda \leq \nu \wedge \mu$.

For $\emptyset \neq \mathcal{F} \subseteq I^X$,

$$< \mathcal{F} > \stackrel{\text{def}}{=} \{\mu \in I^X : \exists\nu \in \mathcal{F} \text{ such that } \nu \leq \mu\}.$$

If $\mu \in I^X$ then

$$< \mu > \stackrel{\text{def}}{=} < \{\mu\} > .$$

We call $\mathcal{B} \subseteq \mathcal{F}$ a *prefilter base for* $\mathcal{F}$ if $< \mathcal{B} >= \mathcal{F}$. So $\mathcal{B}$ is a prefilter base for $\mathcal{F}$ iff $\forall\ \mu \in \mathcal{F},\ \exists\ \nu \in \mathcal{B} : \nu \leq \mu$.

If $\mathcal{F}$ and $\mathcal{G}$ are prefilters and $\mathcal{F} \subseteq \mathcal{G}$ we shall say that $\mathcal{F}$ is *coarser* than $\mathcal{G}$ or $\mathcal{G}$ is *finer* than $\mathcal{F}$.

5.1.2 Proposition

1. If $\mathcal{B}$ is a prefilter base then $<\mathcal{B}>$ is a prefilter.

2. If $\mathcal{F}$ is a prefilter and $\mathcal{B} \subseteq \mathcal{F}$ satisfies $<\mathcal{B}>=\mathcal{F}$ then $\mathcal{B}$ is a prefilter base.

PROOF.

(1) We have $<\mathcal{B}> \neq \emptyset$ and $0 \notin <\mathcal{B}>$.

If $\mu,\ \nu \in <\mathcal{B}>$ then $\exists\, \mu',\ \nu' \in \mathcal{B}$ such that $\mu' \leq \mu$ and $\nu' \leq \nu$. So, $\exists\, \lambda \in \mathcal{B}$ such that $\lambda \leq \mu' \wedge \nu' \leq \mu \wedge \nu$. Therefore, $\mu \wedge \nu \in <\mathcal{B}>$.

If $\mu \in <\mathcal{B}>$ and $\mu \leq \nu$ then $\exists\, \lambda \in <\mathcal{B}>$ such that $\lambda \leq \mu \leq \nu$. So, $\nu \in <\mathcal{B}>$. Hence, $<\mathcal{B}>$ is a prefilter.

(2) $<\mathcal{B}> \neq \emptyset$. So $\mathcal{B} \neq \emptyset$. $0 \notin \mathcal{F} = <\mathcal{B}> \Rightarrow 0 \notin \mathcal{B}$.

Let $\mu,\ \nu \in \mathcal{B}$. Then $\mu,\ \nu \in \mathcal{F} = <\mathcal{B}>$. So, $\mu \wedge \nu \in <\mathcal{B}>$. Therefore, $\exists\, \lambda \in \mathcal{B}$ such that $\lambda \leq \mu \wedge \nu$. Hence, $\mathcal{B}$ is a prefiter base.

□

5.1.3 Definitions

Given a prefilter $\mathcal{F}$ and $\mu \in I^X$ the following subset of I

$$C^\mu(\mathcal{F}) \overset{\text{def}}{=} \{\alpha \in I : \forall\, \nu \in \mathcal{F}, \exists\, x \in X \text{ such that } \nu(x) > \mu(x) + \alpha\}$$

will be called the *characteristic set of $\mathcal{F}$ with respect to μ* and

$$c^\mu(\mathcal{F}) \overset{\text{def}}{=} \sup\ C^\mu(\mathcal{F})$$

will be called the *characteristic value of $\mathcal{F}$ with respect to μ.*

When $\mu = 0$ we shall refer to them just as the *characteristic set* and *characteristic value* of $\mathcal{F}$. We shall denote them by $C(\mathcal{F})$ and $c(\mathcal{F})$.

That is,

$$C(\mathcal{F}) \overset{\text{def}}{=} \{\mu \in I^X : \forall\, \nu \in \mathcal{F}, \exists\, x \in X \text{ such that } \nu(x) > \alpha\}.$$

and

$$c(\mathcal{F}) \overset{\text{def}}{=} \sup\ C(\mathcal{F}).$$

We collect together some basic facts in the following proposition.

5.1.4 Proposition

Let $\mathcal{F}$ be a prefilter and $\mu \in I^X$ then

1. $C^\mu(\mathcal{F}) = \{\alpha \in I : \mu + \alpha \notin \mathcal{F}\}$;

2. $\{\alpha \in I : \mu + \alpha \in \mathcal{F}\} = \{1\}$ or $[c, 1]$ with $c \in I_1$ or $(c, 1]$ with $c \in I_1$;

3. $C^\mu(\mathcal{F}) = \emptyset$ or $[0, c)$ with $c \in I_0$ or $[0, c]$ with $c \in I_1$;

4. $c^\mu(\mathcal{F}) \overset{\text{def}}{=} \sup\ C^\mu(\mathcal{F}) = \inf\ \{\alpha \in I : \mu + \alpha \in \mathcal{F}\}$.

PROOF.

(1) We have

$$\begin{aligned} \mu + \alpha \notin \mathcal{F} &\iff \forall\, \nu \in \mathcal{F},\ \mu + \alpha \ngeq \nu \\ &\iff \forall\, \nu \in \mathcal{F},\ \exists\, x \in X : \nu(x) > \mu(x) + \alpha. \end{aligned}$$

(2) Let $A = \{\alpha \in I : \mu + \alpha \in \mathcal{F}\}$. Then $1 \in A$. If $\alpha \in A$ and $\alpha \leq \beta$ then $\beta \in A$. Therefore $A = \{1\}$ or $[c, 1]$ with $c \in I_1$ or $(c, 1]$ with $c \in I_1$.

(3) Let $B = C^\mu(\mathcal{F}) = \{\alpha \in I : \mu + \alpha \notin \mathcal{F}\}$. Then $A \cap B = \emptyset$ and $A \cup B = I$. So, $B = [0, c)$ with $c \in I_0$ or $\emptyset$ or $[0, c]$ with $c \in I_1$.

(4) Therefore we have

$$\sup\ B = \inf\ A.$$

That is,

$$c^{\mu}(\mathcal{F}) = \sup\ C^{\mu}(\mathcal{F}) = \inf\ \{\alpha \in I : \mu + \alpha \in \mathcal{F}\}.$$

□

Note

1. $C(\mathcal{F}) = \{\alpha \in I : \alpha 1_X \notin \mathcal{F}\}$;
2. $c(\mathcal{F}) \stackrel{\text{def}}{=} \sup\ C(\mathcal{F}) = \inf\ \{\alpha \in I : \alpha 1_X \in \mathcal{F}\}$;
3. $c(\mathcal{F}) = \inf\limits_{\nu \in \mathcal{F}} \sup\ \nu$;
4. For a prefilter base $\mathcal{F}$, $c(\mathcal{F}) = c(< \mathcal{F} >)$.

If $\mu = 0$ then

$$C^{\mu}(\mathcal{F}) = C(\mathcal{F}) = \{\alpha \in I : \alpha 1_X \notin \mathcal{F}\},$$
$$c(\mathcal{F}) = \sup\ C(\mathcal{F}) = \inf\ \{\alpha \in I : \alpha 1_X \in \mathcal{F}\}.$$

We have

$$\{\alpha \in I : \alpha 1_X \in \mathcal{F}\} = \{\sup\ \nu : \nu \in \mathcal{F}\}.$$

Since if $\nu \in \mathcal{F}$ then $(\sup\ \nu)1_X \in \mathcal{F}$ and $\sup\ \alpha 1_X = \alpha$. Therefore

$$c(\mathcal{F}) = \inf_{\nu \in \mathcal{F}} \sup\ \nu.$$

We next see the definition of prime prefilter which is similar to the definition of prime filter.

5.1.5 Definition

A prefilter $\mathcal{F}$ is said to be *prime* if $\forall\ \nu, \mu \in I^X$ such that $\nu \vee \mu \in \mathcal{F}$ we have either $\nu \in \mathcal{F}$ or $\mu \in \mathcal{F}$.

5.1.6 Theorem

If $\mu \in I^X$ then
$< \mu >$ is prime iff $\exists\ \alpha > 0, \exists\ x \in X : \mu = \alpha 1_x$.

Proof.

($\Rightarrow$)

Let $x_1, x_2 \in \mu^0$ with $x_1 \neq x_2$. Then take $\mu_1 = \mu(x_1)1_{x_1}$ and $\nu_2(x) = \begin{cases} \mu(x) & \text{if } x \neq x_1 \\ 0 & \text{if } x = x_1. \end{cases}$

Then $\nu_1 \vee \nu_2 = \mu \in < \mu >$, so $\nu_1 \in < \mu >$ or $\nu_2 \in < \mu >$. Therefore $\nu_1 \geq \mu$ or $\nu_2 \geq \mu$ which is clearly false. Thus μ^0 is a singlton. Hence $\exists\ \alpha > 0,\ \exists\ x \in X : \mu = \alpha 1_x$.

($\Leftarrow$)

Let $\nu_1 \vee \nu_2 \in < \alpha 1_x >$. Then $\nu_1 \vee \nu_2 \geq \alpha 1_x$. So

$$\nu_1 \vee \nu_2 \geq \alpha.$$

Therefore

$$\nu_1(x) \geq \alpha \text{ or } \nu_2(x) \geq \alpha.$$

Consequently,

$$\nu_1 \in < \alpha 1_x > \text{ or } \nu_2 \in < \alpha 1_x > .$$

Therefore $< \alpha 1_x >$ is prime. $\square$

Note
This reveals that prime prefilters are not maximal since if $\beta \leq \alpha \leq \mu(x)$ then $< \mu > \subseteq < \alpha 1_x > \subseteq < \beta 1_x >$ with both $< \alpha 1_x >$ and $< \beta 1_x >$ being prime.

5.1.7 Definitions
If $\mathcal{F}$ and $\mathcal{G}$ are prefilter bases then

$$\mathcal{F} \sim \mathcal{G} \Leftrightarrow \forall\ \nu \in \mathcal{F}, \forall\ \mu \in \mathcal{G},\ \nu \wedge \mu \neq 0.$$

If $\mathcal{F} \sim \mathcal{G}$ we define,

$$\mathcal{F} \vee \mathcal{G} \stackrel{\text{def}}{=} < \{\nu \wedge \mu : \nu \in \mathcal{F},\ \mu \in \mathcal{G}\} > .$$

and it is easy to see that $\mathcal{F} \vee \mathcal{G}$ is the smallest prefilter containing both $\mathcal{F}$ and $\mathcal{G}$.

For prefilter bases $\mathcal{F}$ and $\mathcal{G}$ we define,

$$c(\mathcal{F}, \mathcal{G}) \stackrel{\text{def}}{=} \begin{cases} c(\mathcal{F} \vee \mathcal{G}) & \text{if } \mathcal{F} \sim \mathcal{G} \\ 0 & \text{otherwise.} \end{cases}$$

If $0 \neq \mu \in I^X$ then $< \mu >$ is a prefilter and for a prefilter $\mathcal{F}$ we define,

$$c(\mathcal{F}, \mu) \stackrel{\text{def}}{=} c(\mathcal{F}, < \mu >).$$

We define the *saturation* of a prefilter as follows.

5.1.8 Definition
For a prefilter base $\mathcal{F}$ with $c(\mathcal{F}) > 0$ we define,

$$\hat{\mathcal{F}} \stackrel{\text{def}}{=} \{\sup_{\varepsilon \in I_0} (\nu_\varepsilon - \varepsilon) : (\nu_\varepsilon : \varepsilon \in I_0) \in \mathcal{F}^{I_0}\}.$$

5.1.9 Theorem
If $\mathcal{F}$ and $\mathcal{G}$ are prefilter bases with $c(\mathcal{F}) \wedge c(\mathcal{G}) > 0$ then,

1. $\mathcal{F} \subseteq \hat{\mathcal{F}}$;
2. $(\forall\ \varepsilon \in I_0,\ \nu + \varepsilon \in \mathcal{F}) \Rightarrow \nu \in \hat{\mathcal{F}}$;
3. $\hat{\mathcal{F}}$ is a prefilter base;
4. $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \hat{\mathcal{F}} \subseteq \hat{\mathcal{G}}$;
5. $\tilde{\mathcal{F}} \stackrel{\text{def}}{=} < \hat{\mathcal{F}} > = \widehat{< \mathcal{F} >}$;
6. $\hat{\hat{\mathcal{F}}} \subseteq \tilde{\mathcal{F}}$;
7. $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \tilde{\mathcal{F}} \subseteq \tilde{\mathcal{G}}$;
8. If $\mathcal{F}$ is a prefilter then $\tilde{\mathcal{F}} = \hat{\mathcal{F}}$;
9. If $\mathcal{F}$ is a prefilter then $c(\hat{\mathcal{F}}) = c(\mathcal{F})$.

PROOF.

(1) Let $\nu \in \mathcal{F}$ and $\forall\ \varepsilon \in I_0,\ \nu_\varepsilon = \nu$. Then

$$\sup_{\varepsilon \in I_0} (\nu_\varepsilon - \varepsilon) = \sup_{\varepsilon \in I_0} (\nu - \varepsilon) = \nu \in \hat{\mathcal{F}}.$$

(2) Let $\forall\, \varepsilon \in I_0,\ \nu + \varepsilon \in \mathcal{F}$. Then

$$\sup_{\varepsilon \in I_0} \left((\nu + \varepsilon) - \varepsilon\right) = \nu \in \hat{\mathcal{F}}.$$

(3) We have $\hat{\mathcal{F}} \neq \emptyset$ and $0 \notin \hat{\mathcal{F}}$.

Let $\mu = \sup\limits_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon),\ \nu = \sup\limits_{\varepsilon \in I_0} (\nu_\varepsilon - \varepsilon) \in \hat{\mathcal{F}}$ with $\forall\, \varepsilon \in I_0,\ \mu_\varepsilon, \nu_\varepsilon \in \mathcal{F}$. Then

$$\begin{aligned}
(\mu \wedge \nu)(x) &= \sup_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon)(x) \wedge \sup_{\varepsilon \in I_0} (\nu_\varepsilon - \varepsilon)(x) \\
&= \sup_{\varepsilon \in I_0} \sup_{\delta \in I_0} (\mu_\varepsilon - \varepsilon)(x) \wedge (\nu_\delta - \delta)(x) \\
&\geq \sup_{\varepsilon \in I_0} (\mu_\varepsilon \wedge \nu_\varepsilon - \varepsilon)(x).
\end{aligned}$$

So

$$\mu \wedge \nu \geq \sup_{\varepsilon \in I_0} (\mu_\varepsilon \wedge \nu_\varepsilon - \varepsilon).$$

But we have,

$$\forall\, \varepsilon \in I_0,\ \mu_\varepsilon, \nu_\varepsilon \in \mathcal{F} \Rightarrow \forall\, \varepsilon \in I_0,\ \exists\, \lambda_\varepsilon \in \mathcal{F} \text{ such that } \lambda_\varepsilon \leq \mu_\varepsilon \wedge \nu_\varepsilon.$$

Therefore

$$\mu \wedge \nu \geq \sup_{\varepsilon \in I_0} (\mu_\varepsilon \wedge \nu_\varepsilon - \varepsilon) \geq \sup_{\varepsilon \in I_0} (\lambda_\varepsilon - \varepsilon) \in \hat{\mathcal{F}}.$$

Hence, $\hat{\mathcal{F}}$ is a prefilter base.

(4) Let $\mu = \sup\limits_{\varepsilon \in I_0} (\nu_\varepsilon - \varepsilon) \in \hat{\mathcal{F}}$ with $\forall\, \varepsilon \in I_0,\ \nu_\varepsilon \in \mathcal{F}$. Then

$$\forall\, \varepsilon \in I_0,\ \nu_\varepsilon \in \mathcal{G} \Rightarrow \mu \in \hat{\mathcal{G}}.$$

(5) Let $\mu =< \hat{\mathcal{F}} >$. Then $\exists\, \nu \in \hat{\mathcal{F}}$ such that $\nu \leq \mu$.

Therefore $\exists\, (\nu_\varepsilon : \varepsilon \in I_0) \in \mathcal{F}^{I_0}$ such that $\mu \geq \nu = \sup\limits_{\varepsilon \in I_0} (\nu_\varepsilon - \varepsilon)$.

Let $\forall\, \varepsilon \in I_0,\ \mu_\varepsilon \overset{\text{def}}{=} \mu + \varepsilon \geq \nu_\varepsilon$. Then $\mu_\varepsilon \in < \mathcal{F} >$.

So,

$$\mu = \sup_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon) \in \widehat{< \mathcal{F} >}.$$

Therefore

$$< \hat{\mathcal{F}} > \subseteq \widehat{< \mathcal{F} >}.$$

Let $\mu \in \widehat{< \mathcal{F} >}$. Then $\exists\, (\mu_\varepsilon : \varepsilon \in I_0) \in < \mathcal{F} >^{I_0}$ such that $\mu = \sup\limits_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon)$. So

$$\forall\, \varepsilon \in I_0,\ \exists\, \nu_\varepsilon \in \mathcal{F} \text{ such that } \nu_\varepsilon \leq \mu_\varepsilon.$$

Therefore

$$\sup_{\varepsilon \in I_0} (\nu_\varepsilon - \varepsilon) \leq \mu.$$

Hence

$$\mu \in < \hat{\mathcal{F}} > .$$

Thus

$$\widehat{< \mathcal{F} >} \subseteq < \hat{\mathcal{F}} > .$$

(6) Let $\mu \in \hat{\hat{\mathcal{F}}}$. Then $\exists\ (\mu_\varepsilon : \varepsilon \in I_0) \in \hat{\mathcal{F}}^{I_0}$ such that $\mu = \sup\limits_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon)$. Therefore

$$\mu_\varepsilon = \sup_{\delta \in I_0} (\mu_\delta^\varepsilon - \delta) \text{ for some } (\mu_\delta^\varepsilon : \delta \in I_0) \in \mathcal{F}^{I_0}.$$

So

$$\begin{aligned} \mu &= \sup_{\varepsilon \in I_0} (\sup_{\delta \in I_0} (\mu_\delta^\varepsilon - \delta) - \varepsilon) \\ &= \sup_{\varepsilon \in I_0} \sup_{\delta \in I_0} (\mu_\delta^\varepsilon - \delta - \varepsilon) \\ &= \sup_{\alpha \in I_0} \sup_{\substack{\varepsilon, \delta \in I_0 \\ \varepsilon + \delta = \alpha}} (\mu_\delta^\varepsilon - \alpha) \\ &= \sup_{\alpha \in I_0} (\nu_\alpha - \alpha). \end{aligned}$$

Where,

$$\nu_\alpha = \sup_{\substack{\varepsilon, \delta \in I_0 \\ \varepsilon + \delta = \alpha}} \mu_\delta^\varepsilon \in < \mathcal{F} > .$$

Therefore

$$\mu \in \widehat{< \mathcal{F} >} = \tilde{\mathcal{F}}.$$

(7) We have $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \hat{\mathcal{F}} \subseteq \hat{\mathcal{G}} \Rightarrow < \hat{\mathcal{F}} > \subseteq < \hat{\mathcal{G}} >$.

(8) If $\mathcal{F}$ is a prefilter then $\mathcal{F} = < \mathcal{F} >$. Therfore $\tilde{\mathcal{F}} = \widehat{< \mathcal{F} >} = \hat{\mathcal{F}}$.

(9) We have $c(\mathcal{F}) = \inf\limits_{\nu \in \mathcal{F}} \sup\ \nu$ and so,

$$\mathcal{F} \subseteq \hat{\mathcal{F}} \Rightarrow c(\mathcal{F}) \geq c(\hat{\mathcal{F}}).$$

If $\exists\ \alpha$ such that $c(\hat{\mathcal{F}}) < \alpha < c(\mathcal{F})$ then

$$\inf_{\nu \in \hat{\mathcal{F}}} \sup\ \nu < \alpha \Rightarrow \exists\ \nu \in \hat{\mathcal{F}} \text{ such that } \sup\ \nu < \alpha.$$

So

$$\nu = \sup_{\varepsilon \in I_0} (\nu_\varepsilon - \varepsilon) \text{ for some } (\nu_\varepsilon : \varepsilon \in I_0) \in \mathcal{F}^{I_0}.$$

Define $\mu = \beta 1_X$ where $\sup\ \nu < \beta < \alpha$. then $\nu < \mu$. therefore

$$\exists\ \delta > 0 \text{ such that } \nu < \nu + \delta < \mu.$$

But we have $\forall\ \varepsilon \in I_0,\ \nu + \varepsilon \geq \nu_\varepsilon$ and so, $\mu > \nu + \delta \geq \nu_\delta$. Since $\mathcal{F}$ is a filter. Therfore $\mu \in \mathcal{F}$. That is $\mu \in \mathcal{F}$ and $\sup\ \mu = \beta \Rightarrow c(\mathcal{F}) \leq \beta < \alpha$. This is a contradiction to $c(\mathcal{F}) > \alpha$. Therefore $c(\hat{\mathcal{F}}) = c(\mathcal{F})$.

□

5.1.10 Definition

A prefilter $\mathcal{F}$ with $\mathcal{F} = \hat{\mathcal{F}}$ will be called *saturated prefiter.*

Some useful results regarding saturated prefilters.

5.1.11 Theorem

If $\mathcal{F}$ is a prefilter then

(1)

$$\begin{aligned} \mathcal{F} \text{ is saturated } &\Leftrightarrow (\forall\ \varepsilon \in I_0,\ \nu + \varepsilon \in \mathcal{F} \Rightarrow \nu \in \mathcal{F}) \\ &\Leftrightarrow (\forall\ \varepsilon \in I_0,\ \exists\ \nu_\varepsilon \in \mathcal{F} : \nu_\varepsilon \leq \nu + \varepsilon \Rightarrow \nu \in \mathcal{F}). \end{aligned}$$

(2) $\hat{\mathcal{F}} = \{\mu \in I^X : \forall\ \varepsilon \in I_0,\ \mu + \varepsilon \in \mathcal{F}\}$.

(3) $\hat{\mathcal{F}}$ is a saturated prefilter.

(4) If $\mathcal{F}$ is prefilter base then $\tilde{\tilde{\mathcal{F}}} = \tilde{\mathcal{F}}$.

PROOF.

(1)
(⇒)
We have

$$(\forall\ \varepsilon \in I_0,\ \nu + \varepsilon \in \mathcal{F}) \Rightarrow \nu \in \hat{\mathcal{F}}.$$

But $\mathcal{F} = \hat{\mathcal{F}}$, therefore $\nu \in \mathcal{F}$.
(⇐)
Let $\mu \in \hat{\mathcal{F}}$. Then we have

$$\exists\ (\mu_\varepsilon : \varepsilon \in I_0) \in \mathcal{F}^{I_0} \text{ such that } \mu = \sup_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon).$$

Therefore

$$\forall\ \varepsilon \in I_0,\ \mu + \varepsilon \geq \mu_\varepsilon \in \mathcal{F}.$$

Since $\mathcal{F}$ is a prefilter, we have $\forall\ \varepsilon \in I_0,\ \mu + \varepsilon \in \mathcal{F}$. Hence $\mu \in \mathcal{F}$. Therefore $\mathcal{F} = \hat{\mathcal{F}}$. It is easy to show that,

$$(\forall\ \varepsilon \in I_0,\ \nu + \varepsilon \in \mathcal{F} \Rightarrow \nu \in \mathcal{F}) \Longleftrightarrow (\forall\ \varepsilon \in I_0, \exists\ \nu_\varepsilon \in \mathcal{F} : \nu_\varepsilon \leq \nu + \varepsilon \Rightarrow \nu \in \mathcal{F}).$$

(2) We have

$$\{\mu \in I^X : \forall\ \varepsilon \in I_0,\ \mu + \varepsilon \in \mathcal{F}\} \subseteq \hat{\mathcal{F}}.$$

Let $\mu \in \hat{\mathcal{F}}$. Then we have

$$\exists\ (\mu_\varepsilon : \varepsilon \in I_0) \in \mathcal{F}^{I_0} \text{ such that } \mu = \sup_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon).$$

Therefore

$$\forall\ \varepsilon \in I_0,\ \mu + \varepsilon \geq \mu_\varepsilon \in \mathcal{F}.$$

Since $\mathcal{F}$ is a prefilter, we have $\forall\ \varepsilon \in I_0,\ \mu + \varepsilon \in \mathcal{F}$.

(3) If $\mathcal{F}$ is a prefilter then

$$\tilde{\mathcal{F}} = < \hat{\mathcal{F}} > = \widehat{< \mathcal{F} >} = \hat{\mathcal{F}}.$$

So, $\hat{\mathcal{F}}$ is a prefilter.
Now let $\forall\ \varepsilon \in I_0,\ \mu + \varepsilon \in \hat{\mathcal{F}}$. Then

$$\forall\ \varepsilon \in I_0,\ \forall\ \delta \in I_0,\ \mu + \varepsilon + \delta \in \mathcal{F}$$

$$\Rightarrow \forall\ \alpha \in I_0,\ \mu + \alpha \in \mathcal{F}$$

$$\Rightarrow \mu \in \hat{\mathcal{F}}.$$

Therefore $\hat{\mathcal{F}}$ is a saturated prefilter.

(4) If $\mathcal{F}$ is a prefilter base then $< \mathcal{F} >$ is a prefilter. So

$$\tilde{\mathcal{F}} = \widehat{< \mathcal{F} >} = \widehat{\widehat{< \mathcal{F} >}}.$$

That is, $\tilde{\mathcal{F}}$ is a saturated prefilter. So,

$$\tilde{\tilde{\mathcal{F}}} = \widehat{< \tilde{\mathcal{F}} >} = \hat{\tilde{\mathcal{F}}} = \tilde{\mathcal{F}}.$$

□

We next see characteristic set of a saturated prefilter.

5.1.12 Proposition
Let $\mathcal{F}$ be a prefilter with $c(\mathcal{F}) = \alpha$ then $C(\hat{\mathcal{F}}) = [0, \alpha)$.

PROOF.

We have $c(\mathcal{F}) = \inf \{\beta \in I : \beta 1_X \in \mathcal{F}\}$.
So,

$$\beta 1_X \in \mathcal{F} \text{ for all } \beta > \alpha \text{ and } \beta 1_X \notin \mathcal{F} \text{ for all } \beta < \alpha$$
$$\Rightarrow \forall\, \varepsilon \in I_0,\ \alpha 1_X + \varepsilon = (\alpha + \varepsilon \wedge 1) 1_X \in \mathcal{F}$$
$$\Rightarrow \alpha 1_X \in \hat{\mathcal{F}}.$$

On the other hand if $\beta < \alpha$ then $\exists\, \varepsilon > 0$ such that $\beta + \varepsilon < \alpha$

$$\Rightarrow \beta 1_X + \varepsilon = (\beta + \varepsilon) 1_X \notin \mathcal{F}.$$

Which means $\beta 1_X \notin \hat{\mathcal{F}}$. But

$$C(\hat{\mathcal{F}}) = \{t \in I : t 1_X \notin \hat{\mathcal{F}}\}.$$

Therefore $C(\hat{\mathcal{F}}) = [0, \alpha)$. □

5.2 Prefilters from Filters

Recall from 1.2 that if $\nu \in I^X$ and $\alpha \in I$
$\nu^\alpha \overset{\text{def}}{=} \{x \in X : \nu(x) > \alpha\}$,
$\nu_\alpha \overset{\text{def}}{=} \{x \in X : \nu(x) \geq \alpha\}$.

5.2.1 Definition
Let $\mathbb{F}$ be a filter on X . For $\alpha \in (0, 1]$ define,

$$\mathbb{F}_\alpha \overset{\text{def}}{=} < \{\alpha 1_F : F \in \mathbb{F}\} >,$$

$$\mathbb{F}^\alpha \overset{\text{def}}{=} \{\nu \in I^X : \forall\, \beta < \alpha,\ \nu^\beta \in \mathbb{F}\}.$$

5.2.2 Theorem
Let $\mathbb{F}$ and $\mathbb{G}$ are filters on X. Then we have

1. $\mathbb{F}_\alpha$ and $\mathbb{F}^\alpha$ are prefilters on X with $\mathbb{F}_\alpha \subseteq \mathbb{F}^\alpha$ for any $\alpha \in (0, 1]$,
2. $\mathbb{F}_\alpha \subseteq \mathbb{F}_\beta$ and $\mathbb{F}^\alpha \subseteq \mathbb{F}^\beta$ if $0 < \beta < \alpha$,
3. $\mathbb{F}_\alpha \subseteq \mathbb{G}_\alpha$ and $\mathbb{F}^\alpha \subseteq\ G^\alpha$ if $\mathbb{F} \subseteq \mathbb{G}$,
4. $\hat{\mathbb{F}_\alpha} = \mathbb{F}^\alpha = \hat{\mathbb{F}^\alpha}$,
5. $C(\mathbb{F}_\alpha) = C(\mathbb{F}^\alpha) = [0, \alpha)$ and $c(\mathbb{F}_\alpha) = c(\mathbb{F}^\alpha) = \alpha$.

PROOF.

(1) We have $0 \notin \{\alpha 1_F : F \in \mathbb{F}\} \neq \emptyset$ and

$$\alpha 1_F \wedge \alpha 1_G = \alpha 1_{F \cap G} \in \{\alpha 1_F : F \in \mathbb{F}\}.$$

Therefore $\{\alpha 1_F : F \in \mathbb{F}\}$ is a prefilter base. Hence $\mathbb{F}_\alpha$ is a prefilter.

$$1 \in \mathbb{F}^\alpha \neq \emptyset \text{ and } 0 \notin \mathbb{F}^\alpha.$$

Let $\mu, \nu \in \mathbb{F}^\alpha$ then $\forall\ \beta < \alpha,\ \mu^\beta, \nu^\beta \in \mathbb{F}$. But

$$(\mu \wedge \nu)^\beta = \mu^\beta \cap \nu^\beta \in \mathbb{F}.$$

So $\mu \wedge \nu \in \mathbb{F}^\alpha$.

Let $\mu \in \mathbb{F}^\alpha$ and $\mu \leq \nu$. Then $\forall\ \beta < \alpha,\ \mu^\beta \in \mathbb{F}$ For $\beta < \alpha,\ \nu^\beta \supseteq \mu^\beta \in \mathbb{F}$, and so $\nu^\beta \in \mathbb{F}$ Therefore $\nu \in \mathbb{F}^\alpha$.

Hence $\mathbb{F}^\alpha$ is a prefilter.

Let $\mu \in \mathbb{F}_\alpha$. Then $\exists\ F \in \mathbb{F}$ such that $\mu \geq \alpha 1_F$. For $\beta < \alpha,\ \mu^\beta \supseteq (\alpha 1_F)^\beta = F \in \mathbb{F}$ and so $\mu \in \mathbb{F}^\alpha$. Therefore $\mathbb{F}_\alpha \subseteq \mathbb{F}^\alpha$.

(2) We have $\mu \in \mathbb{F}_\alpha \Rightarrow \exists\ F \in \mathbb{F}$ such that $\mu \geq \alpha 1_F$. If $0 < \beta < \alpha$ then $\mu \geq \beta 1_F$ and so $\mu \in \mathbb{F}_\beta$. Therefore $\mu \in \mathbb{F}^\alpha \Rightarrow \forall\ \gamma < \alpha,\ \mu^\gamma \in \mathbb{F} \Rightarrow \forall\ \gamma < \beta,\ \mu^\gamma \in \mathbb{F} \Rightarrow \mu \in \mathbb{F}^\beta$.

(3)

$$\mu \in \mathbb{F}_\alpha \Rightarrow \exists\ F \in \mathbb{F} : \mu \geq \alpha 1_F \Rightarrow \exists\ F \in \mathbb{G} : \mu \geq \alpha 1_F \Rightarrow \mu \in \mathbb{G}_\alpha.$$

$$\mu \in \mathbb{F}^\alpha \Rightarrow \forall\ \beta < \alpha,\ \mu^\beta \in \mathbb{F} \Rightarrow \forall\ \beta < \alpha,\ \mu^\beta \in \mathbb{G} \Rightarrow \mu \in \mathbb{G}^\alpha.$$

(4) Let $\mu \in \hat{\mathbb{F}^\alpha}$. Then $\exists\ (\mu_\varepsilon : \varepsilon \in I_0) \in (\mathbb{F}^\alpha)^{I_0}$ such that $\mu = \sup\limits_{\varepsilon \in I_o} (\mu_\varepsilon - \varepsilon)$. So

$$\forall\ \varepsilon \in I_o,\ \forall\ \beta < \alpha,\ \mu_\varepsilon^\beta \in \mathbb{F},$$

$$\mu^\beta = (\bigvee_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon))^\beta = \bigcup_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon)^\beta = \bigcup_{\varepsilon \in I_0} \mu_\varepsilon^{\beta+\varepsilon}.$$

If $\beta < \alpha$ then $\exists\ \delta > 0$ such that $\beta + \delta < \alpha \Rightarrow \mu_\varepsilon^{\beta+\varepsilon} \in \mathbb{F}$

$$\Rightarrow \bigcup_{\varepsilon \in I_0} \mu_\varepsilon^{\beta+\varepsilon} \in \mathbb{F} \Rightarrow \mu^\beta \in \mathbb{F}.$$

Therefore $\mu \in \mathbb{F}^\alpha$ and hence $\mathbb{F}^\alpha = \hat{\mathbb{F}^\alpha}$.

$$\mathbb{F}_\alpha \subseteq \mathbb{F}^\alpha \Rightarrow \hat{\mathbb{F}_\alpha} \subseteq \hat{\mathbb{F}^\alpha} = \mathbb{F}^\alpha.$$

Let $\mu \in \mathbb{F}^\alpha$. Then $\forall\ \beta < \alpha,\ \mu^\beta \in \mathbb{F}$. Then for each $\varepsilon \in I_0$, define $\nu_\varepsilon = \alpha 1_{\mu^{\alpha - \alpha\varepsilon}}$. Since $\mu^{\alpha - \alpha\varepsilon} \in \mathbb{F}$ so $\nu_\varepsilon \in \mathbb{F}_\alpha$ and also $\mu + \varepsilon \geq \nu_\varepsilon$. That is, $\forall\ \varepsilon \in I_0,\ \mu \geq \nu_\varepsilon - \varepsilon$

$$\Rightarrow \mu \geq \sup_{\varepsilon \in I_0} (\nu_\varepsilon - \varepsilon) \text{ with } \forall\ \varepsilon \in I_0,\ \nu_\varepsilon \in \mathbb{F}_\alpha.$$

Since $\hat{\mathbb{F}_\alpha}$ is a prefilter and so $\mu \in \hat{\mathbb{F}_\alpha}$. Therefore $\mathbb{F}^\alpha \subseteq \hat{\mathbb{F}_\alpha}$. Hence

$$\hat{\mathbb{F}_\alpha} = \hat{\mathbb{F}^\alpha} = \mathbb{F}^\alpha.$$

(5) We have $C(\mathbb{F}_\alpha) = \{\beta \in I : \beta 1_X \notin \mathbb{F}_\alpha\}$. If $\beta < \alpha$ then $\beta 1_X \ngeq \alpha 1_F$ for any $F \in \mathbb{F} \Rightarrow \beta 1_X \notin \mathbb{F}_\alpha$. But if $\beta \geq \alpha$ then $\beta 1_X \geq \alpha 1_F$ for some $F \in \mathbb{F} \Rightarrow \beta 1_X \in \mathbb{F}_\alpha$. Therefore

$$C(\mathbb{F}_\alpha) = [0, \alpha) \text{ and hence } c(\mathbb{F}_\alpha) = \alpha.$$

We have

$$C(\mathbb{F}^\alpha) = \{\beta \in I : \beta 1_X \notin \mathbb{F}^\alpha\}.$$

If $\beta < \alpha$ then $\exists\ \delta : \beta < \delta < \alpha$ and $(\beta 1_X)^\delta = \emptyset \notin \mathbb{F} \Rightarrow \beta 1_X \notin \mathbb{F}^\alpha$.
If $\beta \geq \alpha$ then $\forall\ \gamma < \alpha,\ (\beta 1_X)^\gamma = X \in \mathbb{F} \Rightarrow \beta 1_X \in \mathbb{F}^\alpha$.
Therefore

$$C(\mathbb{F}^\alpha) = [0, \alpha) \text{ and hence } c(\mathbb{F}^\alpha) = \alpha$$

□

5.3 Filters from Prefilters

5.3.1 Definition

Let $\mathcal{F}$ be a prefilter on X. For $\alpha \in C(\mathcal{F})$, we define,

$$\mathcal{F}_\alpha \stackrel{\text{def}}{=} \{\nu^\alpha : \nu \in \mathcal{F}\} = \{F \subseteq X : \alpha 1_X \vee 1_F \in \mathcal{F}\},$$

and for $\alpha \in (0, c(\mathcal{F})]$ we define,

$$\mathcal{F}^\alpha \stackrel{\text{def}}{=} \{\nu^\beta : \beta < \alpha,\ \nu \in \mathcal{F}\}.$$

5.3.2 Theorem

Let $\mathcal{F}$ and $\mathcal{G}$ are prefilters on X. Then we have

1. $\mathcal{F}_\alpha$ for any $\alpha \in C(\mathcal{F})$ and $\mathcal{F}^\alpha$ for any $\alpha \in (0, c(\mathcal{F})]$ are filters on X,
2. $\mathcal{F}_\beta \subseteq \mathcal{F}^\alpha$ if $0 \leq \beta < \alpha \leq c(\mathcal{F})$,
3. $\mathcal{F}_\alpha \supseteq \mathcal{F}_\beta$ if $0 \leq \beta < \alpha$ and $\alpha \in C(\mathcal{F})$; and $\mathcal{F}^\alpha \supseteq \mathcal{F}^\beta$ if $0 < \beta < \alpha \leq c(\mathcal{F})$,
4. $\mathcal{F}_\alpha \subseteq \mathcal{G}_\alpha$ and $\mathcal{F}^\alpha \subseteq \mathcal{G}^\alpha$ if $\mathcal{F} \subseteq \mathcal{G}$,
5. For $\alpha \in (0, c(\mathcal{F})]$, $\mathcal{F}^\alpha = \bigcup\limits_{0<\gamma<\alpha} \mathcal{F}^\gamma$.

Proof.

(1) Let $\alpha \in C(\mathcal{F})$. Then $\alpha 1_X \notin \mathcal{F}$. So, for $\nu \in \mathcal{F}$, $\nu^\alpha \neq \emptyset$. We also know $\mathcal{F}_\alpha \neq \emptyset$.

Let $\mu^\alpha, \nu^\alpha \in \mathcal{F}_\alpha$. Then $\mu^\alpha \cap \nu^\alpha = (\mu \wedge \nu)^\alpha \in \mathcal{F}_\alpha$.

Let $\mu^\alpha \in \mathcal{F}_\alpha$ and $F \supseteq \mu^\alpha$. Then $1_F \vee \mu \geq \mu$, $1_F \vee \mu \in \mathcal{F}$ and $(1_F \vee \mu)^\alpha = F$ Therefore $F \in \mathcal{F}_\alpha$.

Hence $\mathcal{F}_\alpha$ is a filter.

For $\alpha \in (0, c(\mathcal{F})]$, $\mathcal{F}^\alpha \neq \emptyset$ and $\emptyset \notin \mathcal{F}^\alpha$.

Let $\nu^\beta, \mu^\gamma \in \mathcal{F}^\alpha$ where $\nu, \mu \in \mathcal{F}$ and $\beta, \gamma < \alpha$. Then

$$(\nu \wedge \mu)^{\beta \vee \gamma} \subseteq \nu^\beta \cap \mu^\gamma \text{and } \nu \wedge \mu \in \mathcal{F}.$$

Therefore $\mathcal{F}^\alpha$ is a filter base.

Clearly $\mathcal{F}^\alpha \subseteq < \mathcal{F}^\alpha >$. Let $A \in < \mathcal{F}^\alpha >$. Then $\exists\ \nu \in \mathcal{F}$ and $\beta < \alpha$ such that $A \supseteq \nu^\beta$. Define $\mu = 1_A \vee \nu \geq \nu$ then $\mu \in \mathcal{F}$. $\mu^\beta = A \in \mathcal{F}^\alpha$. Therefore $< \mathcal{F}^\alpha >= \mathcal{F}^\alpha$. Hence $\mathcal{F}^\alpha$ is a filter.

(2) Let $\nu^\beta \in \mathcal{F}_\beta$ then we have $\nu \in \mathcal{F}$.

$$0 \leq \beta < \alpha \leq c(\mathcal{F}) \Rightarrow \nu^\beta \in \mathcal{F}^\alpha.$$

Therefore $\mathcal{F}_\beta \subseteq \mathcal{F}^\alpha$

(3) Let $\nu^\beta \in \mathcal{F}_\beta$. Then we have $\nu \in \mathcal{F}$. Define $\mu = 1_{\nu^\beta} \vee \nu \geq \nu$ then $\mu \in \mathcal{F}$. Since $0 \leq \beta < \alpha$. So $\mu^\alpha = \nu^\beta$. Therefore $\nu^\beta \in \mathcal{F}_\alpha$. Hence $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$.

Let $\nu^\gamma \in \mathcal{F}^\beta$. Then $\gamma < \beta$ and $\nu \in \mathcal{F}$. Since $0 < \beta < \alpha \leq c(\mathcal{F})$ and so $\gamma < \alpha$ and $\nu \in \mathcal{F}$. Therefore $\nu^\gamma \in \mathcal{F}^\alpha$. Hence $\mathcal{F}^\beta \subseteq \mathcal{F}^\alpha$.

(4) Let $\nu^\alpha \in \mathcal{F}_\alpha$. Then $\nu \in \mathcal{F}$ and so $\nu \in \mathcal{G}$. Therefore $\nu^\alpha \in \mathcal{G}_\alpha$.

Let $\nu^\beta \in \mathcal{F}^\alpha$. Then $\beta < \alpha$ and $\nu \in \mathcal{F}$ and so $\beta < \alpha$ and $\nu \in \mathcal{G}$. Therefore $\nu^\beta \in \mathcal{G}^\alpha$.

(5) We have

$$0 < \gamma < \alpha \Rightarrow \mathcal{F}^\gamma \subseteq \mathcal{F}^\alpha \Rightarrow \bigcup_{0<\gamma<\alpha} \mathcal{F}^\gamma \subseteq \mathcal{F}^\alpha.$$

Let $\nu^\beta \in \mathcal{F}^\alpha$. Then $\beta < \alpha$ and $\nu \in \mathcal{F}$. So $\exists\ \gamma : \beta < \gamma < \alpha$ such that $\nu^\beta \in \mathcal{F}^\gamma$. Therefore $\nu^\beta \in \bigcup\limits_{0<\gamma<\alpha} \mathcal{F}^\gamma$. Hence $\mathcal{F}^\alpha = \bigcup\limits_{0<\gamma<\alpha} \mathcal{F}^\gamma$.

□

5.3.3 Theorem

Let $\mathbb{F}$ and $\mathbb{G}$ are filters on X and $\mathcal{F}$ and $\mathcal{G}$ are prefilters on X. Then we have

1. $\mathbb{F} = (\mathbb{F}_\alpha)^\beta = (\mathbb{F}^\alpha)^\beta$ if $0 < \beta \leq \alpha$,
2. $\mathbb{F} = (\mathbb{F}_\alpha)_\beta = (\mathbb{F}^\alpha)_\beta$ if $0 \leq \beta < \alpha$,
3. $(\mathcal{F}^\alpha)^\beta \supseteq \mathcal{F}$ if $0 < \beta \leq \alpha \leq c(\mathcal{F})$,
4. $(\mathcal{F}^\alpha)_\beta \supseteq \mathcal{F}$ if $0 \leq \beta < \alpha \leq c(\mathcal{F})$,
5. $(\mathcal{F}_\alpha)^\beta \supseteq \mathcal{F}$ if $0 < \beta \leq \alpha$ and $\alpha \in C(\mathcal{F})$,
6. $(\mathcal{F}_\alpha)_\beta \supseteq \mathcal{F}$ if $0 \leq \beta < \alpha$ and $\alpha \in (0, c(\mathcal{F})]$,
7. $((\mathcal{F}^\alpha)^\alpha)^\alpha = \mathcal{F}^\alpha$ for any $\alpha \in (0, c(\mathcal{F})]$,
8. if $F \in \mathcal{F}_0$ and $c(\mathcal{F}) < \gamma \leq 1$ then $\gamma 1_F \in \mathcal{F}$.

PROOF.

(1) Let $F \in \mathbb{F}$. Then $\alpha 1_F \in \mathbb{F}_\alpha$ and for $\gamma < \beta \leq \alpha$, $(\alpha 1_F)^\gamma = F \in (\mathbb{F}_\alpha)^\beta$. Therefore $\mathbb{F} \subseteq (\mathbb{F}_\alpha)^\beta$. We have

$$\mathbb{F}_\alpha \subseteq \mathbb{F}^\alpha \Rightarrow (\mathbb{F}_\alpha)^\beta \subseteq (\mathbb{F}^\alpha)^\beta.$$

So, $\mathbb{F} \subseteq (\mathbb{F}_\alpha)^\beta \subseteq (\mathbb{F}^\alpha)^\beta$.

Let $\nu^\gamma \in (\mathbb{F}^\alpha)^\beta$ then $\gamma < \beta$ and $\nu \in \mathbb{F}^\alpha$.

$$\nu \in \mathbb{F}^\alpha \Rightarrow \forall\ \delta < \alpha,\ \nu^\delta \in \mathbb{F}.$$

Since $0 < \beta \leq \alpha,$ so $\nu^\gamma \in \mathbb{F}$. Therefore $(\mathbb{F}^\alpha)^\beta \subseteq \mathbb{F}$.

Hence $\mathbb{F} = (\mathbb{F}_\alpha)^\beta = (\mathbb{F}^\alpha)^\beta$.

(2) Let $F \in \mathbb{F}$. Then $\alpha 1_F \in \mathbb{F}_\alpha$ and for $0 \leq \beta < \alpha$, $(\alpha 1_F)^\beta = F \in (\mathbb{F}_\alpha)_\beta$. Therefore $\mathbb{F} \subseteq (\mathbb{F}_\alpha)_\beta$. We have

$$\mathbb{F}_\alpha \subseteq \mathbb{F}^\alpha \Rightarrow (\mathbb{F}_\alpha)_\beta \subseteq (\mathbb{F}^\alpha)_\beta.$$

So $\mathbb{F} \subseteq (\mathbb{F}_\alpha)_\beta \subseteq (\mathbb{F}^\alpha)_\beta$.

Let $\nu^\beta \in (\mathbb{F}^\alpha)_\beta$. Then $\nu \in \mathbb{F}^\alpha$.

$$\nu \in \mathbb{F}^\alpha \Rightarrow \forall\ \gamma < \alpha,\ \nu^\gamma \in \mathbb{F}.$$

Since $0 \leq \beta < \alpha$ and so $\nu^\beta \in \mathbb{F}$. Therefore $(\mathbb{F}^\alpha)_\beta \subseteq \mathbb{F}$.

Hence $\mathbb{F} = (\mathbb{F}_\alpha)_\beta = (\mathbb{F}^\alpha)_\beta$.

(3) Let $\nu \in \mathcal{F}$. Then since $\alpha \leq c(\mathcal{F})$ so $\forall\ \gamma < \alpha,\ \nu^\gamma \in \mathcal{F}^\alpha$. Since $0 < \beta \leq \alpha$. So $\forall\ \gamma < \beta,\ \nu^\beta \in \mathcal{F}^\alpha$. Therefore $\nu \in (\mathcal{F}^\alpha)^\beta$. Hence $\mathcal{F} \subseteq (\mathcal{F}^\alpha)^\beta$.

(4) Let $\nu \in \mathcal{F}$. Then since $\alpha \leq c(\mathcal{F})$ so $\forall\ \gamma < \alpha,\ \nu^\gamma \in \mathcal{F}^\alpha$. Since $0 \leq \beta < \alpha$. So $\nu \geq \beta 1_{\nu^\beta} \in (\mathcal{F}^\alpha)_\beta$. Therefore $\nu \in (\mathcal{F}^\alpha)_\beta$. Hence $\mathcal{F} \subseteq (\mathcal{F}^\alpha)_\beta$.

(5) Let $\nu \in \mathcal{F}$. Then $\nu^\alpha \in \mathcal{F}_\alpha$. Since $0 < \beta \leq \alpha$. So $\forall\ \gamma < \beta,\ \nu^\gamma \in \mathcal{F}_\alpha$. Therefore $\nu \in (\mathcal{F}_\alpha)^\beta$. Hence $\mathcal{F} \subseteq (\mathcal{F}_\alpha)^\beta$.

(6) Let $\nu \in \mathcal{F}$. Then $\nu^\alpha \in \mathcal{F}_\alpha$. Since $0 \leq \beta < \alpha$. So $\nu \geq \beta 1_{\nu^\alpha} \in (\mathcal{F}_\alpha)_\beta$. Therefore $\nu \in (\mathcal{F}_\alpha)_\beta$. Hence $\mathcal{F} \subseteq (\mathcal{F}_\alpha)_\beta$.

(7) We have

$$(\mathcal{F}^\alpha)^\alpha \supseteq \mathcal{F} \Rightarrow ((\mathcal{F}^\alpha)^\alpha)^\alpha \supseteq \mathcal{F}^\alpha.$$

Let $\nu^\beta \in ((\mathcal{F}^\alpha)^\alpha)^\alpha$. Then $\beta < \alpha$ and $\nu \in (\mathcal{F}^\alpha)^\alpha$. So $\forall\ \gamma < \alpha,\ \nu^\gamma \in \mathcal{F}^\alpha$. Therefore $\nu^\beta \in \mathcal{F}^\alpha$. Hence $((\mathcal{F}^\alpha)^\alpha)^\alpha = \mathcal{F}^\alpha$.

(8) Let $F \in \mathcal{F}_0$. Then $F = \nu_1^0$ for some $\nu_1 \in \mathcal{F}$. We have

$$\gamma > c(\mathcal{F}) = \inf_{\nu \in \mathcal{F}} \sup\ \nu.$$

Therefore $\exists\ \nu_2 \in \mathcal{F}$ such that $\sup\ \nu_2 < \gamma$. Let $\nu = \nu_1 \vee \nu_2$. Then $\nu \in \mathcal{F}$ and $\nu^0 = \nu_1^0 \cap \nu_2^0 \supseteq F$. So $\sup\ \nu \leq \sup\ \nu_2 < \gamma$. Therefore $\nu \leq \gamma 1_F \in \mathcal{F}$. □

The following theorem is very useful when we try to show two saturated prefilters are equal.

5.3.4 Theorem

If $\mathcal{F}$ and $\mathcal{G}$ are saturated prefilters such that $c(\mathcal{F}) = c(\mathcal{G}) = \alpha$ and $\mathcal{F}^\beta = \mathcal{G}^\beta$ for all $\beta \in (0, \alpha]$ then $\mathcal{F} = \mathcal{G}$.

Proof.

Let $\mu \in \mathcal{F}$ and $\varepsilon \in I_0$. We seek $\nu \in \mathcal{G}$ such that $\nu \leq \mu + \varepsilon$. Then, since ε is arbitrary, we will have $\forall\ \varepsilon \in I_0,\ \mu + \varepsilon \in \mathcal{F}$ and since $\mathcal{G}$ is saturated so $\mu \in \mathcal{F}$.

Choose $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n = \alpha$ with $\alpha_i - \alpha_{i-1} < \varepsilon/4$ for each $i \in \{1, 2, \ldots, n\}$.

(i) Since $\mu^{\alpha_0} \in \mathcal{F}^{\alpha_1} = \mathcal{G}^{\alpha_1}$. So $\exists\ \nu_1' \in \mathcal{G}$ and $\beta_1 < \alpha_1$ with $(\nu_1')^{\beta_1} = \mu^{\alpha_0}$. Thus $\exists\ \nu_1 = \nu_1' \wedge \alpha 1_X \in \mathcal{G}$ and $\beta_1 < \alpha_1$ with $\nu_1^{\beta_1} \subseteq (\nu_1')^{\beta_1} = \mu^{\alpha_0}$.

(ii) Since $\mu^{\alpha_1} \in \mathcal{F}^{\alpha_2} = \mathcal{G}^{\alpha_2}$. So $\exists\ \nu_2' \in \mathcal{G}$ and $\beta_2' < \alpha_2$ with $(\nu_2')^{\beta_2'} = \mu^{\alpha_1}$. Thus $\exists\ \nu_2 = \nu_2' \wedge \alpha 1_X \in \mathcal{G}$ and $\beta_2 \beta_2' \vee \alpha_1 < \alpha_2$ with $\nu_2^{\beta_2} \subseteq (\nu_2')^{\beta_2} \subseteq (\nu_2')^{\beta_2'} = \mu^{\alpha_1}$. In general we have:

For each $i \in \{1, 2, \ldots, n\}$, $\exists\ \alpha 1_X \geq \nu_i \in \mathcal{G}$ and $\beta_i \in [\alpha_{i-1}, \alpha_i)$ with $\nu_i^{\beta_i} \subseteq \mu^{\alpha_{i-1}}$. Let $\nu = \bigwedge_{i \in \{1,2,\ldots,n\}} \nu_i$.

Then $\alpha 1_X \geq \nu \in \mathcal{G}$ and $\nu^{\beta_i} \subseteq \nu_i^{\beta_i} \subseteq \mu^{\alpha_{i-1}}$ for each $I \in \{1, 2, \ldots, n\}$.

Thus $\forall\ i \in \{1, 2, \ldots, n\}$ and $\forall\ x \in X$ we have:

$$\nu(x) > \beta_i \Rightarrow \mu(x) > \alpha_{i-1} > \alpha_i 1/4\varepsilon > \beta_i - 1/4\varepsilon.$$

Now we have to show $\nu \leq \mu + \varepsilon$.

Let $x \in X$ and $\nu(x) > \beta$. Then

(a) if $0 \leq \beta < \beta_1$ then $(\mu + \varepsilon)(x) \geq \varepsilon > \alpha_1 > \beta_1 > \beta$.

(b) if $\beta_i \leq \beta < \beta_{i+1}$ for some $i \in \{1, 2, \ldots, n\}$ then

$$\nu(x) > \beta \geq \beta_i \Rightarrow (\mu+\varepsilon)(x) = \mu(x)+\varepsilon > \beta_i - 1/4\varepsilon + \varepsilon = \beta_i + 3/4\varepsilon \geq \alpha_{i-1} + 3/4\varepsilon \geq \alpha_{i+1} > \beta_{i+1} > \beta.$$

(c) if $\beta_n \leq \beta < \alpha$ then

$$\nu(x) > \beta \geq \beta_n \Rightarrow (\mu+\varepsilon)(x) = \mu(x)+\varepsilon \geq \beta_n - 1/4\varepsilon + \varepsilon = \beta_n + 3/4\varepsilon \geq \alpha_{n-1} + 3/4\varepsilon > \alpha > \beta.$$

Therefore in any case

$$\nu(x) > \beta \Rightarrow (\mu + \varepsilon)(x) > \beta.$$

Thus $\mu(x) \leq (\mu + \varepsilon)(x)$. But x is arbitrary therefore $\nu \leq \mu + \varepsilon$.

□

5.3.5 Proposition

Let $\mathcal{F}$ be a saturated prefilter and $c(\mathcal{F}) = \alpha$. Then

$$\mathcal{F} = \bigcap_{\beta < \alpha} (\mathcal{F}^\beta)^\beta.$$

PROOF.

Let $\mathcal{G} = \bigcap\limits_{\beta<\alpha} (\mathcal{F}^{\beta})^{\beta}$. Then $\mathcal{G}$ is a saturated prefilter, as an arbitrary intersection of saturated perfilters.

$$\begin{aligned} t1_X \in \mathcal{G} &\iff \forall\ \beta < \alpha,\ t1_X \in (\mathcal{F}^{\beta})^{\beta} \\ &\iff \forall\ \beta < \alpha,\ \forall\ \gamma < \beta,\ (t1_X)^{\gamma} \in \mathcal{F}^{\beta} \\ &\iff \forall\ \beta < \alpha,\ \forall\ \gamma < \beta,\ \gamma < t \\ &\iff \forall\ \beta < \alpha,\ \beta \leq t \\ &\iff t \geq \alpha. \end{aligned}$$

Therefore $C(\mathcal{G}) = [0, \alpha) = C(\mathcal{G})$. Now we have to show $\forall\ \beta \in (0, \alpha],\ \mathcal{F}^{\beta} = \mathcal{G}^{\beta}$. Since $\mathcal{G} \supseteq \mathcal{F}$, we have $\mathcal{G}^{\beta} \supseteq \mathcal{F}^{\beta}$.

On the other hand $\mathcal{G} \subseteq (\mathcal{F}^{\beta})^{\beta}$ and so $\mathcal{G}^{\beta} \subseteq ((\mathcal{F}^{\beta})^{\beta})^{\beta} = \mathcal{F}^{\beta}$. Therefore by above theorem $\mathcal{F} = \mathcal{G} = \bigcap\limits_{\beta<\alpha} (\mathcal{F}^{\beta})^{\beta}$.

□

5.3.6 Theorem

Let $\mathcal{F}$ be a prefilter on X with $c(\mathcal{F}) = c > 0$ and $\mathbb{F}$ is a filter on X. Then

1. $\mathcal{F}$ is prime $\Leftrightarrow$ $\mathcal{F}_0$ is an ultrafilter,
2. $\mathcal{F}$ is prime $\Rightarrow \mathcal{F}_0 = \mathcal{F}^c$,
3. $\mathbb{F}$ is ultrafilter $\Leftrightarrow \mathbb{F}_c$ is prime, for any $\alpha \in (0, 1]$,
4. $\mathbb{F}$ is ultrafilter $\Leftrightarrow \mathbb{F}^c$ is prime, for any $\alpha \in (0, 1]$,
5. if $\mathcal{F}$ is prime and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G}$ is prime.

PROOF.

(1)
($\Rightarrow$)

Let $A \cup B \in \mathcal{F}_0$. Then $1_{A\cup B} \in \mathcal{F}$. $1_{A\cup B} = 1_A \vee 1_B \in \mathcal{F}$. Since $\mathcal{F}$ is prime. So $1_A \in \mathcal{F}$ or $1_B \in \mathcal{F}$. Therefore $A \in \mathcal{F}_0$ or $B \in \mathcal{F}_0$ and so $\mathcal{F}_0$ is prime. That is $\mathcal{F}_0$ is ultra.

($\Leftarrow$)

Let $\mu \vee \nu \in \mathcal{F}$. Then $(\mu \vee \nu)^0 = \mu^0 \cup \nu^0 \in \mathcal{F}_0$. Since $\mathcal{F}_0$ is an ultrafiter. So $\mu^0 \in \mathcal{F}_0$ or $\nu^0 \in \mathcal{F}_0$. Therefore $\mu \in \mathcal{F}$ or $\nu \in \mathcal{F}$ and hence $\mathcal{F}$ is prime.

(2) $\mathcal{F}_0 \subseteq \mathcal{F}^c$ and $\mathcal{F}_0$ is ultra $\Rightarrow \mathcal{F}_0 = \mathcal{F}^c$.

(3) We have $\mathbb{F} = (\mathbb{F}_{\alpha})_0$ and by (1)

$$\mathbb{F} = (\mathbb{F}_{\alpha})_0 \text{ is an ultrafilter } \iff \mathbb{F}_{\alpha} \text{ is prime }.$$

(4) We have $\mathbb{F} = (\mathbb{F}^{\alpha})_0$ and by (1)

$$\mathbb{F} = (\mathbb{F}^{\alpha})_0 \text{ is an ultrafilter } \iff \mathbb{F}^{\alpha} \text{ is prime }.$$

(5) We have $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{F}_0 \subseteq \mathcal{G}_0$ and $\mathcal{F}$ is prime $\iff \mathcal{F}_0$ is ultra. Therefore $\mathcal{G}_0$ is ultra and hence $\mathcal{G}$ is prime.

□

5.3.7 Lemma

If $\mathcal{F}$ is a prefilter on X and $\mathbb{F}$ is a filter on X then

1. $\mathcal{F} \sim \mathbb{F}_1 \Rightarrow (\mathcal{F} \vee \mathbb{F}_1)_0 = \mathcal{F}_0 \vee \mathbb{F}$,

2. If $\mathbb{F}$ is an ultrafilter then
$\mathcal{F} \sim \mathbb{F}_1 \Leftrightarrow \mathcal{F}_0 \subseteq \mathbb{F} \Leftrightarrow (\mathcal{F} \vee \mathbb{F}_1)_0 = \mathbb{F}$.

Proof.

(1)

$$\begin{aligned} \nu^0 \in (\mathcal{F} \vee \mathbb{F}_1)_0 &\Rightarrow \nu \geq \mu \wedge 1_F \text{ for some } \mu \in \mathcal{F},\ F \in \mathbb{F} \\ &\Rightarrow \nu^0 \supseteq \mu^0 \cap F \text{ with } \mu \in \mathcal{F},\ F \in \mathbb{F} \\ &\Rightarrow \nu^0 \in \mathcal{F}_0 \vee \mathbb{F}. \end{aligned}$$

Let $A \in \mathcal{F}_0 \vee \mathbb{F}$. Then $A \supseteq \mu^0 \cap F$ for some $\mu \in \mathcal{F},\ F \in \mathbb{F}$. So $A \supseteq (\mu \wedge 1_F)^0$ with $\mu \in \mathcal{F},\ F \in \mathbb{F}$. Therefore $A \in (\mathcal{F} \vee \mathbb{F}_1)_0$. Hence $(\mathcal{F} \vee \mathbb{F}_1)_0 = \mathcal{F} \vee \mathbb{F}$.

(2) Let $\mathbb{F}$ is an ultrafilter. Then

$$\begin{aligned} \mathcal{F} \sim \mathbb{F}_1 &\Longleftrightarrow \forall\ \nu \in \mathcal{F},\ \forall\ F \in \mathbb{F},\ \nu \vee 1_F \neq 0 \\ &\Longleftrightarrow \forall\ \nu \in \mathcal{F},\ \forall\ F \in \mathbb{F},\ \nu^0 \cap F \neq \emptyset \\ &\Longleftrightarrow \forall\ \nu \in \mathcal{F},\ \nu^0 \in \mathbb{F} \\ &\Longleftrightarrow \mathcal{F}_0 \subseteq \mathbb{F}. \end{aligned}$$

$\mathcal{F} \sim \mathbb{F}_1 \Rightarrow (\mathcal{F} \vee \mathbb{F}_1)_0 = \mathcal{F}_0 \vee \mathbb{F} = \mathbb{F}$. Clearly, $(\mathcal{F} \vee \mathbb{F}_1)_0 = \mathbb{F} \Rightarrow \mathcal{F} \sim \mathbb{F}_1$. □

The following definition aims to characterises the collection of minimal prime prefilters.

5.3.8 Definitions
If $\mathcal{F}$ is a prefilter and $\mathbb{F}$ is filter on X we define,

$$\mathbb{P}(\mathbb{F}) \stackrel{\text{def}}{=} \{\mathbb{K} : \mathbb{K} \text{ is an ultrafilter and } \mathbb{F} \subseteq \mathbb{K}\};$$

$$\mathcal{P}(\mathcal{F}) \stackrel{\text{def}}{=} \{\mathcal{G} : \mathcal{G} \text{ is a prime prefilter and } \mathcal{F} \subseteq \mathcal{G}\};$$

$$\mathcal{P}_m(\mathcal{F}) \stackrel{\text{def}}{=} \{\mathcal{G} : \mathcal{G} \in \mathcal{P}(\mathcal{F}) \text{ and } \mathcal{G} \text{ is minimal }\}.$$

5.3.9 Lemma

1. $\mathcal{P}(\mathcal{F})$ has minimal elements,

2. $\mathcal{F} = \bigcap\limits_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$,

3. $\mathcal{F} = \bigcap\limits_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \mathcal{G}$.

Proof.

(1) For $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{P}(\mathcal{F})$ define,

$$\mathcal{G}_1 \preceq \mathcal{G}_2 \Longleftrightarrow \mathcal{G}_1 \supseteq \mathcal{G}_2$$

Then $\mathcal{P}(\mathcal{F})$ is a partially ordered set. Let $\mathcal{C}$ be a non-empty chain in $\mathcal{P}(\mathcal{F})$. Then $\mathcal{H} = \bigcap\limits_{\mathcal{G} \in \mathcal{C}} \mathcal{G}$ is a prime prefilter and $\mathcal{F} \subseteq \mathcal{H}$.
Clearly $\mathcal{F} \subseteq \mathcal{H}$,
We have $\mathcal{H} \neq \emptyset$ and $0 \notin \mathcal{H}$.
Let $\mu, \nu \in \mathcal{H}$. Then $\forall\ \mathcal{G} \in \mathcal{C};\ \mu, \nu\mathcal{G} \Rightarrow \mu \wedge \nu \in \mathcal{H}$.
Let $\mu \in \mathcal{H}$ and $\mu \leq \nu$ then $\forall\ \mathcal{G} \in \mathcal{C},\ \mu \in \mathcal{G}$ and $\mu \leq \nu \Rightarrow \nu \in \mathcal{H}$.
Therefore $\mathcal{H}$ is a prefilter.
Let $\mu \vee \nu \in \mathcal{H}$ then $\forall\ \mathcal{G} \in \mathcal{C},\ \mu \vee \nu \in \mathcal{G}$. So

$$\forall\ \mathcal{G} \in \mathcal{C},\ (\mu \in \mathcal{G} \text{ or } \nu \in \mathcal{G}).$$

If $\forall\ \mathcal{G} \in \mathcal{C},\ \mu \in \mathcal{G}$ then $\mu \in \mathcal{H}$.

If $\exists\ \mathcal{G}' \in \mathcal{C}$ such that $\mu \notin \mathcal{G}'$ then $\nu \in \mathcal{G}'$

Therefore $\forall\ \mathcal{G} \in \mathcal{C}$ such that $\mathcal{G} \supseteq \mathcal{G}'$ we have $\nu \in \mathcal{G}$ and if $\mathcal{G} \in \mathcal{C}$ such that $\mathcal{G} \subseteq \mathcal{G}'$ then $\mu \notin \mathcal{G}$, so we must again have $\nu \in \mathcal{G}$. Therefore $\nu \in \mathcal{H}$. Hence $\mathcal{H}$ is a prime prefilter.

So, we have $\mathcal{H} \in \mathcal{P}(\mathcal{F})$ and an upper bound for $\mathcal{C}$. By Zorn's Lemma, $(\mathcal{P}(\mathcal{F}),\ \preceq)$ has a maximal element. That is $(\mathcal{P}(\mathcal{F}),\ \subseteq)$ has a minimal element. Therefore $\mathcal{P}_m(\mathcal{F}) \neq \emptyset$.

(2) Firat we have to show that

$$\mathcal{G} \in \mathcal{P}(\mathcal{F}) \Longleftrightarrow \mathcal{G}_0 \in \mathbb{P}(\mathcal{F}_0) \text{ and } \mathcal{F}_0 = \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}_0.$$

$$\begin{aligned} \mathcal{G} \in \mathcal{P}(\mathcal{F}) &\Longleftrightarrow \mathcal{G} \text{ is prime prefilter and } \mathcal{F} \subseteq \mathcal{G} \\ &\Longleftrightarrow \mathcal{G}_0 \text{ is ultra and } \mathcal{F}_0 \subseteq \mathcal{G}_0 \\ &\Longleftrightarrow \mathcal{G}_0 \in \mathbb{P}(\mathcal{F}_0). \end{aligned}$$

We have

$$\mathcal{F}_0 = \bigcap_{\mathbb{F} \in \mathbb{P}(\mathcal{F}_0)} \mathbb{F}.$$

$$\begin{aligned} \mathbb{F} \in \mathbb{P}(\mathcal{F}_0) &\Rightarrow \mathbb{F} \text{ is an ultra filter and } \mathcal{F}_0 \subseteq \mathbb{F} \\ &\Rightarrow (\mathcal{F} \vee \mathbb{F}_1)_0 = \mathbb{F} \text{ and } \mathbb{F} \text{ is ultra.} \\ &\Rightarrow \mathcal{F} \vee \mathbb{F}_1 \text{ is prime and } \mathcal{F} \subseteq \mathcal{F} \vee \mathbb{F}_1 \\ &\Rightarrow \mathcal{F} \vee \mathbb{F}_1 \in \mathcal{P}(\mathcal{F}). \end{aligned}$$

That is we have

$$\mathcal{G} \in \mathcal{P}(\mathcal{F}) \Rightarrow \mathcal{G}_0 \in \mathcal{P}(\mathcal{F}_0) \text{ and}$$

$$\mathbb{F} \in \mathbb{P}(\mathcal{F}_0) \Rightarrow (\mathcal{F} \vee \mathbb{F}_1 \in \mathcal{P}(\mathcal{F}) \text{ and } (\mathcal{F} \vee \mathbb{F}_1)_0 = \mathbb{F}).$$

Therefore

$$\mathcal{F}_0 = \bigcap_{\mathbb{F} \in \mathbb{P}(\mathcal{F}_0)} \mathbb{F} = \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}_0.$$

Clearly $\mathcal{F} \subseteq \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$

Let $\mu \in \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$. Then $\forall\, \mathcal{G} \in \mathcal{P}(\mathcal{F}),\ \mu \in \mathcal{G}$. So $\forall\, \mathcal{G} \in \mathcal{P}(\mathcal{F}),\ \mu^0 \in \mathcal{G}_0$. Therefore $\mu^0 \in \mathcal{F}_0$ and so $\mu \in \mathcal{F}$. Hence $\mathcal{F} = \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$.

(3) If $\mathcal{G} \in \mathcal{P}_m(\mathcal{F})$ then $\mathcal{G} \in \mathcal{P}(\mathcal{F})$ and $\mathcal{G}$ is minimal.

So obviously, $\mathcal{F} = \bigcap_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \mathcal{G}$.

□

The following crucial theorem due to Lowen [48] characterises the mimimal prime prefilters.

5.3.10 Theorem

If $\mathcal{F}$ is a prefilter then

$$\mathcal{P}_m(\mathcal{F}) = \{\mathcal{F} \vee \mathbb{F}_1 : \mathbb{F} \in \mathbb{P}(\mathcal{F}_0)\}.$$

Proof.

Let $\mathcal{G} \in \mathcal{P}_m(\mathcal{F})$. Then $\mathcal{G} \in \mathcal{P}(\mathcal{F})$ and $\mathcal{G}$ is minimal. We have

$$\mathcal{G} \in \mathcal{P}(\mathcal{F}) \Rightarrow \mathcal{G}_0 \in \mathbb{P}(\mathcal{F}_0).$$

and

$$\mathcal{G}_0 \text{ is ultra } \Rightarrow (\mathcal{G}_0)_1 \text{ is prime } \Rightarrow \mathcal{F} \vee (\mathcal{G}_0)_1 \text{ is prime .}$$

Therefore $\mathcal{F} \vee (\mathcal{G}_0)_1 \in \mathcal{P}(\mathcal{F})$. We have

$$\mathcal{G}_0 = \{F \subseteq X : 1_F \in \mathcal{G}\}.$$

and so

$$(\mathcal{G}_0)_1 =< \{1_F : F \in \mathcal{G}_0\} >=< \{1_F : F \subseteq X,\ 1_F \in \mathcal{G}\} >\subseteq \mathcal{G}.$$

So $(\mathcal{G}_0)_1 \subseteq \mathcal{G}$ and $\mathcal{F} \subseteq \mathcal{G}$. Therefore $\mathcal{F} \vee (\mathcal{G}_0)_1 \subseteq \mathcal{G}$. But $\mathcal{G}$ is minimal. Hence $\mathcal{G} = \mathcal{F} \vee (\mathcal{G}_0)_1$. That is, if $\mathcal{G} \in \mathcal{P}_m(\mathcal{F})$ then $\mathcal{G} = \mathcal{F} \vee (\mathcal{G}_0)_1$ and $\mathcal{G}_0 \in \mathbb{P}(\mathcal{F}_0)$. Therefore

$$\mathcal{P}_m(\mathcal{F}) \subseteq \{\mathcal{F} \vee \mathbb{F}_1 : \mathbb{F} \in \mathbb{P}(\mathcal{F}_0)\}.$$

Let $\mathbb{F} \in \mathbb{P}(\mathcal{F}_0)$. Then $\mathcal{F} \vee \mathbb{F}_1 \in \mathcal{P}(\mathcal{F})$.

If $\exists \mathcal{G} \in \mathcal{P}_m(\mathcal{F})$ such that $\mathcal{G} \subset \mathcal{F} \vee \mathbb{F}_1$ then

$$\mathcal{G} = \mathcal{F} \vee (\mathcal{G}_0)_1 \subset \mathcal{F} \vee \mathbb{F}_1.$$

So $(\mathcal{G}_0)_1 \subset \mathbb{F}_1 \Rightarrow \mathcal{G}_0 \subset \mathbb{F}$. This is a contradiction, since $\mathbb{F}, \mathcal{G}_0 \in \mathbb{P}(\mathcal{F}_0)$. Therefore $\mathcal{F} \vee \mathbb{F}_1 \in \mathcal{P}_m(\mathcal{F})$. Hence

$$\mathcal{P}_m(\mathcal{F}) = \{\mathcal{F} \vee \mathbb{F}_1 : \mathbb{F} \in \mathbb{P}(\mathcal{F}_0)\}.$$

□

5.3.11 Theorem

If $\mathcal{F}$ is a prefilter then $\exists \mathcal{G} \in \mathcal{P}_m(\mathcal{F})$ with $c(\mathcal{F}) = c(\mathcal{G})$.

PROOF.

If $c = c(\mathcal{F}) > 0$ then choose $\mathbb{F} \in \mathbb{P}(\mathcal{F}^c)$. Then we have $\mathbb{F} \in \mathbb{P}(\mathcal{F}_0)$ and so $\mathcal{F} \vee \mathbb{F}_1 \in \mathcal{P}_m(\mathcal{F})$. Therefore $c(\mathcal{F}) \geq c(\mathcal{F} \vee \mathbb{F}_1)$. Let $0 < \alpha < c(\mathcal{F})$ and $\mu \in \mathcal{F} \vee \mathbb{F}_1$. Then $\exists \nu \in \mathcal{F},\ F \in \mathbb{F}$ such that $\mu \geq \nu \wedge 1_F$. Now $\nu^\alpha \in \mathcal{F}^c \subseteq \mathbb{F}$ and so $\nu^\alpha \cap F \neq \emptyset$. Let $x \in \nu^\alpha \cap F$. Then $\sup \mu \geq \sup \nu \wedge 1_F \geq \nu(x) > \alpha$. Since μ is arbitrary we have $c(\mathcal{F} \vee \mathbb{F}_1) \geq \alpha$ and since α is arbitrary we conclude that $c(\mathcal{F} \vee \mathbb{F}_1) \geq c(\mathcal{F})$. Therefore $c(\mathcal{F}) = c(\mathcal{F} \vee \mathbb{F}_1)$. If $c(\mathcal{F}) = 0$ then choose $\mathbb{F} \in \mathbb{P}(\mathcal{F}_0)$. Therefore $\mathcal{F} \vee \mathbb{F}_1 \in \mathcal{P}_m(\mathcal{F})$ and so $0 = c(\mathcal{F}) \geq c(\mathcal{F} \vee \mathbb{F}_1)$. Therefore $c(\mathcal{F}) = c(\mathcal{F} \vee \mathbb{F}_1)$.
□

For a prefilter we now see the definition of lower characteristic.

5.3.12 Definition

For a prefilter $\mathcal{F}$ we define the *lower characteristic* of $\mathcal{F}$ by

$$\bar{c}(\mathcal{F}) \stackrel{\text{def}}{=} \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{G})$$

and it is easy to see that $\bar{c}(\mathcal{F}) = c(\mathcal{F})$ when $\mathcal{F}$ is prime.

5.3.13 Lemma

If $\mathcal{F}$ is a prefilter with $\bar{c}(\mathcal{F}) > 0$ then

$$0 < \alpha < \bar{c}(\mathcal{F}) \Leftrightarrow \forall\ \nu \in \mathcal{F},\ \nu^\alpha \in \mathcal{F}_0.$$

PROOF.

We have

$$\begin{aligned} \bar{c}(\mathcal{F}) &= \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{F}) = \inf_{\mathbb{F} \in \mathbb{P}(\mathcal{F}_0)} c(\mathcal{F} \vee \mathcal{F}_1) \\ &= \inf_{\mathbb{F} \in \mathbb{P}(\mathcal{F}_0)} \inf_{\nu \in \mathcal{F}} \inf_{F \in \mathbb{F}} \sup(\nu \wedge 1_F). \end{aligned}$$

So

$$\begin{aligned} 0<\alpha<\bar{c}(\mathcal{F}) &\iff \forall\, \mathbb{F}\in\mathbb{P}(\mathcal{F}_0),\ \forall\,\nu\in\mathcal{F},\ \forall\,F\in\mathbb{F},\ \nu^\alpha\cap F\neq\emptyset \\ &\iff \forall\,\nu\in\mathcal{F},\ \forall\,\mathbb{F}\in\mathbb{P}(\mathcal{F}_0),\ \nu^\alpha\in\mathbb{F} \\ &\iff \forall\,\nu\in\mathcal{F},\ \nu^\alpha\in\bigcap_{\mathbb{F}\in\mathbb{P}(\mathcal{F}_0)}\mathbb{F}=\mathcal{F}_0. \end{aligned}$$

□

5.3.14 Proposition

If $\mathcal{F}$ is a prefilter with $\bar{c}(\mathcal{F})>0$ and $\mathbb{F}$ is a filter then

1. $\bar{c}(\mathcal{F})=\inf\limits_{\nu\in\mathcal{F}}\sup\{\alpha:\nu^\alpha\in\mathcal{F}_0\}=\sup\{\alpha:\mathcal{F}_\alpha=\mathcal{F}_0\}$,
2. If $\mathcal{F}=\hat{\mathcal{F}}$ then $(\mathcal{F}^{\bar{c}})_c\subseteq\mathcal{F}$, where $\bar{c}=\bar{c}(\mathcal{F})$ and $c=c(\mathcal{F})$,
3. $\bar{c}(\hat{\mathcal{F}})\geq\bar{c}(\mathcal{F})$,
4. $\bar{c}(\mathbb{F}_\alpha)=\alpha$ and $\bar{c}(\mathbb{F}^\alpha)\geq\alpha$, for $\alpha\in(0,1]$.

Proof.

(1) Let $\beta=\inf\limits_{\nu\in\mathcal{F}}\ \sup\{\alpha:\nu^\alpha\in\mathcal{F}\}$. Then

$$\begin{aligned} 0<\alpha<\bar{c}(\mathcal{F}) &\iff \forall\,\nu\in\mathcal{F},\ \nu^\alpha\in\mathcal{F}_0 \\ &\iff \forall\,\nu\in\mathcal{F},\ \alpha\leq\sup\{\delta:\nu^\delta\in\mathcal{F}_0\} \\ &\iff \alpha\leq\inf_{\nu\in\mathcal{F}}\ \sup\{\delta:\nu^\delta\in\mathcal{F}_0\}=\beta. \end{aligned}$$

That is $\alpha<\bar{c}(\mathcal{F})\iff\alpha\leq\beta$. So $\bar{c}(\mathcal{F})=\beta$. Now we have

$$\begin{aligned} \alpha<\bar{c}(\mathcal{F}) &\iff \forall\,\nu\in\mathcal{F},\ \nu^\alpha\in\mathcal{F}_0 \\ &\iff \mathcal{F}_\alpha\subseteq\mathcal{F}_0 \\ &\iff \alpha\leq\sup\{\delta:\mathcal{F}_\delta\subseteq\mathcal{F}_0\}=\sup\{\delta:\mathcal{F}_\delta=\mathcal{F}_0\}. \end{aligned}$$

So $\bar{c}=\sup\{\delta:\mathcal{F}_\delta=\mathcal{F}_0\}$.

(2) let $\mu\in(\mathcal{F}^{\bar{c}})_c$. Then $\mu\geq c1_{\nu^\beta}$ where $\beta<\bar{c}$ and $\nu\in\mathcal{F}$. $\beta<\bar{c}\Rightarrow\nu^\beta\in\mathcal{F}_0$. We intend to show that $\mu\in\hat{\mathcal{F}}$.

Let $\varepsilon\in I_0$ and define $\gamma_\varepsilon=(c+\varepsilon/2)\wedge 1$ and $\nu_\varepsilon=\gamma_\varepsilon 1_{\nu^\beta}$. Then $\mu+\varepsilon\geq\nu_\varepsilon$ and $\nu_\varepsilon\in\mathcal{F}$. Since $\nu^\beta\in\mathcal{F}_0$ and $c(\mathcal{F})<\gamma_\varepsilon\leq 1$. Therefore $\forall\,\varepsilon\in I_0$, $\exists\nu_\varepsilon\in\mathcal{F}$ such that $\nu_\varepsilon\leq\mu+\varepsilon$. Hence $\mu\in\hat{\mathcal{F}}=\mathcal{F}$.

(3) Let $0<\alpha<\bar{c}(\mathcal{F})$, $\mathcal{G}\in\mathcal{P}_m(\hat{\mathcal{F}})$ and $\mu\in\mathcal{G}$. Then $\exists\mathbb{F}\in\mathbb{P}(\hat{\mathcal{F}}_0)$ such that $\mathcal{G}=\hat{\mathcal{F}}\vee\mathbb{F}_1$ and hence $\mu\geq\nu\wedge 1_F$ where $\nu\in\hat{\mathcal{F}}$ and $F\in\mathbb{F}$.

Now let $\nu=\sup\limits_{\varepsilon\in I_0}\ (\nu_\varepsilon-\varepsilon)$ for some family $(\nu_\varepsilon:\varepsilon\in I_0)\in\mathcal{F}^{I_0}$. Choose β such that $\alpha<\beta<\bar{c}(\mathcal{F})$ and let $\delta=\beta-\alpha$. Then we have $\nu_\delta\in\mathcal{F}$ and $\nu\geq\nu_\delta-\delta$. $\beta<\bar{c}(\mathcal{F})\Rightarrow\nu_\delta^\beta\in\mathcal{F}_0\subseteq(\hat{\mathcal{F}})_0\subseteq\mathbb{F}$ and hence $\nu_\delta^\beta\cap F\neq\emptyset$. Chosse $x\in\nu_\delta^\beta\cap F$,

$$\sup\ \mu\geq\sup\ \nu\wedge 1_F=\sup_{y\in F}\ \nu(y)\geq\nu(x)\geq\nu_\delta(x)-\delta>\beta-\delta=\alpha.$$

Since μ is arbitrary, $c(\mathcal{G})\geq\alpha$. Since $\mathcal{G}$ is arbitrary $\bar{c}(\hat{\mathcal{F}})\geq\alpha$ and since α is arbitrary,

$$\bar{c}(\hat{\mathcal{F}})\geq\bar{c}(\mathcal{F}).$$

(4) Let $\mathcal{G}\in\mathcal{P}_m(\mathbb{F}_\alpha)$. Then $\exists\mathbb{K}\in\mathbb{P}((\mathbb{F}_\alpha)_0)$ such that $\mathcal{G}=\mathbb{F}_\alpha\vee\mathbb{K}_1$. But $(\mathbb{F}_\alpha)_0=\mathbb{F}$. So $\mathbb{K}\in\mathbb{P}(\mathbb{F})$ and

$$\begin{aligned} c(\mathcal{G})=c(\mathbb{F}_\alpha\vee\mathbb{K}_1) &= \inf_{F\in\mathbb{F}}\ \inf_{K\in\mathbb{K}}\ \sup\ (\alpha 1_F\wedge 1_K) \\ &= \alpha.\ [\text{ since } F\cap K\neq\emptyset] \end{aligned}$$

and so $\bar{c}(\mathbb{F}_\alpha) = \alpha$. Therefore $\bar{c}(\mathbb{F}^\alpha) = \bar{c}(\hat{\mathbb{F}_\alpha}) \geq \bar{c}(\mathbb{F}_\alpha) = \alpha$.

□

It is worth noting that inequality in 3 may be strict. This can be seen by letting $\mathcal{F} =< \{\alpha \vee 1_F : \alpha > 0, F \in \mathbb{F}\} >$. Where $\mathbb{F}$is a filter on a set X. Then $\hat{\mathcal{F}} =< \{1_F : F \in \mathbb{F}\} >= \mathbb{F}^1$ and hence $\bar{c}(\mathcal{F}) = 0$ but $\bar{c}(\hat{\mathcal{F}}) = 1$.

5.4 Images and Preimages

If $f : X \longrightarrow Y$,$\mathcal{F}$ is a prefilter on X and $\mathcal{G}$ is a prefilter on Y then

$$f[\mathcal{F}] \stackrel{\text{def}}{=} \{f[\mu] : \mu \in \mathcal{F}\} \text{ and } f^{-1}[\mathcal{G}] \stackrel{\text{def}}{=} \{f^{-1}[\nu] : \nu \in \mathcal{G}\}.$$

5.4.1 Lemma

Let $f : X \longrightarrow Y$,$\mathcal{F}$ is a prefilter on X and $\mathcal{G}$ is a prefilter on Y. Then

1. $f[\mathcal{F}]$ is a prefilter base on Y,
2. f is sujective then $f^{-1}[\mathcal{G}]$ is a prefilter base on X,
3. If $c(\mathcal{F}) = c(f[\mathcal{F}])$,
4. If f is injective then $c(f^{-1}[\mathcal{G}]) = c(\mathcal{G})$,
5. If f is injective and $\mu \in I^X$ then $c(\mathcal{G}, f[\mu]) = c(f^{-1}[\mathcal{G}], \mu)$,
6. If $\mathcal{F}$ is prime then $< f[\mathcal{F}] >$ is prime,
7. $\bar{c}(f[\mathcal{F}]) \geq \bar{c}(\mathcal{F})$.

Proof.

(1) Clearly $f[\mathcal{F}] \neq \emptyset$ and $0 \notin f[\mathcal{F}]$.
Let $f[\mu_1], f[\mu_2] \in f[\mathcal{F}]$ with $\mu_1, \mu_2 \in \mathcal{F}$. Then $f[\mu_1] \wedge f[\mu_2] \geq f[\mu_1 \wedge \mu_2]$ with $\mu_1 \wedge \mu_2 \in \mathcal{F}$. Therefore $f[\mathcal{F}]$ is a prefilter base on Y.

(2) Clearly $f[\mathcal{F}] \neq \emptyset$. Since f is surjective. We have $0 \notin f[\mathcal{F}]$.
Let $f^{-1}[\nu_1], f^{-1}[\nu_2] \in f^{-1}[\mathcal{G}]$ with $\nu_1, \nu_2 \in \mathcal{G}$. Then $f^{-1}[\nu_1] \wedge f^{-1}[\nu_2] = f^{-1}[\nu_1 \wedge \nu_2] \in f^{-1}[\mathcal{G}]$.
Therefore $f^{-1}[\mathcal{G}]$ is a prefilter base on X.

(3) We have

$$c(\mathcal{F}) = \inf_{\mu \in \mathcal{F}} \sup \ \mu \text{ and } c(f[\mathcal{F}]) = \inf_{\mu \in \mathcal{F}} \sup \ f[\mu].$$

and

$$\begin{aligned} \sup \ f[\mu] &= \sup_{y \in Y} f[\mu](y) = \sup_{y \in Y} \sup_{f(x)=y} \mu(x) \\ &= \sup_{x \in X} \mu(x) \ \ [\text{since } X = \bigcup_{y \in Y} f^{\leftarrow}\{y\}] \\ &= \sup \ \mu. \end{aligned}$$

Therefore $c(\mathcal{F}) = c(f[\mathcal{F}])$.

(4) we have

$$\begin{aligned} c(f^{-1}[\mathcal{G}] &= \inf_{\nu \in \mathcal{G}} \sup \ f^{-1}[\nu] = \inf_{\nu \in \mathcal{G}} \sup_{x \in X} \nu(f(x)) \\ &= \inf_{\nu \in \mathcal{G}} \sup_{y \in Y} \nu(y) \\ &= c(\mathcal{G}). \end{aligned}$$

(5) We have

$$\begin{aligned} \mathcal{G} \sim f[\mu] &\iff \forall\, \nu \in \mathcal{F},\ \nu \wedge f[\mu] \neq 0 \\ &\iff \forall\, \nu \in \mathcal{F},\ f^{-1}[\nu] \wedge \mu \neq 0 \quad [\text{ since } f \text{ is injective}] \\ &\iff f^{-1}[\mathcal{G}] \sim \mu. \end{aligned}$$

Since f is injective. So $\nu \wedge f[\mu] = f[f^{-1}[\nu] \wedge \mu]$. Therefore

$$\sup \quad \nu \wedge f[\mu] = \sup \quad f[f^{-1}[\nu] \wedge \mu] = \sup \quad f^{-1}[\nu] \wedge \mu.$$

Hence

$$c(\mathcal{G}, f[\mu]) = \inf_{\nu \in \mathcal{F}} \ \sup \quad \nu \wedge f[\mu] = \inf_{\nu \in \mathcal{F}} \ \sup \quad f^{-1}[\nu] \wedge \mu = c(f^{-1}[\mathcal{G}], \mu).$$

(6) Let $\nu_1 \vee \nu_2 \in < f[\mathcal{F}] >$. Then $\exists \mu \in \mathcal{F}$ such that $\nu_1 \vee \nu_2 \geq f[\mu]$. So $f^{-1}[\nu_1 \vee \nu_2] \geq f^{-1}[f[\mu]] \geq \mu$. Therefore $\mu \leq f^{-1}[\nu_1] \vee f^{-1}[\nu_2] \in \mathcal{F}$. Since $\mathcal{F}$ is prime. So $f^{-1}[\nu_1] \in \mathcal{F}$ or $f^{-1}[\nu_2] \in \mathcal{F}$. Therefore $\nu_1 \geq f[f^{-1}[\nu_1]] \in f[\mathcal{F}]$ or $\nu_2 \geq f[f^{-1}[\nu_2]] \in f[\mathcal{F}]$. Hence $\nu_1 \in < f[\mathcal{F}] >$ or $\nu_2 \in < f[\mathcal{F}] >$.

(7) We have

$$\bar{c}(\mathcal{F}) = \inf_{\mathcal{H} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{F}) = \inf_{\mathcal{H} \in \mathcal{P}_m(\mathcal{F})} c(f[\mathcal{H}]).$$

and

$$\bar{c}(f[\mathcal{H}]) = \inf_{\mathcal{G} \in \mathcal{P}_m(f[\mathcal{F}])} c(\mathcal{G}).$$

But

$$\mathcal{P}_m(f[\mathcal{F}]) \subseteq \{f[\mathcal{H}] : \mathcal{H} \in \mathcal{P}_m(\mathcal{F})\}.$$

Therefore $\bar{c}(\mathcal{F}) \leq \bar{c}(f[\mathcal{F}])$.

□

5.5 Convergence in Fuzzy Topological Space

Topological spaces provide the appropriate setting for the abstract study of continuity and convergence. In [48] Lowen extended the theory of continuity and convergence in topological spaces to the realm of fuzzy topological spaces.

We define the *adherence* and *limit* of a prefilter in a fuzzy topological space as follows.

Let $(X, \mathcal{T})$ be a fuzzy topological space. Then

5.5.1 Definition

If $\mathcal{F}$ is a prefilter, then we define the *adherence* of $\mathcal{F}$

$$\text{adh } \mathcal{F} = \inf_{\nu \in \mathcal{F}} \bar{\nu}.$$

where $\bar{\nu}$ is the fuzzy topological closure of ν.

5.5.2 Definition

If $\mathcal{F}$ is a prefilter then we define the *limit* of $\mathcal{F}$,

$$\lim \mathcal{F} = \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \text{adh } \mathcal{G}.$$

Note $\inf_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \text{adh } \mathcal{G} = 0.$

5.5.3 Proposition

Let $\mathcal{F}$ and $\mathcal{G}$ be prefilters. Then

1. If $\mathcal{F} \supseteq \mathcal{G}$ then adh $\mathcal{F} \leq$ adh $\mathcal{G}$,
2. $\lim \mathcal{F} \leq$ adh $\mathcal{F}$,
3. If $\mathcal{F}$ is prime then $\lim \mathcal{F} =$ adh $\mathcal{F}$,
4. If $\mathcal{F}$ is a prefilter base then
   adh $<\mathcal{F}>=$ adh $\mathcal{F}$.

Proof.

(1) We have

$$\mathcal{F} \supseteq \mathcal{G} \Rightarrow \inf_{\nu \in \mathcal{F}} \bar{\nu} \leq \inf_{\nu \in \mathcal{G}} \bar{\nu}.$$

Therefore

$$\mathcal{F} \supseteq \mathcal{G} \Rightarrow \text{adh } \mathcal{F} \leq \text{adh } \mathcal{G}.$$

(2) We have

$$\mathcal{G} \in \mathcal{P}_m(FF) \Rightarrow \text{adh } \mathcal{G} \leq \text{adh } \mathcal{F}.$$

Therefore

$$\lim \mathcal{F} = \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \text{adh } \mathcal{G} \leq \text{adh } \mathcal{F}.$$

(3) If $\mathcal{F}$ is prime then $\mathcal{P}_m(\mathcal{F}) = \{\mathcal{F}\}$. Therefore $\lim \mathcal{F} =$ adh $\mathcal{F}$.

(4) we have

$$\text{adh } <\mathcal{F}>= \inf_{\nu \in <\mathcal{F}>} \bar{\nu} \text{ and adh } \mathcal{F} = \inf_{\nu \in \mathcal{F}} \bar{\nu}.$$

Clearly adh $<\mathcal{F}>\leq$ adh $\mathcal{F}$. But $\forall\ \mu \in <\mathcal{F}>$, $\exists \nu \in \mathcal{F}$ such that $\nu \in \mu$ and so $\bar{\nu} \leq \bar{\mu}$. Therefore

$$\inf_{\mu \in <\mathcal{F}>} \bar{\mu} \geq \inf_{\nu \in \mathcal{F}} \bar{\nu}.$$

Thus

$$\text{adh } <\mathcal{F}>\geq \text{adh } \mathcal{F}.$$

Therefore

$$\text{adh } <\mathcal{F}>= \text{adh } \mathcal{F}.$$

□

5.5.4 Theorem

Let $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ be a function. Then the following are equivalent

1. f is continuous,
2. For each prefilter on X, $f[\text{adh } \mathcal{F}] \leq \text{adh } f[\mathcal{F}]$,
3. For each prefilter on X, $f[\lim \mathcal{F}] \leq \lim f[\mathcal{F}]$.

Proof.

(1) $\Longleftrightarrow$ (2)
We have

$$\text{adh } \mathcal{F} = \inf_{\mu \in \mathcal{F}} \bar{\mu} \text{ and adh } f[\mathcal{F}] = \inf_{\mu \in \mathcal{F}} \overline{f[\mu]}.$$

and

$$f[\text{adh } \mathcal{F}] = f[\inf_{\mu \in \mathcal{F}} \bar{\mu} \leq \inf_{\mu \in \mathcal{F}} f[\bar{\mu}].$$

But

$$f \text{ is continuous } \Longleftrightarrow \forall\ \mu \in I^X,\ f[\bar{\mu}] \leq \overline{f[\mu]}.$$

Therefore consequently

$$f \text{ is continuous } \iff f[\text{adh } \mathcal{F}] \leq \text{adh } f[\mathcal{F}].$$

The rest of the proof can be found in [48].

More information regarding prefilters can be found in [33, 64, 74].

Chapter 6

Fuzzy Uniform Spaces

6.1 Introduction

In [52] Lowen introduced and studied the notion of a fuzzy uniform space. To define a fuzzy uniform space first we have to define some basic definitions which are generalisations of the standard notion.

If $\sigma, \psi \in I^{X\times X}$ we define,

$$\sigma_s(x,y) \stackrel{\text{def}}{=} \sigma(y,x),$$

$$(\sigma \circ \psi)(x,y) \stackrel{\text{def}}{=} \sup_{z\in X} \psi(x,z) \wedge \sigma(z,y).$$

If $U, V \subseteq X \times X$ and $\sigma = 1_U, \psi = 1_V$ then

$$\begin{aligned}(1_U)_s(x,y) = 1 &\iff 1_U(y,x) = 1\\ &\iff (y,x) \in U\\ &\iff (x,y) \in U_s\\ &\iff (1_{U_s})(x,y) = 1.\end{aligned}$$

Therefore $(1_U)_s = 1_{U_s}$.

$$\begin{aligned}(1_V \circ 1_U)(x,y) = 1 &\iff \sup_{z\in X} 1_U(x,z) \wedge 1_V(z,y) = 1\\ &\iff \exists z \in X :\ (x,z) \in U \text{ and } (z,y) \in V\\ &\iff (x,y) \in V \circ U\\ &\iff 1_{V\circ U}(x,y) = 1.\end{aligned}$$

Therefore $1_V \circ 1_U = 1_{V\circ U}$.
Therefore the above definitions are natural generalisations of the standard notions.

6.1.1 Definitions

If X is a set and $\mathcal{D} \subseteq I^{X\times X}$ is called a *fuzzy uniformity* on X iff

1. $\mathcal{D}$ is a prefilter and $\hat{\mathcal{D}} = \mathcal{D}$;
2. $\forall\ \sigma \in \mathcal{D},\ \forall\ x \in X,\ \sigma(x,x) = 1$;
3. $\forall\ \sigma \in \mathcal{D},\ \sigma_s \in \mathcal{D}$;
4. $\forall\ \sigma \in \mathcal{D},\ \forall\ \varepsilon \in I_0,\ \exists \psi \in \mathcal{D} : \psi \circ \psi \leq \sigma + \varepsilon$.

We call $(X, \mathcal{D})$ a *fuzzy uniform space*.

If X is a set and $\mathcal{B} \subseteq I^{X\times X}$ is called a *fuzzy uniform base* on X iff

1. $\mathcal{B}$ is a prefilter base;
2. $\forall\, \sigma \in \mathcal{B},\ \forall\, x \in X,\ \sigma(x,x) = 1$;
3. $\forall\, \sigma \in \mathcal{B},\ \forall\, \varepsilon > 0,\ \exists \psi \in \mathcal{B} : \psi \leq \sigma_s + \varepsilon$;
4. $\forall\, \sigma \in \mathcal{B},\ \forall\, \varepsilon > 0,\ \exists \psi \in \mathcal{B} : \psi \circ \psi \leq \sigma + \varepsilon$.

If $\mathcal{D}$ is a fuzzy uniformity on X, then we call that $\mathcal{B}$ is a *base for* $\mathcal{D}$ iff $\mathcal{B}$ is a prefilter base and $\tilde{\mathcal{B}} = \mathcal{D}$.

6.1.2 Proposition
(1) If $\mathcal{B}$ is a fuzzy uniform base then $\tilde{\mathcal{B}}$ is a fuzzy uniformity.
(2) If $\mathcal{B}$ is a base for a fuzzy uniformity $\mathcal{D}$, then $\mathcal{B}$ is a fuzzy uniform base.

Proof.

(1) Let $\mathcal{B}$ is a fuzzy uniform base.
Then $\mathcal{B}$ is a prefilter base. So $\tilde{\mathcal{B}}$ is a saturated prefilter.
Clearly $\forall\, \sigma \in \tilde{\mathcal{B}},\ \forall\, x \in X,\ \sigma(x,x) = 1$.
Let $\sigma \in \tilde{\mathcal{B}}$ and $\varepsilon \in I_0$. Then $\exists \sigma_\varepsilon \in \mathcal{B}$ such that $\sigma \geq \sigma_\varepsilon - \varepsilon/2$. So $\sigma_s \geq (\sigma_\varepsilon)_s - \varepsilon/2$. Since $\sigma_\varepsilon \in \mathcal{B}$, we have $\exists \psi_\varepsilon \in \mathcal{B}$ such that $\psi_\varepsilon \leq (\sigma_\varepsilon)_s + \varepsilon/2$. So $\sigma_s \geq (\sigma_\varepsilon)_s - \varepsilon/2 \geq \psi_\varepsilon - \varepsilon$. Therefore $\forall\, \varepsilon \in I_0,\ \sigma_s + \varepsilon \in \tilde{\mathcal{B}}$ and so $\sigma_s \in \tilde{\mathcal{B}}$.

Let $\sigma \in \tilde{\mathcal{B}}$ and $\varepsilon \in I_0$. Then $\exists \sigma_\varepsilon \in \mathcal{B}$ such that $\sigma \geq \sigma_\varepsilon - \varepsilon/2$. Since $\sigma_\varepsilon \in \mathcal{B}$. So $\exists \psi \mathcal{B}$ such that $\psi \circ \psi \leq \sigma_\varepsilon + \varepsilon/2$. Therefore $\exists \psi \in \tilde{\mathcal{B}}$ such that $\psi \circ \psi \leq \sigma_\varepsilon + \varepsilon$.
Hence $\tilde{\mathcal{B}}$ is a fuzzy uniformity.

(2) We have $\mathcal{B}$ is a prefilter base and $\tilde{\mathcal{B}}$ is a fuzzy uniformity.
Clearly $\forall\, \sigma \in \mathcal{B},\ \forall\, x \in X,\ \sigma(x,x) = 1$.
Let $\sigma \in \mathcal{B}$ and $\varepsilon \in I_0$. Then $\sigma + \varepsilon/2 \in \tilde{\mathcal{B}}$ and so $\sigma_s + \varepsilon/2 \in \tilde{\mathcal{B}}$. Therefore $\exists \psi \in \mathcal{B}$ such that $\sigma_s + \varepsilon/2 \geq \psi - \varepsilon/2 \Rightarrow \psi \leq \sigma_s + \varepsilon$.

Let $\sigma \in \mathcal{B}$ and $\varepsilon \in I_0$. Then $\sigma + \varepsilon/3 \in \tilde{\mathcal{B}}$ and so $\exists \psi' \in \tilde{\mathcal{B}}$ such that $\psi' \circ \psi' \leq (\sigma + \varepsilon/3) + \varepsilon/3$. We have

$$\psi' \in \tilde{\mathcal{B}} \Rightarrow \exists \psi \in \mathcal{B} \text{ such that } \psi' \geq \psi - \varepsilon/3.$$

So $\psi \leq \psi' + \varepsilon/3 \Rightarrow \psi \circ \psi \leq \psi' \circ \psi' + \varepsilon/3 \leq \sigma + \varepsilon$. That is $\exists \psi \in \mathcal{B}$ such that $\psi \circ \psi \leq \sigma + \varepsilon$. □

6.1.3 Proposition
If $\mathcal{D}$ is a fuzzy uniformity on X then

$$\mathcal{B} \overset{\text{def}}{=} \{\sigma \in \mathcal{D} : \sigma = \sigma_s\}$$

is a fuzzy uniform base for $\mathcal{D}$.

Proof.

Clearly $0 \notin \mathcal{B}$ and $\mathcal{B} \neq \emptyset$.
Let $\sigma, \psi \in \mathcal{B}$. Then $\sigma = \sigma_s$ and $\psi = \psi_s$. So $(\sigma \wedge \psi)_s = (\sigma \wedge \psi)$.
Therefore $\mathcal{B}$ is a prefilter base.
Now we have to show $\tilde{\mathcal{B}} = \mathcal{D}$.
Let $\sigma \in \tilde{\mathcal{B}}$. Then $\exists (\psi_\varepsilon : \varepsilon \in I_0) \in \mathcal{B}^{I_0}$ such that

$$\sigma \geq \sup_{\varepsilon \in I_0} (\psi_\varepsilon - \varepsilon).$$

But $\mathcal{B} \subseteq \mathcal{D}$ and $\mathcal{D}$ is a saturated prefilter. Therefore $\sigma \in \mathcal{D}$.

Conversly let $\sigma \in \mathcal{D}$. Then $\sigma_s \in \mathcal{D}$. We have $(\sigma \wedge \sigma_s)_s = \sigma \wedge \sigma_s$. Therefore $\sigma \wedge \sigma_s \in \mathcal{B}$. But $\sigma \geq \sigma \wedge \sigma_s$ and so $\sigma \in <\mathcal{B}> \subseteq \tilde{\mathcal{B}}$. Hence $\tilde{\mathcal{B}} = \mathcal{D}$.

□

The proof of the following proposition is straightforward.

6.1.4 Proposition

If $\sigma \in \mathcal{D}$ then
$\sigma \leq \sigma \circ \sigma$ and
$\sigma \leq \sigma^n$ for any $n \in \mathbb{N}$
where $\sigma^n = \sigma \circ \sigma \circ \ldots \circ \sigma$ (n factors).

6.2 Fuzzy Neighbourhood Spaces

It is shown in [50] that a fuzzy topology can be defined using a fuzzy neighbourhood system. Here we only assemble some facts regarding neighbourhood spaces which are essential for defining a fuzzy topology. More information regarding fuzzy neighbourhood spaces can be found in [50].

6.2.1 Definitions

A collection $(\mathcal{N}_x)_{x \in X}$ of prefilters on X is called a *fuzzy neighbourhood system* iff the following conditions are fulfilled:

1. $\forall\, x \in X,\ \forall\, \mu \in \mathcal{N}_x,\ \mu(x) = 1;$
2. $\forall\, x \in X,\ \mathcal{N}_x = \hat{\mathcal{N}_x};$
3. $\forall\, x \in X,\ \forall\, \mu \in \mathcal{N}_x,\ \forall\, \varepsilon \in I_0, \exists(\nu_z : z \in X)$ such that $\forall\, z \in X,\ \nu_x \in \mathcal{N}_z$ and $\forall\, y \in X$,
$$\sup_{z \in X} \nu_x(z) \wedge \nu_z(y) \leq \mu(y) + \varepsilon.$$

$\mathcal{N}_x$ is called a *fuzzy neighbourhood prefilter in* X and the elements of $\mathcal{N}_x$ are called *fuzzy neighbourhoods of* x.

A collection of prefilter bases $(\beta_x)_{x \in X}$ on X is called a *fuzzy neighbourhood base* iff the following conditions are fulfilled:

1. $\forall\, x \in X,\ \forall\, \mu \in \beta_x,\ \mu(x) = 1;$
2. $\forall\, x \in X,\ \forall\, \mu \in \beta_x,\ \forall\, \varepsilon \in I_0,\ \exists(\nu_z; z \in X)$ such that $\forall\, z \in X, \nu_z \in \beta_z$ and $\forall\, y \in X$
$$\sup_{z \in X} \nu_x(z) \wedge \nu_z(y) \leq \mu(y) + \varepsilon.$$

β_x is called a *fuzzy neighbourhood base in* x and elements of β_x are called *basic fuzzy neighbourhoods of* x.

If $\mathcal{N} = (\mathcal{N}_x)_{x \in X}$ is a fuzzy neighbourhood system then we call $\beta = (\beta_x)_{x \in X}$ is a *base for* $\mathcal{N}$ iff $\forall\, x \in X,\ \beta_x$ is a prefilter base and $\tilde{\beta_x} = \mathcal{N}_x$.

6.2.2 Proposition

If $(\beta_x)_{x \in X}$ is a fuzzy neighbourhood base then $(\tilde{\beta_x})_{x \in X}$ is a fuzzy neighbourhood system with $(\beta_x)_{x \in X}$ as a base.

Proof.

(i) Clearly $\forall\ x \in X,\ \forall \mu \in \tilde{\beta_x} =< \hat{\beta_x} >,\ \mu(x) = 1$.
(ii) We also have $\forall\ x \in X,\ \tilde{\beta_x}$ is saturated.
(iii) Let $x \in X,\ \mu \in \tilde{\beta_x}$ and $\varepsilon \in I_0$ then $\exists(\lambda_\delta : \delta \in I_0) \in (\beta_x)^{I_0}$ such that

$$\mu \geq \sup_{\delta \in I_0} (\lambda_\delta - \delta)$$

$$\Rightarrow \mu \geq \lambda_{\varepsilon/2} - \varepsilon/2.$$

We have $x \in X,\ \lambda_{\varepsilon/2} \in \beta_x$ and $\varepsilon \in I_0$ and so $\exists(\nu_z : z \in X)$ such that $\forall\ z \in X,\ \nu_z \in \beta_z$ and $\forall\ y \in X$,

$$\sup_{z \in X} \nu_x(z) \wedge \nu_z(y) \leq \lambda_{\varepsilon/2}(y) + \varepsilon/2 \leq \mu(y) + \varepsilon.$$

Therefore
$\forall\ x \in X,\ \forall \mu \in \tilde{\beta_x},\ \forall\ \varepsilon \in I_0,\ \exists(\nu_z : z \in X)$ such that $\forall\ z \in X,\ \nu_z \in \tilde{\beta_z}$ and $\forall\ y \in X$,

$$\sup_{z \in X} \nu_x(z) \wedge \nu_z(y) \leq \mu(y) + \varepsilon.$$

Hence $(\tilde{\beta_x})_{x \in X}$ is a fuzzy neighbourhood system with $(\beta_x)_{x \in X}$ as a basis.

□

6.2.3 Proposition

If $(\beta_x)_{x \in X}$ is a base for the fuzzy neighbourhood system $(\mathcal{N}_x)_{x \in X}$ then $(\beta_x)_{x \in X}$ is a fuzzy neighbourhood base.

Proof.

Clearly $\forall\ x \in X,\ \forall\ \mu \in \beta_x \subseteq \mathcal{N}_x,\ \mu(x) = 1$.
Let $x \in X,\ \mu \in \beta_x$ and $\varepsilon \in I_0$ then $\exists(\nu_z : z \in X)$ such that $\forall\ z \in X,\ \nu_z \in \tilde{\beta_z} = \mathcal{N}_z$ and $\forall y \in X$,

$$\sup_{z \in X} \nu_x(z) \wedge \nu_z(y) \leq \mu(y) + \varepsilon/2.$$

We have $\forall\ z \in X,\ \nu_z \in \tilde{\beta_x}$ and so

$$\forall\ z \in X, \exists \lambda_z \in \beta_z \text{ such that } \nu_z \geq \lambda_z - \varepsilon/2.$$

So

$$\sup_{z \in X} \lambda_x(z) \wedge \lambda_z(y) \leq \sup_{z \in X} \nu_x(z) \wedge \nu_z(y) + \varepsilon/2 \leq \mu(y) + \varepsilon$$

Therefore $\forall\ x \in X,\ \forall\ \mu \in \beta_x,\ \forall\ \varepsilon \in I_0,\ \exists(\lambda_z : z \in X)$ such that $\forall\ z \in X,\ \lambda_z \in \beta_z$ and $\forall\ y \in X$

$$\sup_{z \in X} \lambda_x(z) \wedge \lambda_z(y) \leq \mu(y) + \varepsilon.$$

Hence $(\beta_x)_{x \in X}$ is a fuzzy neighbourhood base.

□

6.2.4 Theorem

If $\mathcal{N} = (\mathcal{N}_x)_{x \in X}$ is a fuzzy neighbourhood system on X then the operation

$$^-: I^X \longrightarrow I^X \text{ for } \mu \in I^X \text{ and } x \in X$$

$$\bar{\mu}(x) = \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} \mu(y) \wedge \nu(y) = \inf_{\nu \in \mathcal{N}_x} \sup\ \mu \wedge \nu$$

is a fuzzy closure operator.

PROOF.

We have

$$\bar{0} = \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} 0(y) \wedge \nu(y) = 0.$$

So $\bar{0} = 0$.

We have

$$\bar{\mu}(x) = \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} \mu(y) \wedge \nu(y).$$

Since $\nu(x) = 1$, $\bar{\mu}(x) \geq \mu(x)$.

We have

$$\begin{aligned}
\overline{\mu \vee \lambda}(x) &= \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} (\mu \vee \lambda(y) \wedge \nu(y)) \\
&= \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} ((\mu(y) \wedge \nu(y)) \vee (\lambda(y) \wedge \nu(y))) \\
&= \inf_{\nu \in \mathcal{N}_x} (\sup_{y \in X} \mu(y) \wedge \nu(y) \ \vee \ \sup_{y \in X} \lambda(y) \wedge \nu(y)) \\
&\geq \inf_{\nu \in \mathcal{N}_x} \mu(y) \wedge \nu(y) \ \vee \ \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} \lambda(y) \wedge \nu(y) \\
&= \bar{\mu}(x) \vee \bar{\lambda}(x) = (\bar{\mu} \vee \bar{\lambda})(x)
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mu} \vee \bar{\lambda}(x) &= \inf_{\nu, \nu' \in \mathcal{N}_x} (\sup_{y \in X} \mu(y) \wedge \nu(y) \ \vee \ \sup_{y \in X} \lambda(y) \wedge \nu'(y)) \\
&\geq \inf_{\nu, \nu' \in \mathcal{N}_x} \sup_{y \in X} (\mu \wedge \nu) \vee (\lambda \wedge \nu')(y).
\end{aligned}$$

But $\nu, \nu' \in \mathcal{N}_x \Rightarrow \nu \wedge \nu' \in \mathcal{N}_x$. So we have

$$\bar{\mu} \vee \bar{\lambda}(x) \geq \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} ((\mu \vee \lambda) \wedge \nu)(y) = \overline{\mu \vee \lambda}(x).$$

We have

$$\begin{aligned}
\bar{\bar{\mu}}(x) &= \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} \bar{\mu} \wedge \nu(y) \\
&= \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} (\inf_{\nu' \in \mathcal{N}_y} \sup_{z \in X} \mu(z) \wedge \nu'(z)) \wedge \nu(y)) \\
&= \inf_{\nu \in \mathcal{N}_x} \sup_{y \in X} \inf_{\nu' \in \mathcal{N}_y} \sup_{z \in X} \mu(z) \wedge \nu'(z) \wedge \nu(y).
\end{aligned}$$

For $\nu \in \mathcal{N}_x$ and $\varepsilon \in I_0$, $\exists (\nu_z : z \in X)$ such that $\forall\ z \in X$, $\nu_z \in \mathcal{N}_z$ and $\forall\ z \in X$

$$\sup_{y \in X} \nu_x(y) \wedge \nu_y(z) \leq \nu(z) + \varepsilon.$$

Then

$$\begin{aligned}
\sup_{z \in X} \mu(z) \wedge \nu(z) \ + \varepsilon &\geq \sup_{z \in X} \mu(z) \wedge (\nu(z) + \varepsilon) \\
&\geq \sup_{z \in X} \mu(z) \wedge (\sup_{y \in X} \nu_x(y) \wedge \nu_y(z)) \\
&= \sup_{z, y \in X} \mu(z) \wedge \nu_x(y) \wedge \nu_y(z).
\end{aligned}$$

So

$$\begin{aligned}
\bar{\bar{\mu}}(x) &\leq \sup_{y \in x} \inf_{\nu' \in \mathcal{N}_y} \sup_{z \in X} \mu(z) \wedge \nu'(z) \wedge \nu_x(y) \\
&\leq \sup_{y, z \in X} \mu(z) \wedge \nu_y(z) \wedge \nu_x(y) \\
&\leq \sup_{z \in X} \mu(z) \wedge \nu(z) \ + \varepsilon.
\end{aligned}$$

Therefore

$$\bar{\bar{\mu}}(x) \leq \sup_{z \in X} \mu(z) \wedge \nu(z) \ + \varepsilon.$$

is true for all $\nu \in \mathcal{N}_x$ and $\varepsilon \in I_0$ it follows that

$$\bar{\bar{\mu}}(x) \leq \bar{\mu}(x).$$

But we have shown that $\bar{\bar{\mu}}(x) \geq \bar{\mu}(x)$. Hence $\bar{\bar{\mu}} = \bar{\mu}$. $\square$

6.2.5 Proposition
If $\beta = (\beta_x)_{x\in X}$ is a base for the fuzzy neighbourhood system $\mathcal{N} = (\mathcal{N}_x)_{x\in X}$ then $\forall\ \mu \in I^X,\ \forall\ x \in X$ we have

$$\begin{aligned} \bar{\mu}(x) &= \inf_{\nu\in\tilde{\beta}_x} \sup\ \mu\wedge\nu = \inf_{\nu\in<\beta_x>} \sup\ \mu\wedge\nu \\ &= \inf_{\nu\in\beta_x} \sup \mu\wedge\nu. \end{aligned}$$

PROOF.

Let $\mu \in I^X$ and $x \in X$. Then

$$\bar{\mu}(x) = \inf_{\nu\in\mathcal{N}_x} \sup\ \mu\wedge\nu.$$

Since $\beta_x \subseteq \hat{\beta}_x \subseteq \tilde{\beta}_x = \mathcal{N}_x$ and $\beta_x \subseteq <\beta_x> \subseteq \tilde{\beta}_x = \mathcal{N}_x$. So we have

$$\bar{\mu}(x) \leq \inf_{\nu\in\hat{\beta}_x} \sup\ \mu\wedge\nu \leq \inf_{\nu\in\beta_x} \sup\ \mu\wedge\nu$$

and

$$\bar{\mu}(x) \leq \inf_{\nu\in<\beta_x>} \sup\ \mu\wedge\nu \leq \inf_{\nu\in\beta_x} \sup\ \mu\wedge\nu.$$

But $\forall\ \lambda \in \mathcal{N}_x$ and $\forall\ \varepsilon \in I_0,\ \exists\nu \in \beta_x$ such that

$$\lambda \geq \nu - \varepsilon.$$

Therefore

$$\begin{aligned} \bar{\mu}(x) &= \inf_{\lambda\in\mathcal{N}_x} \sup\ \mu\wedge\lambda \\ &\geq \inf_{\nu\in\beta_x} \sup\ \mu\wedge(\nu-\varepsilon) \\ &\geq \inf_{\nu\in\beta_x} \sup\ \mu\wedge\nu\ -\varepsilon. \end{aligned}$$

This is true for all $\varepsilon \in I_0$ and hence

$$\bar{\mu}(x) = \inf_{\nu\in\beta_x} \sup\ \mu\wedge\nu.$$

$\square$

If $\mathcal{N} = (\mathcal{N}_x)_{x\in X}$ is a fuzzy neighbourhood system then the above fuzzy closure operator generates a fuzzy topology and is denoted by $\tau_{\mathcal{N}}$.

The fuzzy topology which is generated by a fuzzy neighbourhood system, will be called a *fuzzy neighbourhood space.*

Not every fuzzy topological space is a fuzzy neighbourhood space. The reader can be found more facts regarding fuzzy neighbourhood spaces in [50].

6.3 Fuzzy Uniform Topology

A fuzzy topology can be defined using a fuzzy closure operator. Here we find two fuzzy closure operators one directly from a fuzzy uniform space and and the other from a neighbiurhood system which is generated from a fuzzy uniform space. Eventually we can see that if the fuzzy uniform space is same then fuzzy topologies generated from two fuzzy closure operators are same.

First we define some natural generalisation of the standard notions.
For $\sigma \in I^{X\times X},\ \mu \in I^X$ and $x \in X$ we define $\sigma < x >$ by

$$\sigma < x > (y) \stackrel{\text{def}}{=} \sigma(y,x)$$

and $\sigma < \mu >$ by

$$\sigma < \mu > (x) \stackrel{\text{def}}{=} \sup \mu \wedge \sigma < x >= \sup_{y\in X} \mu(y) \wedge \sigma(y,x).$$

If $A \subseteq X$ and $U \subseteq X \times X$ then

$$\begin{aligned} 1_U < 1_A > (x) = 1 \quad &\Longleftrightarrow \sup_{y\in X} 1_A(y) \wedge 1_U(y,x) = 1 \\ &\Longleftrightarrow \exists y \in A : (y,x) \in U \\ &\Longleftrightarrow x \in U(A) \\ &\Longleftrightarrow 1_{U(A)}(x) = 1. \end{aligned}$$

Therefore $1_U < 1_A >= 1_{U(A)}$. Thus the definition of $\sigma < \mu >$ is a natural generalisation of the standard notion.

Let $\sigma \in I^{X\times X}$ and $\beta \in I_1$. Then

$$\sigma^\beta \stackrel{\text{def}}{=} \{(x,y) : \sigma(x,y) > \beta\}.$$

In the following lemma we collect some basic facts.

6.3.1 Lemma
Let $\sigma, \psi \in I^{X\times X};\ \nu, \mu \in I^X,\ \varepsilon \in I,\ \beta \in I_1,\ x \in X$ and $n \in \mathbb{N}$. Then
(1) $\nu \leq \sigma < \nu >$,
(2) $(\sigma + \varepsilon) < \nu >\leq \sigma < \nu > +\varepsilon$,
(3) $\sigma < \mu \vee \nu >= \sigma < \mu > \vee \sigma < \nu >$,
(4) $\sigma < \psi < \nu >>= (\sigma \circ \psi) < \nu >$,
(5) $\sup\ \sigma < \nu > \wedge \mu = \sup\ \mu \wedge \sigma_s < \mu >$,
(6) $(\sigma^\beta)_s = (\sigma_s)^\beta$,
(7) $(\sigma < \nu >)^\beta = \sigma^\beta(\nu^\beta)$
(8) $\sigma < x >^\beta = \sigma_s^\beta(x)$,
(9) $(\sigma^\beta)^n = (\sigma^n)^\beta$.

Proof.

(1)
$$\sigma < \nu > (x) = \sup\ \nu \wedge \sigma < x >= \sup_{y\in X} \nu(y) \wedge \sigma(y,x) \geq \nu(x).$$

(2)
$$\begin{aligned} (\sigma + \varepsilon) < \nu > (x) \quad &= \sup_{y\in X} \nu(y) \wedge (\sigma + \varepsilon)(y,x) \\ &\leq \sup_{y\in X} (\nu(y) + \varepsilon) \wedge (\sigma(y,x) + \varepsilon) \\ &= (\sup_{y\in X} \nu(y) \wedge \sigma(y,x)) + \varepsilon \\ &= \sigma < \nu > (x) + \varepsilon. \end{aligned}$$

(3)
$$\begin{aligned} \sigma < \mu \vee \nu > (x) \quad &= \sup\ (\mu \vee \nu) \wedge \sigma < x > \\ &= \sup\ (\mu \wedge \sigma < x >) \vee (\nu \wedge \sigma < x >) \\ &= \sup\ \mu \wedge \sigma < x >\ \vee\ \sup\ \nu \wedge \sigma < x > \\ &= \sigma < \mu > (x) \vee \sigma < \nu > (x) \\ &= (\sigma < \mu > \vee \sigma < \nu >)(x). \end{aligned}$$

(4)

$$\begin{aligned}\sigma < \psi < \nu >> (x) &= \sup_{y\in X} \psi < \nu > (y) \wedge \sigma < x > (y)\\ &= \sup_{y\in X} (\sup_{z\in X} \nu(z) \wedge \psi(z,y)) \wedge \sigma(y,x)\\ &= \sup_{y\in X} \sup_{z\in X} \nu(z) \wedge \psi(z,y) \wedge \sigma(y,x).\end{aligned}$$

$$\begin{aligned}(\sigma \circ \psi) < \nu > (x) &= \sup_{z\in X} \nu(z) \wedge (\sigma \circ \psi) < x > (z)\\ &= \sup_{z\in X} \nu(z) \wedge (\sup_{y\in X} \psi(z,y) \wedge \sigma(y,x))\\ &= \sup_{z\in X} \sup_{y\in X} \nu(z) \wedge \psi(z,y) \wedge \sigma(y,x).\end{aligned}$$

(5)

$$\begin{aligned}\sup_{x\in X} \sigma < \nu > \wedge \mu(x) &= \sup_{x\in X}(\sup_{y\in X} \nu(y) \wedge \sigma(y,x)) \wedge \mu(x)\\ &= \sup_{x\in X} \sup_{y\in X} \nu(y) \wedge \sigma(y,x) \wedge \mu(x)\\ &= \sup_{y\in X}(\sup_{x\in X} \mu(x) \wedge \sigma_s(x,y)) \wedge \nu(y)\\ &= \sup_{y\in X} \sigma_s < \mu > (y) \wedge \nu(y)\\ &= \sup_{y\in X} \sigma_s < \mu > \wedge \nu(y).\end{aligned}$$

(6)

$$\begin{aligned}(x,y) \in (\sigma^\beta)_s &\Longleftrightarrow (y,x) \in \sigma^\beta\\ &\Longleftrightarrow \sigma(y,x) > \beta\\ &\Longleftrightarrow \sigma_s(x,y) > \beta\\ &\Longleftrightarrow (x,y) \in (\sigma_s)^\beta.\end{aligned}$$

(7)

$$\begin{aligned}x \in (\sigma < \nu >)^\beta &\Longleftrightarrow \sigma < \nu > (x) = \sup_{y\in X} \nu(y) \wedge \sigma(y,x) > \beta\\ &\Longleftrightarrow \exists y \in X : \nu(y) \wedge \sigma(y,x) > \beta\\ &\Longleftrightarrow \exists y \in \nu^\beta : (y,x) \in \sigma^\beta\\ &\Longleftrightarrow x \in \sigma^\beta(\nu^\beta).\end{aligned}$$

(8)

$$\begin{aligned}y \in \sigma < x >^\beta &\Longleftrightarrow \sigma < x > (y) > \beta\\ &\Longleftrightarrow \sigma(y,x) = \sigma_s(x,y) > \beta\\ &\Longleftrightarrow (x,y) \in \sigma_s^\beta\\ &\Longleftrightarrow y \in \sigma_s^\beta(x).\end{aligned}$$

(9)

$$\begin{aligned}(x,y) \in (\sigma^\beta)^n &\Longleftrightarrow \exists x_1, x_2, \ldots, x_{n-1} : (x,x_1) \in \sigma^\beta, (x_1,x_2) \in \sigma^\beta, \ldots, (x_{n-1},y) \in \sigma^\beta\\ &\Longleftrightarrow \sup\{\sigma(x,x_1) \wedge \sigma(x_1,x_2) \wedge \ldots \wedge \sigma(x_{n-1},y) : x_i \in X, i \in \{1,2,\ldots,n-1\}\} > \beta\\ &\Longleftrightarrow \sigma^n(x,y) > \beta\\ &\Longleftrightarrow (x,y) \in (\sigma^n)^\beta.\end{aligned}$$

□

6.3.2 Theorem
Let $(X, \mathcal{D})$ be a fuzzy uniform space. Then the map$^-$: $I^X \longrightarrow I^X$ defined by

$$\bar{\mu} = \inf_{\sigma\in\mathcal{D}} \sigma < \mu >$$

is a fuzzy closure operator.

PROOF.

we have
$$\bar{0}(x) = \inf_{\sigma\in\mathcal{D}} \sigma < 0 > (x) = \inf_{\sigma\in\mathcal{D}} \sup_{y\in X} 0(y) \wedge \sigma(y,x) = 0.$$

Therefore $\bar{0} = 0$.

We have
$$\bar{\mu} = \inf_{\sigma\in\mathcal{D}} \sigma < \mu > \geq \mu.$$

We have
$$\begin{aligned} \overline{\mu \vee \nu} &= \inf_{\sigma\in\mathcal{D}} \sigma < \mu \vee \nu > \\ &= \inf_{\sigma\in\mathcal{D}} \sigma < \mu > \vee \sigma < \nu > \\ &\geq \inf_{\sigma\in\mathcal{D}} \sigma < \mu > \ \vee \inf_{\sigma'\in\mathcal{D}} \sigma' < \nu > \\ &= \bar{\mu} \vee \bar{\nu} \end{aligned}$$

and
$$\begin{aligned} \bar{\mu} \vee \bar{\nu} &= \inf_{\sigma\in\mathcal{D}} \sigma < \mu > \ \vee \inf_{\sigma'\in\mathcal{D}} \sigma' < \nu > \\ &= \inf_{\sigma,\sigma'\in\mathcal{D}} \sigma < \mu > \vee \sigma' < \nu > . \end{aligned}$$

But $\sigma, \sigma' \in \mathcal{D} \Rightarrow \sigma \wedge \sigma' \in \mathcal{D}$ and $\sigma < \mu > \vee \sigma' < \nu > \geq \sigma \wedge \sigma' < \mu \vee \nu >$. Therefore
$$\bar{\mu} \vee \bar{\nu} \geq \inf_{\sigma\in\mathcal{D}} \sigma < \mu \vee \nu > = \overline{\mu \vee \nu}.$$

Hence
$$\overline{\mu \vee \nu} = \bar{\mu} \vee \bar{\nu}.$$

Let $\sigma \in \mathcal{D}$ and $\varepsilon \in I_0$. Then $\exists \sigma' \in \mathcal{D}$ such that $\sigma' \circ \sigma' \leq \sigma + \varepsilon$.

Therefore for any $x \in X$ we have,
$$\begin{aligned} \sigma < \mu > (x) &= \sup_{y\in X} \mu(y) \wedge \sigma(y,x) \\ &\geq \sup_{y\in X} \mu(y) \wedge (\sigma' \circ \sigma'(y,x) - \varepsilon) \\ &= \sup_{y\in X} \sup_{z\in X} \mu(y) \wedge \sigma'(y,z) \wedge \sigma'(z,x) - \varepsilon \\ &\geq \sup_{z\in X} \sigma'(z,x) \wedge (\inf_{\sigma''\in\mathcal{D}} \sup_{y\in X} \mu(y) \wedge \sigma''(y,z)) - \varepsilon \\ &= \sup_{z\in X} \sigma'(z,x) \wedge \bar{\mu}(z) - \varepsilon \\ &= \sigma' < \bar{\mu} > (x) - \varepsilon. \end{aligned}$$

Thus for any $\varepsilon \in I_0$ and $\sigma \in \mathcal{D}$ there exists $\sigma' \in \mathcal{D}$ such that $\sigma < \mu > (x) \geq \sigma' < \bar{\mu} > (x) - \varepsilon$. Therefore
$$\inf_{\sigma\in\mathcal{D}} \sigma < \mu > \geq \inf_{\sigma'\in\mathcal{D}} \sigma' < \bar{\mu} > .$$

That is $\bar{\mu}(x) \geq \bar{\bar{\mu}}(x)$. Hence $\bar{\mu} = \bar{\bar{\mu}}$. □

6.3.3 Proposition

If $\mathcal{B}$ is a base for the fuzzy uniformity $\mathcal{D}$ then for all $\mu \in I^X$ we have
$$\begin{aligned} \bar{\mu} &= \inf_{\sigma\in\hat{\mathcal{B}}} \sigma < \mu > = \inf_{\sigma\in<\mathcal{B}>} \sigma < \mu > \\ &= \inf_{\sigma\in\mathcal{B}} \sigma < \mu > . \end{aligned}$$

Proof.

Since $\mathcal{B} \subseteq \hat{\mathcal{B}} \subseteq \mathcal{D}$ and $\mathcal{B} \subseteq < \mathcal{B} > \subseteq \mathcal{D}$. So for $\mu \in I^X$ we have

$$\bar{\mu} \leq \inf_{\sigma \in \hat{\mathcal{B}}} \sigma < \mu > \leq \inf_{\sigma \in \mathcal{B}} \sigma < \mu >$$

and

$$\bar{\mu} \leq \inf_{\sigma \in < \mathcal{B} >} \sigma < \mu > \leq \inf_{\sigma \in \mathcal{B}} \sigma < \mu > .$$

Let $\varepsilon \in I_0$ and $\sigma \in \mathcal{D}$ then $\exists \psi \in \mathcal{B}$ such that

$$\sigma \geq \psi - \varepsilon.$$

Therefore

$$\begin{aligned} \bar{\mu} &= \inf_{\sigma \in \mathcal{D}} \sigma < \mu > \\ &\geq \inf_{\psi \in \mathcal{B}} (\psi - \varepsilon) < \mu > \\ &\geq \inf_{\psi \in \mathcal{B}} \psi < \mu > - \varepsilon. \end{aligned}$$

which proves that

$$\bar{\mu} \geq \inf_{\psi \in \mathcal{B}} \psi < \mu > .$$

Hence

$$\bar{\mu} = \inf_{\psi \in \mathcal{B}} \psi < \mu > .$$

□

The above fuzzy closure operatoer defines a fuzzy topology

$$\tau_{\mathcal{D}} = \{\sigma' : \sigma = \bar{\sigma}\} = \{1 - \sigma : \sigma = \bar{\sigma}\}$$

associateed with $\mathcal{D}$ and $\tau_{\mathcal{D}}$ is called the *fuzzy uniform topology.*

6.3.4 Theorem

Let $(X, \mathcal{D})$ be a fuzzy uniform space. For $x \in X$ define,

$$\mathcal{D}_x \stackrel{\text{def}}{=} \{\sigma < x >: \sigma \in \mathcal{D}\}.$$

Then $(\mathcal{D}_x)_{x \in X}$ is a fuzzy neighbourhood system.

Proof.

First we have to show $\mathcal{D}_x$ is a prefilter.

$$\sigma < x > (x) = \sigma(x, x) = 1 \text{ so } 0 \notin \mathcal{D}_x \text{ and } \mathcal{D}_x \neq \emptyset.$$

Let $\sigma < x >, \psi < x > \in \mathcal{D}_x$. Then $\sigma, \psi \in \mathcal{D}$ and so $\sigma \wedge \psi \in \mathcal{D}$

$$\sigma < x > \wedge \psi < x > = (\sigma \wedge \psi) < x > \in \mathcal{D}_x.$$

Let $\sigma < x > \in \mathcal{D}_x$ and $\sigma < x > \leq \mu$. Then $\sigma \in \mathcal{D}$ and $\forall\ y \in X,\ \sigma < x > (y) = \sigma(y, x) \leq \mu(y)$.

Now define $\psi \in I^{X \times X}$ by

$$\psi(y, z) = \begin{cases} \sigma(y, z) & \text{if } z \neq x \\ \mu(y) & \text{if } z = x. \end{cases}$$

So $\psi \geq \sigma$. Therefore $\psi \in \mathcal{D}$ such that $\psi < x > = \mu \in \mathcal{D}_x$. Hence $\mathcal{D}_x$ is a prefilter.

we have $\forall\ x \in X,\ \forall\ \sigma \in \mathcal{D},\ \sigma < x > (x) = 1$.

Let $\forall\ \varepsilon \in I_0,\ \sigma < x > +\varepsilon \in \mathcal{D}_x$. Then $\forall\ \varepsilon \in I_0,\ (\sigma + \varepsilon) < x >= \sigma < x > +\varepsilon \in \mathcal{D}_x$. So $\forall\ \varepsilon \in I_0,\ (\sigma + \varepsilon) \in \mathcal{D}$. Since $\mathcal{D}$ is saturated prefiter and therefore $\sigma \in \mathcal{D}$. Hence $\mathcal{D}_x$ is saturated.

Let $x \in X,\ \sigma \in \mathcal{D}$ and $\varepsilon \in I_0$. Then $\exists \psi \in \mathcal{D}$ such that $\psi \circ \psi \leq \sigma + \varepsilon$.
Therefore $(\psi < z >: z \in X)$ such that $\forall\ z \in X,\ \psi < z > \in \mathcal{D}_x$.
and for $y \in X$,

$$\begin{aligned}&\sup_{z\in X}\ \psi < x > (z) \wedge \psi < z > (y)\\ &= \sup_{z\in X}\ \psi(y,z) \wedge \psi(z,x)\\ &= \psi \circ \psi(y,x)\\ &\leq \sigma(y,x) + \varepsilon\\ &= \sigma < x > (y) + \varepsilon.\end{aligned}$$

Hence $(\mathcal{D}_x)_{x\in X}$ is a fuzzy neighbourhood system. □

Thus $(\mathcal{D}_x)_{x\in X}$ is a fuzzy neighbourhood system and therefore the operator

$$\bar{\mu}(x) = \inf_{\nu\in\mathcal{D}_x} \sup\ \mu \wedge \nu$$

is a fuzzy closure operator. This is precisely we have earlier because

$$\begin{aligned}\bar{\mu}(x) &= \inf_{\sigma\in\mathcal{D}} \sigma < \mu > (x) = \inf_{\sigma\in\mathcal{D}} \sup\ \mu \wedge \sigma < x >\\ &= \inf_{\nu\in\mathcal{D}_x} \sup\ \mu \wedge \nu.\end{aligned}$$

Therefore a fuzzy uniform topology is a fuzzy neighbourhood space.

6.3.5 Lemma

Let $(X, \mathcal{D})$ be a fuzzy uniform space and $\nu \in I^X$. Then

1. $\bar{\nu}(x) = c(\mathcal{D}_x, \nu)$,
2. $\sup\ \bar{\nu} = \sup\ \nu$,
3. $\bar{\nu} = \inf_{\sigma\in\mathcal{D}} \overline{\sigma < \nu >}$.

PROOF.

(1) We have

$$\begin{aligned}\bar{\nu}(x) &= \inf_{\sigma\in\mathcal{D}} \sigma < \nu > (x) = \inf_{\sigma\in\mathcal{D}} \sup\ \nu \wedge \sigma < x >\\ &= \inf_{\mu\in\mathcal{D}_x} \sup\ \nu \wedge \mu\end{aligned}$$

and

$$\begin{aligned}c(\mathcal{D}_x, \nu) &= c(\mathcal{D}_x \vee < \nu >)\\ &\inf_{\mu\in\mathcal{D}_x} \sup\ \mu \wedge \nu.\end{aligned}$$

(2) We have

$$\nu \leq \bar{\nu} \Rightarrow \sup\ \nu \leq \sup\ \bar{\nu}.$$

For each $x \in X$,

$$\bar{\nu}(x) = c(\mathcal{D}_x, \nu) \leq \sup\ \nu.$$

Therefore

$$\sup\ \bar{\nu} \leq \sup\ \nu.$$

(3) We have

$$\bar{\nu} = \inf_{\sigma\in\mathcal{D}} \sigma < \nu > \leq \inf_{\sigma\in\mathcal{D}} \overline{\sigma < \nu >}.$$

Let $\sigma \in \mathcal{D}$ and $\varepsilon \in I_0$. Then $\exists \psi \in \mathcal{D}$ such that $\psi \circ \psi \leq \sigma + \varepsilon$ and so

$$\begin{aligned}\overline{\psi < \nu >} &= \inf_{\xi \in \mathcal{D}} \xi < \psi < \nu >> \\ &\leq \psi < \psi < \nu >>= \psi \circ \psi < \nu > \\ &\leq (\sigma + \varepsilon) < \nu > \\ &\leq \sigma < \nu > +\varepsilon.\end{aligned}$$

We have show that $\forall\ \varepsilon \in I_0,\ \forall\ \sigma \in \mathcal{D},\ \exists \psi \in \mathcal{D}$ such that

$$\overline{\psi < \nu >} \leq \sigma < \nu > +\varepsilon.$$

Therefore

$$\inf_{\psi \in \mathcal{D}} \overline{\psi < \nu >} \leq \inf_{\sigma \in \mathcal{D}} \sigma < \nu >= \bar{\nu}.$$

□

6.4 Convergence in Fuzzy Uniform Topology

We have seen convergence in a fuzzy topological space and here we see convergence in a fuzzy uniform space. For this we simply use the fuzzy topology which is associated with the particular fuzzy uniform space.

Let $(X, \mathcal{D})$ be a fuzzy uniform space then we have fuzzy topology $\tau_{\mathcal{D}}$ associated with $\mathcal{D}$. If $\mathcal{F}$ is a prefiler on X then we have

$$\text{adh } \mathcal{F} = \inf_{\nu \in \mathcal{F}} \bar{\nu}$$

and

$$\lim\ \mathcal{F} = \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \text{adh } \mathcal{G}.$$

6.4.1 Lemma

Let $(X, \mathcal{D})$ be a fuzzy uniform space, $\mathcal{F}$ a prefilter on X and $x \in X$. Then

1. $(\text{adh } \mathcal{F})(x) = c(\mathcal{D}_x, \mathcal{F})$,
2. $(\lim\ \mathcal{F})(x) = \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{D}_x, \mathcal{G})$,
3. $\sup\ \text{adh } \mathcal{F} \leq c(\mathcal{F})$,
4. $\sup\ \lim\ \mathcal{F} \leq \bar{c}(\mathcal{F})$,
5. $\text{adh } \mathcal{F} = \sup_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \text{adh } \mathcal{G}$.

PROOF.

For $x \in X$ we have (1)

$$\begin{aligned}(\text{adh } \mathcal{F})(x) &= \inf_{\nu \in \mathcal{F}} \bar{\nu}(x) = \inf_{\nu \in \mathcal{F}} \inf_{\sigma \in \mathcal{D}} \sigma < \nu > (x) \\ &= \inf_{\nu \in \mathcal{F}} \inf_{\sigma \in \mathcal{D}} \sup\ \nu \wedge \sigma < x > \\ &= c(\mathcal{F} \vee \mathcal{D}_x) = c(\mathcal{D}_x, \mathcal{F}).\end{aligned}$$

(2) For $x \in X$ we have

$$(\lim\ \mathcal{F})(x) = \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \text{adh } \mathcal{F}(x) = \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{D}_x, \mathcal{G}).$$

(3) If $x \in X$ then

$$(\text{adh } \mathcal{F})(x) = c(\mathcal{D}_x, \mathcal{F}) \leq c(\mathcal{F})$$

$$\Rightarrow \sup \text{ adh } \mathcal{F} \leq c(\mathcal{F})$$

(4) If $x \in X$ then

$$\begin{aligned}(\lim \mathcal{F})(x) &= \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{D}_x, \mathcal{G}) \\ &\leq \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{F}) = \bar{c}(\mathcal{F})\end{aligned}$$

$$\Rightarrow \sup \lim \mathcal{F} \leq \bar{c}(\mathcal{F}).$$

(5) For each $\mathcal{G} \in \mathcal{P}_m(\mathcal{F})$, adh $\mathcal{G} \leq$ adh $\mathcal{F}$ and so $\sup\limits_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})}$ adh $\mathcal{G} \leq$ adh $\mathcal{F}$.

To prove the reverse inequality let $x \in X$.

If $(\text{adh } \mathcal{F})(x) = 0$ then clearly $(\text{adh } \mathcal{F})(x) \leq (\sup\limits_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \text{adh } \mathcal{G})(x)$.

If $(\text{adh } \mathcal{F})(x) > 0$ then choose α such that $(\text{adh } \mathcal{F})(x) > \alpha > 0$. Therefore $c(\mathcal{D}_x, \mathcal{F}) = c(\mathcal{D}_x \vee \mathcal{F}) > \alpha$. So choose an ultra filter $\mathbb{F} \supseteq (\mathcal{D}_x \vee \mathcal{F})_\alpha$. Now we have $\mathcal{F}_0 \subseteq \mathcal{F}_\alpha \subseteq (\mathcal{D}_x \vee \mathcal{F})_\alpha \subseteq \mathbb{F}$.

Let $\mathcal{G} = \mathcal{F} \vee \mathbb{F}_1$. Then $\mathcal{G} \in \mathcal{P}_m(\mathcal{F})$ and

$$\begin{aligned}(\text{adh } \mathcal{G})(x) &= c(\mathcal{G}, \mathcal{D}_x) = c((\mathcal{F} \vee \mathbb{F}_1) \vee \mathcal{D}_x) \\ &= \inf_{\nu \in \mathcal{F}} \inf_{F \in \mathbb{F}} \inf_{\sigma \in \mathcal{D}} \sup \ \nu \wedge 1_F \wedge \sigma < x > \\ &= \inf_{\nu \in \mathcal{F}} \inf_{F \in \mathbb{F}} \inf_{\sigma \in \mathcal{D}} \sup_{y \in F} (\nu \wedge \sigma < x >)(y).\end{aligned}$$

Therefore

$$\nu \wedge \sigma < x > \in \mathcal{F} \vee \mathcal{D}_x \text{ and so } F \cap (\nu \wedge \sigma < x >)^\alpha \neq \emptyset.$$

It follows that $\sup\limits_{y \in F} (\nu \wedge \sigma < x >)(y) > \alpha$ and hence $(\text{adh } \mathcal{G})(x) \geq \alpha$. Thus $\sup\limits_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} (\text{adh } \mathcal{G})(x) \geq \alpha$ and since α is arbitrary,

$$(\text{adh } \mathcal{F})(x) \leq \sup_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} (\text{adh } \mathcal{G})(x).$$

Since x is arbitrary, the result follows.

□

6.5 Uniformly Continuous Functions

We extend the notion of uniformly continuity in uniform spaces in a natural way as follows:

6.5.1 Definition

Let $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ be fuzzy uniform spaces and $f : X \longrightarrow Y$ a mapping. f is said to be uniformly uniformly continuous

$$\begin{aligned}&\text{iff } \forall\ \psi \in \mathcal{E},\ \exists \sigma \in \mathcal{D},\ : (f \times f)[\sigma] \leq \psi \\ &\text{iff } \forall\ \psi \in \mathcal{E},\ (f \times f)^{-1}[\psi] \in \mathcal{D}.\end{aligned}$$

6.5.2 Proposition

Let $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ are fuzzy uniform spaces and $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathcal{D}$ and $\mathcal{E}$ respectively. If $f : X \longrightarrow Y$ is a mapping. Then f is uniformly continuous if and only if

$$\forall\ \psi \in \mathcal{E},\ \forall\ \varepsilon \in I_0,\ \exists \sigma \in \mathcal{B} \text{ such that } \sigma - \varepsilon \leq (f \times f)^{-1}[\psi].$$

Proof.

We have

$$\begin{aligned} f \text{ is uniformly continuous} \quad &\Longleftrightarrow \forall\ \psi \in \mathcal{E},\ \exists \sigma \in \mathcal{D}:\ (f \times f)[\sigma] \leq \psi \\ &\Longleftrightarrow \forall\ \psi \in \mathcal{E},\ \exists \sigma \in \mathcal{D}:\ \sigma \leq (f \times f)^{-1}[\psi]. \end{aligned}$$

But we have $\tilde{\mathcal{B}} = \mathcal{D}$ and $\tilde{\mathcal{C}} = \mathcal{E}$.
($\Rightarrow$)

Let $\psi \in \mathcal{C}$ and $\varepsilon \in I_0$. Then $\psi + \varepsilon/2 \in \mathcal{E}$. Therefore $\exists \sigma' \in \mathcal{D}:\ \sigma' \leq (f \times f)^{-1}[\psi + \varepsilon/2] = (f \times f)^{-1}[\psi] + \varepsilon/2$. Since $\sigma' \in \mathcal{D}$. So $\exists \sigma \in \mathcal{B}$ such that $\sigma' \geq \sigma - \varepsilon/2$. Therefore $\sigma - \varepsilon/2 \leq \sigma' \leq (f \times f)^{-1}[\psi] + \varepsilon/2$. Hence $\sigma - \varepsilon \leq (f \times f)^{-1}[\psi]$.
($\Leftarrow$)

Let $\psi \in \mathcal{E}$. Then $\forall\ \delta \in I_0,\ \exists \psi_\delta \in \mathcal{C}:\ \psi \geq \psi_\delta - \delta$. For $\varepsilon \in I_0,\ \exists \sigma_\varepsilon \in \mathcal{B}$ such that $\sigma_\varepsilon - \varepsilon \leq (f \times f)^{-1}[\psi_\varepsilon]$. Therefore $\sigma_\varepsilon \leq (f \times f)^{-1}[\psi_\varepsilon + \varepsilon \leq (f \times f)^{-1}[\psi + 2\varepsilon] = (f \times f)^{-1}[\psi] + 2\varepsilon$. Thus $\forall\ \varepsilon \in I_0,\ \exists \sigma_\varepsilon \in \mathcal{B}$ such that $\sigma_\varepsilon - 2\varepsilon \leq (f \times f)^{-1}[\psi]$. Therefore $\sigma = \sup\limits_{\varepsilon \in I_0} (\sigma_\varepsilon - 2\varepsilon) \leq (f \times f)^{-1}[\psi]$. Hence $\sigma \in \mathcal{D}$ such that $\sigma \leq (f \times f)^{-1}[\psi]$.

□

6.5.3 Corollary
Let $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ are fuzzy uniform spaces and $f : X \longrightarrow Y$ a mapping. Then f is uniformly continuous if and only if

$$\forall\ \psi \in \mathcal{E},\ \forall\ \varepsilon \in I_0,\ \exists \sigma \in \mathcal{D} \text{ such that } \sigma - \varepsilon \leq (f \times f)^{-1}[\psi].$$

Proof.

Since each fuzzy uniformity is a basis for itself, this follows the result.

6.5.4 Theorem
Let $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ are fuzzy uniform spacs and $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is a uniformly continuous function. Then $f : (X, \tau_{\mathcal{D}}) \longrightarrow (Y, \tau_{\mathcal{E}})$ is continuous.

Proof.

Let $\mu \in I^X$ and $y \in f^{\rightarrow}(X)$. Then

$$\begin{aligned} \overline{f[\mu]}(y) &= \inf_{\psi \in \mathcal{E}} \psi < f[\mu] > (y) \\ &= \inf_{\psi \in \mathcal{E}} \sup_{z \in Y} (f[\mu] \wedge \psi < y >)(z) \\ &= \inf_{\psi \in \mathcal{E}} \sup_{z \in Y} (\sup_{f(x)=z} \mu(x) \wedge \psi(z, y)) \\ &= \inf_{\psi \in \mathcal{E}} \sup_{x \in X} \mu(x) \wedge \psi(f(x), y). \end{aligned}$$

Therefore $\forall\ x' \in f^{\leftarrow}(\{y\})$,

$$\overline{f[\mu]}(y) = \inf_{\psi \in \mathcal{E}} \sup_{x \in X} \mu(x) \wedge \psi(f(x), f(x')).$$

Since f is uniformly continuous we have $\forall\ \psi \in \mathcal{E},\ \exists \sigma \in \mathcal{D}$ such that $\sigma \leq (f \times f)^{-1}[\psi]$. Thus $\forall\ x, x' \in X,\ \sigma(x, x') \leq \psi(f(x), f(x'))$. Hence

$$\overline{f[\mu]}(y) \geq \inf_{\psi \in \mathcal{E}} \sup_{x \in X} \mu(x) \wedge \sigma(x, x') = \bar{\mu}(x').$$

Therefore

$$\overline{f[\mu]}(y) \geq \sup_{x' \in f^{\leftarrow}(\{y\})} \bar{\mu}(x') = f[\bar{\mu}](y).$$

Hence $\overline{f[\mu]} \geq f[\bar{\mu}]$. Since $\mu \in I^X$ is arbitrary we have f is continuous.

□

6.5.5 Lemma
If $f : (X, \mathcal{D}_1) \longrightarrow (Y, \mathcal{D}_2)$ and $g : (Y, \mathcal{D}_2) \longrightarrow (Z, \mathcal{D}_3)$ are two uniformly continuous functions. Then $(g \circ f) : (X, \mathcal{D}_1) \longrightarrow (Z, \mathcal{D}_3)$ is uniformly continuous.

Proof.

Let $\psi \in \mathcal{D}_3$. Then $(g \times g)^{-1}[\psi] \in \mathcal{D}_2$ and

$$(f \times f)^{-1}[(g \times g)^{-1}[\psi]] = ((g \times g) \circ (f \times f))^{-1}[\psi] = ((g \circ f) \circ (g \circ f))^{-1}[\psi] \in \mathcal{D}_1.$$

Therefore $g \circ f$ is uniformly continuous.

□

6.6 The α-Level Uniformities

Investigating a prefilter by its α-levels is a very useful device used in [49, 62, 9]. First we see the α-level uniformities from a fuzzy uniformity.

If $(X, \mathcal{D})$ is a fuzzy uniform space then for each $\alpha \in I_0$ we define

$$\mathcal{D}^\alpha \overset{\text{def}}{=} \{\sigma^\beta : 0 \leq \beta < \alpha,\ \sigma \in \mathcal{D}\}.$$

6.6.1 Proposition
Let $(X, \mathcal{D})$ be a fuzzy uniform space and $\alpha \in I_0$. Then $\mathcal{D}^\alpha$ is a uniformity on X.

Proof.

(i) Since $\mathcal{D}$ is a prefilter. Therefore $\mathcal{D}^\alpha$ is a filter.

(ii) Let $U \in \mathcal{D}^\alpha$. Then $\exists \sigma \in \mathcal{D}$ and $0 \leq \beta < \alpha$ such that $U = \sigma^\beta$. Since $\forall\ x \in X,\ \sigma(x, x) = 1$ and so $\Delta \subseteq U$.

(iii) Let $U \in \mathcal{D}^\alpha$. Then $\exists \sigma \in \mathcal{D}$ and $0 \leq \beta < \alpha$ such that $U = \sigma^\beta$. We have $U_s = (\sigma^\beta)_s = (\sigma_s)^\beta$. But $\sigma_s \in \mathcal{D}$ and $0 \leq \beta < \alpha$. Therefore $U_s \in \mathcal{D}$.

(iv) Let $U \in \mathcal{D}^\alpha$. Then $\exists \sigma \in \mathcal{D}$ and $0 \leq \beta < \alpha$ such that $U = \sigma^\beta$. Let $\varepsilon = (\alpha - \beta)/2$ and $\gamma = (3\alpha + 2\beta)/5$. Then $\varepsilon > 0$. So $\exists \psi \in \mathcal{D}$ such that $\psi \circ \psi \leq \sigma + \varepsilon$. We have $\beta < \gamma < \alpha$. So $V = \psi^\gamma \in \mathcal{D}^\alpha$ and $V \circ V \subseteq U$ since:

$$\begin{aligned}
(x, y) \in V \circ V \quad &\Longleftrightarrow \exists z :\ (x, z), (z, y) \in V \\
&\Longleftrightarrow \sup_{z \in X}\ \psi(x, z) \wedge \psi(z, y) = \psi \circ \psi(x, y) > \gamma \\
&\Rightarrow (\sigma + \varepsilon)(x, y) > \gamma \\
&\Longleftrightarrow \sigma(x, y) > (\alpha + 9\beta)/10 > \beta \\
&\Longleftrightarrow (x, y) \in \sigma^\beta = U.
\end{aligned}$$

□

The uniformity $\mathcal{D}^\alpha$ will be referred to as the *α-level uniformity of $\mathcal{D}$*.

We also have for $0 < \beta \leq \alpha \leq 1$

$$\mathcal{D}^\beta \subseteq \mathcal{D}^\alpha \subseteq \mathcal{D}^1 \text{ and } \mathcal{D}^\alpha = \bigcup_{0<\gamma<\alpha} \mathcal{D}^\gamma.$$

Thus a fuzzy uniformity $\mathcal{D}$ generates a family $(\mathcal{D}^\alpha : \alpha \in I_0)$ of uniformities which become stronger as α increases.

We intend to build a fuzzy uniformity with a predetermined α-level uniformities.

6.6.2 Theorem
Let $(\mathbb{D}(\alpha) : \alpha \in (0, 1))$ be a family of uniformities on X satisfying

(a) $0 < \beta \leq \alpha < 1 \Rightarrow \mathbb{D}(\beta) \subseteq \mathbb{D}(\alpha)$,

(b) $\mathbb{D}(\alpha) = \underset{0<\gamma<\alpha}{\cup} \mathbb{D}(\gamma)$ for each $\alpha \in (0,1)$.

Let
$$\mathcal{D} = \{\sigma \in I^{X\times X} : \forall\ \alpha \in (0,1),\ \forall\ \beta < \alpha,\ \sigma^\beta \in \mathbb{D}(\alpha)\}.$$
Then $\mathcal{D}$ is the unique fuzzy uniformity on X such that $\mathcal{D}^\alpha = \mathbb{D}(\alpha)$ for each $\alpha \in (0,1)$.

Proof.
(i) $\mathcal{D}$ is a saturated prefilter.

Let $\sigma \in \mathcal{D}$ and $0 \leq \beta < 1$. Choose α such that $\beta < \alpha < 1$. Then $\sigma^\beta \in \mathbb{D}(\alpha)$ and so $\{(x,x) : x \in X\} \subseteq \sigma^\beta$ and hence $\sigma(x,x) > \beta$ for each $x \in X$. Since β is arbitrary we have
$$\forall\ x \in X,\ \sigma(x,x) = 1.$$
In particular, $\sigma \neq 0$.

Let $\sigma_1, \sigma_2 \in \mathcal{D}$ and $0 \leq \beta < \alpha < 1$. Then $\sigma_1^\beta, \sigma_2^\beta \in \mathbb{D}(\alpha)$ and so $\sigma_1^\beta \cap \sigma_2^\beta = (\sigma_1 \wedge \sigma_2)^\beta \in \mathbb{D}(\alpha)$. Thus $\sigma_1 \wedge \sigma_2 \in \mathcal{D}$.

Let $\sigma \in \mathcal{D}$. Then $\sigma_s \in \mathcal{D}$. Since if $0 \leq \beta < \alpha < 1$ then $(\sigma_s)^\beta = (\sigma^\beta)_s \in \mathbb{D}(\alpha)$.

Let $\sigma \in \mathcal{D},\ \sigma \leq \psi$ and $0 \leq \beta < \alpha < 1$. Then $\sigma^\beta \subseteq \psi^\beta$ and since $\sigma^\beta \in \mathbb{D}(\alpha),\ \psi^\alpha \in \mathbb{D}(\alpha)$. Consequently $\psi \in \mathcal{D}$.

Let $\sigma = \sup\limits_{\varepsilon \in I_0} (\sigma_\varepsilon - \varepsilon) \in \hat{\mathcal{D}}$ with each $\sigma_\varepsilon \in \mathcal{D}$ and let $0 \leq \beta < \alpha < 1$. We note that
$$\sigma(x,y) > \beta \iff \exists \varepsilon \in I_o : \sigma_\varepsilon(x,y) - \varepsilon > \beta \iff (x,y) \in \underset{\varepsilon \in I_0}{\cup} \sigma_\varepsilon^{\varepsilon+\beta}.$$
In otherwords, $\sigma^\beta = \underset{\varepsilon \in I_0}{\cup} \sigma_\varepsilon^{\varepsilon+\beta}$.

Choose $\varepsilon \in I_0$ such that $\beta < \beta + \varepsilon < \alpha$. Then $\sigma_\varepsilon^{\varepsilon+\beta} \subseteq \sigma^\beta$ with $\sigma_\varepsilon^{\varepsilon+\beta} \in \mathbb{D}(\alpha)$ and so $\sigma^\beta \in \mathbb{D}(\alpha)$. Thus $\sigma \in \mathcal{D}$ and we have shown that $\hat{\mathcal{D}} \subseteq \mathcal{D}$ from which it follows that $\hat{\mathcal{D}} = \mathcal{D}$.

(ii) $\forall\ \sigma \in \mathcal{D},\ \forall\ \varepsilon > 0,\ \exists \psi \in \mathcal{D}$ such that $\psi \circ \psi \leq \sigma + \varepsilon$.

Let $\sigma \in \mathcal{D},\ \varepsilon > 0$ and choose $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n = 1$ and $\alpha_i - \alpha_{i-1} < \varepsilon$ for each $i \in \{1,2,\ldots,n\}$. For $i = 0,1,2,\ldots,(n-1)$ we have $\sigma^{\alpha_i} \in \mathbb{D}(\alpha_{i+1}$ and so $\exists U_{\alpha_{i+1}} \in \mathbb{D}(\alpha_{i+1})$ such that $U_{\alpha_{i+1}} \circ U_{\alpha_{i+1}} \subseteq \sigma^{\alpha_i}$. Let $U'_{\alpha_1} = U_{\alpha_1}$ and $U'_{\alpha_i} = \underset{j \geq I}{\cap} U_{\alpha_j}$. Then since for each $j \leq I$, $U_{\alpha_j} \in \mathbb{D}(\alpha_j) \subseteq \mathbb{D}(\alpha_i$, we have $U'_{\alpha_i} \in \mathbb{D}(\alpha_i)$ and $U'_{\alpha_1} \sup\ seteq U'_{\alpha_i} \sup\ seteq \ldots \sup\ seteq U'_{\alpha_n}$. So we can state:

$\forall\ i \in \{1,2,\ldots,n\},\ \exists U_{\alpha_i} \in \mathbb{D}(\alpha_i : U_{\alpha_i} \circ U_{\alpha_i} \subseteq \sigma^{\alpha_{i-1}}$ and $U_{\alpha_1} \sup\ seteq U_{\alpha_2} \sup\ seteq \ldots \sup\ seteq U_{\alpha_n}$

Let
$$U_{\alpha_0} = X \times X$$
and let
$$\psi = \sup_{i \in \{1,2,\ldots,n\}} \alpha_i 1_{U_{\alpha_{i-1}}}.$$

Then $\psi \in \mathcal{D}$ since if $0 \leq \beta < \alpha < 1$ then $\alpha_i \leq \alpha < \alpha_{i+1}$ for some I. Thus $\beta < \alpha_{i+1}$ and hence $\psi^\beta \sup\ seteq \psi^{\alpha_{i+1}} = U_{\alpha_i} \in \mathbb{D}(\alpha_i) \subseteq \mathbb{D}(\alpha)$. It follows that $\psi^\beta \in \mathbb{D}(\alpha)$ and so $\psi \in \mathcal{D}$.

If $\sigma(x,y) > \alpha_{n-2}$ then $\sigma(x,y) + \varepsilon > \alpha_{n-2} + (\alpha_n - \alpha_{n-2}) = \alpha_n = 1$ and hence we have $(\psi \circ \psi)(x,y) \leq \sigma(x,y) + \varepsilon$.

If $\sigma(x,y) \leq \alpha_{n-2}$ then $\exists i \leq n-2 : \alpha_{i-1} \leq \sigma(x,y) \leq \alpha_i$. Since $(x,y) \notin \sigma^{\alpha_i}$ we have $(x,y) \notin U_{\alpha_{i+1}} \circ U_{\alpha_{i+1}}$ and for no z do we have $(x,z) \in U_{\alpha_{i+1}}$ and $(z,y) \in U_{\alpha_{i+1}}$. In otherwords
$$\forall\ z\ ((x,z) \notin U_{\alpha_{i+1}} \text{ or } (z,y) \notin U_{\alpha_{i+1}}).$$
Thus $\forall\ z$
$$\psi(x,z) \leq \alpha_{i+1} \text{ or } \psi(z,y) \leq \alpha_{i+1}.$$

Consequently,

$$\psi\circ\psi(x,y)=\sup_z \ \psi(x,z)\wedge\psi(z,y)\le\alpha_{i+1}<\alpha_{i-1}+\varepsilon\le\sigma(x,y)+\varepsilon.$$

(iii) $\mathcal{D}^\alpha=\mathbb{D}(\alpha)$ for each $\alpha\in(0,1)$.

Let $U\in\mathcal{D}^\alpha$. Then $U=\sigma^\beta$ for some $\sigma\in\mathcal{D}$ and $\beta<\alpha$ and so $\sigma^\beta\in\mathbb{D}(\alpha)$. Thus we have $\mathcal{D}^\alpha\subseteq\mathbb{D}(\alpha)$.

On the other hand, if $U\in\mathbb{D}(\alpha 0$ then $U\in\mathbb{D}(\beta)$ for some $\beta<\alpha$ since $\mathbb{D}(\alpha)=\bigcup_{\beta<\alpha}\mathbb{D}(\beta)$. Let $\sigma=\beta 1_{X\times X}\vee 1_U$. To show that $\sigma\in\mathcal{D}$ we let $\delta<1$ and $0\le\gamma<\delta$ and show that $\sigma^\gamma\in\mathbb{D}(\delta)$. If $\gamma<\beta$ we have $\sigma^\gamma=X\times X$ and if $\gamma\ge\beta$ then $\sigma^\gamma=U\in\mathbb{D}(\beta)\subseteq\mathbb{D}(\gamma)$. So in both cases we have $\sigma^\gamma\mathbb{D}(\gamma)\subseteq\mathbb{D}(\delta)$. Thus $U=\sigma^\beta$ and hence $U\in\mathcal{D}^\alpha$ and we have shown that $\mathbb{D}(\alpha)\subseteq\mathcal{D}^\alpha$.

(iv) $\mathcal{D}$ is unique.

We invoke (5.3.4) and claim that there is precisely one fuzzy uniformities whose α-level are the $\mathbb{D}(\alpha)$'s.

□

It follows that a fuzzy uniformity is uniquely determined by its family of α-level uniformities.

It is shown that the convergence of a prefilter can be expressed in terms of the convergence of its α-levels.

6.6.3 Theorem

Let $(X,\mathcal{D})$ be a fuzzy uniform space, $\mathcal{F}$ a prefiter on X, $x\in X$ and $\alpha<\bar{c}(\mathcal{F})$. Then

$$(\lim\ \mathcal{F})(x)\ge\alpha\Longleftrightarrow\mathcal{F}_0\longrightarrow x \text{ w.r.t } \mathcal{D}^\alpha.$$

Proof.

$\mathcal{D}^\alpha$ is a uniformity on X. Therefore $x\in X$

$$\mathcal{D}^\alpha_x=\{U(x):U\in\mathcal{D}^\alpha\}=\{\sigma^\beta(x):0\le\beta<\alpha,\ \sigma\in\mathcal{D}\}$$

is a neighbourhood base at x. We can also write

$$\mathcal{D}^\alpha_x=\{\sigma^\beta_s(x):0\le\beta<\alpha,\ \sigma\in\mathcal{D}\}.$$

we first prove the result for a prime prefilter $\mathcal{F}$. In this case we have $\lim\ \mathcal{F}=adh\,\mathcal{F}$, $\bar{c}(\mathcal{F})=c(\mathcal{F})$ and $\mathcal{F}_0$ is an ultrafilter.

We have

$$\begin{aligned}(adh\ \mathcal{F})(x)\ge\alpha\ &\Longleftrightarrow \inf_{\nu\in\mathcal{F}}\inf_{\sigma\in\mathcal{D}}\ \sigma<\nu>(x)\ge\alpha\\ &\Longleftrightarrow\forall\ \beta<\alpha,\ \forall\ \nu\in\mathcal{F},\ \forall\ \sigma\in\mathcal{D},\nu\wedge\sigma<x>>\beta\\ &\Longleftrightarrow\forall\ \sigma\in\mathcal{D},\ \forall\ \beta<\alpha,\ \forall\ \nu\in\mathcal{F},\ \nu^\beta\cap(\sigma<x>)^\beta\ne\emptyset\\ &\Longleftrightarrow\forall\ \sigma\in\mathcal{D},\ \forall\ \beta<\alpha,\ \forall\ F\in\mathcal{F}_0,\ F\cap\sigma^\beta_s(x)\ne\emptyset\ [\text{ since } \mathcal{F}_0=\mathcal{F}^\alpha\\ &\Longleftrightarrow\forall\ V\in\mathcal{D}^\alpha_x,\ \forall\ F\in\mathcal{F}_0,\ F\cap V\ne\emptyset\\ &\Longleftrightarrow\forall\ V\in\mathcal{D}^\alpha_x,\ U\in\mathcal{F}_0\\ &\Longleftrightarrow\mathcal{D}^\alpha_x\subseteq\mathcal{F}_0\\ &\Longleftrightarrow\mathcal{F}_0\longrightarrow x\text{ w.r.t }\mathcal{D}^\alpha.\end{aligned}$$

We have $\mathcal{F}_0=\bigcap_{\mathcal{G}\in\mathcal{P}_m(\mathcal{F})}\mathcal{G}_0$. Now

$$\begin{aligned}(\lim\ \mathcal{F})(x)\ge\alpha\ &\Longleftrightarrow\forall\ \mathcal{G}\in\mathcal{P}_m(\mathcal{F}),\ (adh\ \mathcal{G})(x)\ge\alpha\\ &\Longleftrightarrow\forall\ \mathcal{G}\in\mathcal{P}_m(\mathcal{F}),\ \mathcal{G}_0\longrightarrow x\text{ w.r.t }\mathcal{D}^\alpha\\ &\Longleftrightarrow\forall\ \mathcal{G}\in\mathcal{P}_m(\mathcal{F}),\ \mathcal{D}^\alpha_x\subseteq\mathcal{G}_0\\ &\Longleftrightarrow\mathcal{D}^\alpha_x\subseteq\bigcap_{\mathcal{G}\in\mathcal{P}_m(\mathcal{F})}\mathcal{G}_0=\mathcal{F}_0\\ &\Longleftrightarrow\mathcal{F}_0\longrightarrow x\text{ w.r.t }\mathcal{D}^\alpha.\end{aligned}$$

□

Next we characterise closedness in terms of the α-levels.

6.6.4 Theorem

If $(X, \mathcal{D})$ is a fuzzy uniform space and $\mu \in I^X$ then

$$\mu \text{ is } \mathcal{D} \text{ -closed } \iff \forall\ \alpha \in I_0, \mu_\alpha \text{ is } \mathcal{D}^\alpha \text{ -closed.}$$

Proof.

Let μ be a $\mathcal{D}$-closed, $\alpha \in I_0$ and let $x \in cl_\alpha(\nu_\alpha)$, the $\mathcal{D}^\alpha-$closure of μ_α. We have to show that $x \in \mu_\alpha$.

So let $\beta < \alpha$ be a arbitrary and show $\mu(x) \geq \beta$. Now since μ is $\mathcal{D}-$closed and so

$$\mu(x) = \bar{\mu}(x) = \inf_{\sigma \in \mathcal{D}} \sigma < \mu > (x) = \inf_{\sigma \in \mathcal{D}} \sup \quad \mu \wedge \sigma < x > .$$

Let $\sigma \in \mathcal{D}$. Then $\sigma^\beta \in \mathcal{D}^\alpha$. So we have $x \in \sigma^\beta(\mu_\alpha) \subseteq \sigma^\beta(\mu^\beta)$ and hence $\exists y \in \mu^\beta$ such that $\sigma(y, x) > \beta$. Thus

$$\mu \wedge \sigma < x > \geq \mu(y) \wedge \sigma(y, x) > \beta.$$

Since $\sigma \in \mathcal{D}$ is arbitrary, $\mu(x) \geq \beta$. Therefore $\mu(x) \geq \alpha$.

$(\Leftarrow)$

Let $x \in X$ and $\alpha \leq \bar{\mu}(x)$. Then

$$\begin{aligned}
\alpha \leq \bar{\mu}(x) = \inf_{\sigma \in \mathcal{D}} \sup \quad \mu \wedge \sigma < x > \quad & \iff \forall\ \beta < \alpha,\ \forall\ \sigma \in \mathcal{D},\ \beta < \ \mu \wedge \sigma < x > \\
& \iff \forall\ \beta < \alpha,\ \forall\ \sigma \in \mathcal{D},\ \exists y \in \mu^\beta \subseteq \mu_\beta :\ \sigma(y, x) > \beta \\
& \iff \forall\ \beta < \alpha,\ \forall\ \sigma \in \mathcal{D},\ x \in \sigma^\beta(\mu_\beta) \\
& \iff \forall\ \beta < \alpha,\ \forall\ U \in \mathcal{D}^\alpha,\ x \in U(\mu_\beta) \\
& \iff \forall\ \beta < \alpha,\ x \in cl_\alpha(\mu_\beta) = \mu_\beta \\
& \iff \forall\ \beta < \alpha,\ \mu(x) \geq \beta \\
& \iff \mu(x) \geq \alpha.
\end{aligned}$$

Therefore $\mu(x) \geq \bar{\mu}(x)$. Since x is arbitrary. Therefore $\mu \geq \bar{\mu}$ and hence $\mu = \bar{\mu}$.

□

From the following theorem we establish a fuzzy uniformity from a uniformity.

6.6.5 Theorem

Let $(X, \mathbb{D})$ be a uniform space. Then

$$\mathbb{D}^1 = \{\sigma \in I^{X \times X} : \forall\ \alpha \in I_1,\ \sigma^\alpha \in \mathbb{D}\}$$

is a fuzzy uniformity on X.

Proof.

(i) Since $\mathbb{D}$ is a filter. So $\mathbb{D}^1$ is a saturated prefiter.

(ii) Let $\sigma \in \mathbb{D}^1$ and $x \in X$. Then $\forall\ \alpha \in I_1,\ \sigma^\alpha \in \mathbb{D}$. So $\forall\ \alpha \in I_1,\ (x, x) \in \sigma^\alpha$. Therefore $\sigma(x, x) = 1$.

(iii) Let $\sigma \in \mathbb{D}^1$. Then $\forall\ \alpha \in I_1,\ \sigma^\alpha \in \mathcal{D}$ and so $(\sigma^\alpha)_s \in \mathbb{D}$. Therefore $\forall\ \alpha \in I_1,\ (\sigma_s)^\alpha = (\sigma^\alpha)_s \in \mathcal{D}$. Hence $\sigma_s \in \mathbb{D}^1$.

(iv) Let $\sigma \in \mathbb{D}^1$ and $\varepsilon \in I_0$. Then $\forall\ \alpha \in I_1,\ \sigma^\alpha \in \mathbb{D}$. Take $\delta = 1 - \varepsilon \in I_1$. Therefore $\sigma^\delta \in \mathbb{D}$. So $\exists U \in \mathbb{D}$ such that $U \circ U \subseteq \sigma^\delta$. Now take $\psi = 1_U$ then $\psi \in \mathbb{D}^1$.

If $\psi \circ \psi(x, y) = 0$ then clearly $\psi \circ \psi \leq \sigma + \varepsilon$.

If $\psi \circ \psi(x,y) = 1$ then $\exists z: \ (x,z),(z,y) \in U$ and so $(x,y) \in U \circ U \Rightarrow (x,y) \in \sigma^{\delta} \Rightarrow \sigma(x,y) > \delta$..

Therefore $\sigma(x,y) + \varepsilon > \delta + \varepsilon = 1$. So $\psi \circ \psi \leq \sigma + \varepsilon$. Hence the result.

□

6.6.6 Theorem

Let $(X, \mathbb{D})$ and $(Y, \mathbb{E})$ be uniform spaces. Then

$f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ is uniformly continuous $\iff f : (X, \mathbb{D}^1) \longrightarrow (Y, \mathbb{E}^1)$ is uniformly continuous.

Proof.

Let $\psi \in X \times X$ and $\alpha \in I_1$. Then

$$\begin{aligned}(x,y) \in (f \times f)^{\leftarrow}(\psi^{\alpha}) &\iff (f \times f)(x,y) \in \psi^{\alpha} \\ &\iff \psi((f \times f)(x,y)) > \alpha \\ &\iff (f \times f)^{-1}[\psi](x,y) > \alpha \\ &\iff (x,y) \in ((f \times f)^{-1}[\psi])^{\alpha}.\end{aligned}$$

Therefore $(f \times f)^{\leftarrow}(\psi^{\alpha}) = ((f \times f)^{-1}[\psi])^{\alpha}$.

$(\Rightarrow)$

Let $f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ be uniformly continuous. Then $\forall\ U \in \mathbb{E},\ (f \times f)^{\leftarrow}(U) \in \mathbb{D}$. Let $\psi \in \mathbb{E}^1$. Then $\forall\ \alpha \in I_1,\ \psi^{\alpha} \in \mathbb{E}$. So

$$((f \times f)^{-1}[\psi])^{\alpha} = (f \times f)^{\leftarrow}(\psi^{\alpha}) \in \mathbb{D}.$$

Thus $\forall\ \alpha \in I_1,\ (f \times f)^{-1}[\psi])^{\alpha} \in \mathbb{D}$. Therefore $(f \times f)^{-1}[\psi] \in \mathbb{D}^1$. Hence f is $(\mathbb{D}^1 - \mathbb{E}^1)$ uniformly continuous.

$(\Leftarrow)$

Let $f : (X, \mathbb{D}^1) \longrightarrow (Y, \mathbb{E}^1)$ be uniformly continuous. Then $\forall\ \psi \in \mathbb{E}^1,\ (f \times f)^{-1}[\psi] \in \mathbb{D}^1$. Let $U \in E$. Then $1_U \in \mathbb{E}^1$. Therefore $(f \times f)^{-1}[1_U \in \mathbb{D}^1$. Take $\alpha \in I_1$ then

$$((f \times f)^{-1}[1_U])^{\alpha} = (f \times f)^{\leftarrow}((1_U)^{\alpha}) = (f \times f)^{\leftarrow}(U) \in \mathbb{D}.$$

Therefore f is $(\mathbb{D} - \mathbb{E})$ uniformly continuous.

□

We obtain an α-level theorem for uniform continuity.

6.6.7 Theorem

Let $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ be fuzzy uniform spaces. Then

(1) $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is uniformly continuous $\Rightarrow \forall\ \alpha \in (0,1),\ f : (X, \mathcal{D}^{\alpha}) \longrightarrow (Y, \mathcal{E}^{\alpha})$ is uniformly continuous,

(2) $\forall\ \alpha \in (0,1),\ f : (X, \mathcal{D}^{\alpha}) \longrightarrow (Y, \mathcal{E}^{\alpha})$ is uniformly continuous $\Rightarrow f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is uniformly continuous.

Proof.

(1) Let $\alpha \in I_0$ and $U \in \mathcal{E}^{\alpha}$. Then $\exists \psi \in \mathcal{E}$ and $\beta < \alpha$ such that $U = \psi^{\beta}$. Since f is uniformly continuous and $\psi \in \mathcal{E}$. So

$$(f \times f)^{\leftarrow}(U) = (f \times f)^{\leftarrow}(\psi^{\beta}) = ((f \times f)^{-1}[\psi])^{\beta} \in \mathcal{D}^{\alpha}$$

Therefore f is $\mathcal{D}^{\alpha} - \mathcal{E}^{\alpha}$ uniformly continuous.

(2) Let $\psi \in \mathcal{E},\ \alpha \in (0,1)$ and $\beta < \alpha$. Then $\psi^{\beta} \in \mathcal{E}^{\alpha}$ and so

$$(f \times f)^{\leftarrow}(\psi^{\beta}) = ((f \times f)^{-1}[\psi])^{\beta} \in \mathcal{D}^{\alpha}.$$

Thus

(a) $\forall\ \alpha \in (0,1), \forall\ \beta < \alpha,\ ((f \times f)^{\leftarrow}[\psi])^{\beta} \in \mathcal{D}^{\alpha}$.

Now $\mathcal{D}$ is completely determined by its α-level; in other words

$$\mathcal{D} = \{\sigma \in I^{X \times X} : \forall\ \alpha \in (0,1),\ \forall\ \beta < \alpha,\ \sigma^{\beta} \in \mathcal{D}^{\alpha}\}$$

and so

(b) $\sigma \in \mathcal{D} \Longleftrightarrow \forall\ \alpha \in (0,1),\ \forall\ \beta < \alpha,\ \sigma^{\beta} \in \mathcal{D}^{\alpha}$.

It follows from (a) and (b) that $(f \times f)^{\leftarrow}[\psi] \in \mathcal{D}$ and hence f is $\mathcal{D} - \mathcal{E}$ uniformly continuous.

□

More information regarding fuzzy uniform space can be found in [31, 34, 35, 36, 41, 42, 44] and regarding fuzzy neighbourhood space can be found in [**?**, **?**, 5, **?**, 7, 38, 39, 68, 69, 70, 72, 73, 78].

Chapter 7

Generalised Filters

7.1 Introduction

In [16] the notion of generalised filter is introduced and studied. We summarize in in this chapter most of the results from [16].

7.1.1 Definition
We call a non-zero function $f : 2^X \to I$ a *generalised filter* (or a *g-filter*) on X iff

1. $f(\emptyset) = 0$;
2. $\forall A, B \subseteq X,\ f(A \cap B) \geq f(A) \wedge f(B)$;
3. $\forall A, B \subseteq X,\ A \subseteq B \Rightarrow f(A) \leq f(B)$.

Of course, the requirement that f be non-zero is equivalent to requiring that $f(X) > 0$.

For $f : 2^X \to I$ and $A \subseteq X$, we define

$$\langle f \rangle(A) \stackrel{\text{def}}{=} \sup_{B \subseteq A} f(B)$$

If f is non-zero and satisfies:

1. $f(\emptyset) = 0$;
2. $\forall A, B \subseteq X,\ f(A) \wedge f(B) \leq \langle f \rangle(A \cap B)$.

we shall call f a *generalised filter base* (or a *g-filter base*) on X.

Naturally, a g-filter is a g-filter base. Furthermore:

7.1.2 Theorem
If X is a set and f is a g-filter base on X then $\langle f \rangle$ is a g-filter.

PROOF.
(i) $\langle f \rangle(\emptyset) = \sup_{A \subseteq \emptyset} f(A) = f(\emptyset) = 0$, $\langle f \rangle(X) = \sup_{A \subseteq X} f(A) > 0$.

(ii) Let $A, B \subseteq X$. If $\langle f \rangle(A) \wedge \langle f \rangle(B) = 0$ then $\langle f \rangle(A) \wedge \langle f \rangle(B) \leq \langle f \rangle(A \cap B)$. So let $\alpha < \langle f \rangle(A) \wedge \langle f \rangle(B)$.
Then:

$$\begin{aligned}
\alpha < \sup_{U \subseteq A} f(U) \wedge \sup_{V \subseteq B} f(V) &\Rightarrow \exists U \subseteq A,\ \exists V \subseteq B,\ \alpha < f(U) \wedge f(V) \leq \langle f \rangle(U \cap V) \\
&\Rightarrow \exists W \subseteq U \cap V \subseteq A \cap B,\ \alpha < f(W) \\
&\Rightarrow \alpha < \langle f \rangle(A \cap B).
\end{aligned}$$

Thus $\langle f\rangle(A) \wedge \langle f\rangle(B) \leq \langle f\rangle(A \cap B)$.

(iii) If $A \subseteq B$ then:

$$\langle f\rangle(A) = \sup_{U \subseteq A} f(U) \leq \sup_{U \subseteq B} f(U) = \langle f\rangle(B).$$

$\square$

7.1.3 Definition

If f is a g-filter base on X, we define the *characteristic*, $c(f)$, of f by:

$$c(f) = \sup_{A \subseteq X} f(A).$$

It follows from definition that $c(f) > 0$.

Just as for prefilters, we have:

7.1.4 Lemma

If X is a set and f is a g-filter base on X then:

$$c(f) = c(\langle f\rangle).$$

Proof.

$$c(\langle f\rangle) = \sup_{A \subseteq X} \langle f\rangle(A) = \sup_{A \subseteq X} \sup_{B \subseteq A} f(B) = \sup_{B \subseteq X} f(B) = c(f).$$

$\square$

The proof of the following lemma is straightforward.

7.1.5 Lemma

Let f be a g-filter on X and let $A, B \subseteq X$. Then

1. $c(f) = f(X)$;
2. $f(A \cap B) = f(A) \wedge f(B)$.

If f is a g-filter (base) on X with $c(f) = c$ then for $0 \leq \alpha < c$, we define the *(upper) α-level filter (base)*, f^α, associated with f by:

$$f^\alpha \stackrel{\text{def}}{=} \{F \subseteq X : f(F) > \alpha\}$$

and for $0 < \alpha \leq c$, we define the *(lower) α-level filter (base)*, f_α, associated with f by:

$$f_\alpha \stackrel{\text{def}}{=} \{F \subseteq X : f(F) \geq \alpha\}.$$

7.1.6 Theorem

If f is a g-filter (base) on X with $c(f) = c$ and:
(a) $0 \leq \alpha < c$, then f^α is a filter (base) on X;
(b) $0 < \alpha \leq c$, then f_α is a filter (base) on X.

Proof.

(a) Let f be a g-filter on X. $f(X) = c > \alpha \Rightarrow X \in f^\alpha$. Thus $f^\alpha \neq \emptyset$.
If $F \in f^\alpha$ then $f(F) > \alpha \geq 0$ and hence $F \neq \emptyset$.
If $A, B \in f^\alpha$ then $f(A) \wedge f(B) = f(A \cap B) > \alpha$ and hence $A \cap B \in f^\alpha$.
Finally, if $A \in f^\alpha$ and $A \subseteq B$ then $f(B) \geq f(A) > \alpha$ and hence $B \in f^\alpha$.

The proofs of the remaining three assertions are equally simple.

□

It is an easy exercise to show that the α-level filters decrease as α increases and we record this as a lemma.

7.1.7 Lemma

If f is a g-filter (base) with $c(f) = c$ and $0 \leq \alpha \leq \beta < c$ then

$$f_c \subseteq f^\beta \subseteq f^\alpha \subseteq f^0.$$

The notion of a g-filter is a strict extension of the notion of a filter in the sense that we can associate a g-filter with every filter and there are g-filters which are not merely copies of filters. More precisely:

7.1.8 Theorem

Let X be a set, let $F(X)$ denote the collection of all filters on X and let $G(X)$ denote the collection of all g-filters on X. Let

$$\psi : F(X) \to G(X),\ \mathbb{F} \mapsto 1_{\mathbb{F}}.$$

Then ψ is injective but not surjective.

Proof.

The proof that $1_{\mathbb{F}}$ is a g-filter is left as an exercise.

To see that ψ is not surjective, let $X = \{1, 2, 3\}$ and consider

$$f : 2^X \to I$$

where

$$\begin{aligned} f(\emptyset) &= 0 \\ f(\{2\}) = f(\{2,3\}) = f(\{3\}) &= 0 \\ f(\{1\}) = f(\{1,2\}) = f(\{1,3\}) = f(X) &= \tfrac{3}{4} \end{aligned}$$

Alternatively, let

$$g(F) \stackrel{\text{def}}{=} \begin{cases} \frac{3}{4} & \text{if } F = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

and let $f = \langle g \rangle$. Then f is a g-filter and, of course, f cannot be the characteristic function of a filter.

□

We note the following examples of g-filters, leaving the checking to the reader.

7.1.9 Examples

(a) Let $X = \{1, 2, 3\}$ and define f by

$$\begin{aligned} f(F) &= 0 \text{ if } 1 \notin F \\ f(\{1\}) = f(\{1,3\}) &= \tfrac{1}{4} \\ f(\{1,2\}) &= \tfrac{3}{4} \\ f(\{1,2,3\}) &= 1 \end{aligned}$$

(b) For $n \in \mathbb{N}$ let $U_n \stackrel{\text{def}}{=} \{m \in \mathbb{N} : m \geq n\}$. Let $f : 2^{\mathbb{N}} \to I$ be defined by

$$f(F) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{\min\{n : U_n \subseteq F\}} & \text{if } \{n : U_n \subseteq F\} \neq \emptyset \\ 0 \text{ otherwise} \end{cases}$$

Alternatively, let

$$g(F) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{n} \text{ if } F = U_n \\ 0 \text{ otherwise} \end{cases}$$

and let $f = \langle g \rangle$.

(c) Let X be a set, $0 < \alpha \leq 1$ and let $a \in X$. Define $f_{\alpha,a}$ by

$$f_{\alpha,a}(F) \stackrel{\text{def}}{=} \begin{cases} \alpha \text{ if } a \in F \\ 0 \text{ otherwise} \end{cases}$$

In other words

$$f_{\alpha,a} = \alpha 1_{\langle \{a\} \rangle}.$$

Alternatively, let

$$g(F) \stackrel{\text{def}}{=} \begin{cases} \alpha \text{ if } F = \{a\} \\ 0 \text{ otherwise} \end{cases}$$

and let $f = \langle g \rangle$.

This g-filter has the special property that, for all $A, B \subseteq X$

$$f_{\alpha,a}(A \cup B) = f_{\alpha,a}(A) \vee f_{\alpha,a}(B).$$

If $h : X \to Y$ is a function and $f \in I^{2^X}$ is a g-filter on X then we define the *direct image of f,* denoted $h(f)$ by

$$h(f) : 2^Y \to I, \quad B \mapsto f(h^{-1}[B]).$$

In other words

$$h(f)(B) \stackrel{\text{def}}{=} f(h^{-1}[B]).$$

7.1.10 Theorem

If $h : X \to Y$ is a function and f is a g-filter on X then $h(f)$ is a g-filter on Y.

Proof.

(i) $h(f)(\emptyset) = f(h^{-1}[\emptyset]) = f(\emptyset) = 0$, $h(f)(Y) = f(h^{-1}[Y]) = f(X) > 0$.

(ii) If $A, B \in 2^Y$ then:

$$\begin{aligned} h(f)(A \cap B) &= f(h^{-1}[A \cap B]) \\ &= f(h^{-1}[A] \cap h^{-1}[B]) \\ &= f(h^{-1}[A]) \wedge f(h^{-1}[B]) \\ &= h(f)(A) \wedge h(f)(B). \end{aligned}$$

(iii) If $A \subseteq B$ then:

$$h(f)(A) = f(h^{-1}[A]) \leq f(h^{-1}[B]) = h(f)(B).$$

□

7.2 G-filters from Prefilters

Let $\mathcal{F}$ be a prefilter on X with $c(\mathcal{F}) = c > 0$. For $F \subseteq X$ define

$$S_{\mathcal{F}}(F) \stackrel{\text{def}}{=} \{\alpha \in (0, c] : F \in \mathcal{F}^\alpha\}.$$

7.2.1 Lemma
Let $\mathcal{F}$ be a prefilter with $c(\mathcal{F}) = c > 0$. Then $S_{\mathcal{F}}(F) = \emptyset$ or $S_{\mathcal{F}}(F)$ is an interval of form $(\beta, c]$.

PROOF.

If $S_{\mathcal{F}}(F) \neq \emptyset$ then there exists some $\alpha \in S_{\mathcal{F}}(F)$.
If $\alpha \leq \gamma \leq c$ then $F \in \mathcal{F}^\alpha \subseteq \mathcal{F}^\gamma$ and so $\gamma \in S_{\mathcal{F}}(F)$.
Since

$$\mathcal{F}^\alpha = \bigcup_{0<\beta<\alpha} \mathcal{F}^\beta,$$

we have:

$$\begin{aligned} \alpha \in S_{\mathcal{F}}(F) \quad &\Rightarrow F \in \mathcal{F}^\alpha = \bigcup_{0<\beta<\alpha} \mathcal{F}^\beta \\ &\Rightarrow \exists \beta < \alpha,\ F \in \mathcal{F}^\beta \\ &\Rightarrow \exists \beta < \alpha,\ \beta \in S_{\mathcal{F}}(F). \end{aligned}$$

□

This lemma allows us to define, for $F \subseteq X$:

$$f_{\mathcal{F}}(F) \stackrel{\text{def}}{=} \begin{cases} c - \inf S_{\mathcal{F}}(F) & \text{if } S_{\mathcal{F}}(F) \neq \emptyset \\ 0 & \text{if } S_{\mathcal{F}}(F) = \emptyset. \end{cases}$$

We now need to check that the object defined above is indeed a g-filter.

7.2.2 Theorem
If $\mathcal{F}$ is a prefilter with $c(\mathcal{F}) = c > 0$ then $f_{\mathcal{F}}$ is a g-filter.

PROOF.

(a)

$$\begin{aligned} \forall \alpha \leq c(\mathcal{F}),\ \emptyset \notin \mathcal{F}^\alpha \quad &\Rightarrow \forall \alpha \leq c(\mathcal{F}),\ \alpha \notin S_{\mathcal{F}}(\emptyset) \\ &\Rightarrow S_{\mathcal{F}}(\emptyset) = \emptyset \\ &\Rightarrow f_{\mathcal{F}}(\emptyset) = 0. \end{aligned}$$

(b) Let $A, B \subseteq X$. Then:

$$\begin{aligned} \alpha < f_{\mathcal{F}}(A) \wedge f_{\mathcal{F}}(B) \quad &\Rightarrow \inf S_{\mathcal{F}}(A) < c - \alpha \text{ and } \inf S_{\mathcal{F}}(B) < c - \alpha \\ &\Rightarrow c - \alpha \in S_{\mathcal{F}}(A) \text{ and } c - \alpha \in S_{\mathcal{F}}(B) \\ &\Rightarrow A, B \in \mathcal{F}^{c-\alpha} \\ &\Rightarrow A \cap B \in \mathcal{F}^{c-\alpha} \\ &\Rightarrow c - \alpha \in S_{\mathcal{F}}(A \cap B) \\ &\Rightarrow \inf S_{\mathcal{F}}(A \cap B) \leq c - \alpha \\ &\Rightarrow c - \inf S_{\mathcal{F}}(A \cap B) = f_{\mathcal{F}}(A \cap B) \geq \alpha. \end{aligned}$$

Thus

$$f_{\mathcal{F}}(A \cap B) \geq f_{\mathcal{F}}(A) \wedge f_{\mathcal{F}}(B).$$

(c) Let $A \subseteq B \subseteq X$. Then:

$$\begin{aligned} f_{\mathcal{F}}(A) > \alpha \quad & \Rightarrow c - \inf\, S_{\mathcal{F}}(A) > \alpha \\ & \Rightarrow \inf\, S_{\mathcal{F}}(A) < c - \alpha \\ & \Rightarrow c - \alpha \in S_{\mathcal{F}}(A) \\ & \Rightarrow A \in \mathcal{F}^{c-\alpha} \\ & \Rightarrow B \in \mathcal{F}^{c-\alpha} \\ & \Rightarrow c - \alpha \in S_{\mathcal{F}}(B) \\ & \Rightarrow \inf\, S_{\mathcal{F}}(B) \leq c - \alpha \\ & \Rightarrow c - \inf\, S_{\mathcal{F}}(B) = f_{\mathcal{F}}(B) \geq \alpha. \end{aligned}$$

It follows that $f_{\mathcal{F}}(A) \leq f_{\mathcal{F}}(B)$.

□

7.3 Prefilters from G-filters

Our next task is to show that a g-filter gives rise to a prefilter. However, we first discover the connection between the characteristic of a prefilter and the g-filter that it generates.

7.3.1 Lemma
If $\mathcal{F}$ is a prefilter on X with $c(\mathcal{F}) = c > 0$ then:

$$c(f_{\mathcal{F}}) = c(\mathcal{F}).$$

Proof.

Let $c = c(\mathcal{F})$. Then:

$$c(f_{\mathcal{F}}) = f_{\mathcal{F}}(X) = c - \inf\, S_{\mathcal{F}}(X).$$

Now

$$\begin{aligned} \forall \alpha \leq c,\ X \in \mathcal{F}^{\alpha} \quad & \Rightarrow \forall \alpha \leq c,\ \alpha \in S_{\mathcal{F}}(X) \\ & \Rightarrow \inf\, S_{\mathcal{F}}(X) = 0 \\ & \Rightarrow c(f_{\mathcal{F}}) = f_{\mathcal{F}}(X) = c \end{aligned}$$

□

For a g-filter f with $c(f) = c > 0$ we define:

$$\mathcal{F}_f \stackrel{\text{def}}{=} \{\nu \in I^X : \forall 0 < \alpha \leq c,\ \forall \beta < \alpha,\ \nu^{\beta} \in f^{c-\alpha}\}$$

Of course, we need to check that this does produce a prefilter.

7.3.2 Theorem
If f is a g-filter with $c(f) > 0$ then $\mathcal{F}_f$ is a prefilter.

Proof.

(a) We observe that

$$\forall \beta \leq c(f),\ 0^{\beta} = \{x \in X : 0(x) > \beta\} = \emptyset.$$

It follows that

$$\forall 0 < \alpha \leq c,\ \forall \beta < \alpha,\ 0^{\beta} \notin f^{c-\alpha}$$

and this means that $0 \notin \mathcal{F}_f$.

On the other hand, since $c = c(f) = f(X) > 0$ we have

$$\forall 0 < \alpha \leq c,\ \forall \beta < \alpha,\ (1_X)^{\beta} = X \in f^{c-\alpha}$$

and so $1_X \in \mathcal{F}_f$.

(b) Let $\mu, \nu \in \mathcal{F}_f$. It follows from:

$$\forall \beta < c(f),\ \nu^\beta \cap \mu^\beta = (\nu \wedge \mu)^\beta$$

that $\nu \wedge \mu \in \mathcal{F}_f$.

(c) Let $\nu \in \mathcal{F}$ and $\nu \leq \mu$. Let $0 < \alpha \leq c,\ \beta < \alpha$. Then $\nu^\beta \in f^{c-\alpha}$. Since $\nu^\beta \subseteq \mu^\beta$ we have

$$f(\mu^\beta) \geq f(\nu^\beta) \geq c - \alpha.$$

Thus $\mu \in \mathcal{F}_f$.

□

The following lemma simplifies some of the work later on.

7.3.3 Lemma

If f is a g-filter with $c(f) = c > 0$ then:

$$\mathcal{F}_f = \{v \in I^X : \forall 0 \leq \gamma < c,\ \nu^\gamma \in f_{c-\gamma}\}.$$

Proof.

Let us define

$$\mathcal{G} \stackrel{\text{def}}{=} \{v \in I^X : \forall 0 \leq \gamma < c,\ \nu^\gamma \in f_{c-\gamma}\}.$$

Let $\nu \in \mathcal{F}_f$. To show that $\nu \in \mathcal{G}$ let $0 \leq \gamma < c$. Choose α such that $\gamma < \alpha < c$. Then $\nu^\gamma \in f^{c-\alpha}$. Since α is arbitrary, we have

$$\forall \gamma < \alpha < c,\ f(\nu^\gamma) > c - \alpha$$

and hence $f(\nu^\gamma) \geq c - \gamma$. In other words, $\nu^\gamma \in f_{c-\gamma}$.

Conversely, let $\nu \in \mathcal{G}$. To show that $\nu \in \mathcal{F}_f$ let $0 < \alpha \leq c, 0 \leq \beta < \alpha$. Then we have $0 \leq \beta < c$. Thus $\nu^\beta \in f_{c-\beta}$ and so $f(\nu^\beta) \geq c - \beta > c - \alpha$. Therefore $\nu^\beta \in f^{c-\alpha}$.

□

The correlation between g-filters and prefilters is not completely straightforward. In fact, as we shall see, the prefilter associated with a g-filter is rather special.

7.3.4 Theorem

If f is a g-filter then the associated prefilter $\mathcal{F}_f$ is saturated.

Proof.

Suppose that

$$\forall \varepsilon > 0,\ \nu + \varepsilon \in \mathcal{F}_f.$$

We show that $\nu \in \mathcal{F}_f$. To this end, we let $\alpha \leq c(f)$ and $\beta < \alpha$ and show that $\nu^\beta \in f^{c-\alpha}$. Choose γ such that $\beta < \gamma < \alpha$ and let $\varepsilon = \gamma - \beta$. Then, since $\nu + \varepsilon \in \mathcal{F}_f$ we have

$$(\nu + \varepsilon)^\gamma = (\nu + \varepsilon)^{\beta+\varepsilon} = \nu^\beta \in f^{c-\alpha}.$$

□

We saw, in Lemma 7.3.1, the connection between the characteristic of a prefilter and the g-filter that it generates. Let us now find the connection between a g-filter and the prefilter that it generates.

7.3.5 Theorem
Let f be a g-filter on X. Then:
$$c(\mathcal{F}_f) = c(f).$$

PROOF.

Let $c(f) = c$. Then:

$$\begin{aligned}
\forall \nu \in \mathcal{F}_f,\ \forall 0 \le \beta < \alpha \le c,\ \nu^\beta \in f^{c-\alpha} &\Rightarrow \forall \nu \in \mathcal{F}_f,\ \forall 0 \le \beta < \alpha \le c,\ \nu^\beta \neq \emptyset \\
&\Rightarrow \forall \nu \in \mathcal{F}_f,\ \forall 0 \le \beta < \alpha \le c,\ \sup \nu > \beta \\
&\Rightarrow \forall \nu \in \mathcal{F}_f,\ \sup \nu \ge c \\
&\Rightarrow \inf_{\nu \in \mathcal{F}_f} \sup \nu = c(\mathcal{F}_f) \ge c.
\end{aligned}$$

On the other hand,

$$\forall \alpha \le c,\ \forall \beta < \alpha,\ (c1_X)^\beta = X \in f^{c-\alpha}$$

and so $c1_X \in \mathcal{F}_f$. Thus
$$c(\mathcal{F}_f) = \inf_{\nu \in \mathcal{F}_f} \sup \nu \le \sup c1_X = c.$$

□

The use of α-level theorems has proved to be very useful in various situations. See, for example [82, 83, 39]. We therefore investigate the α-levels of g-filters.

7.3.6 Lemma
Let f be a g-filter with $c(f) = c$ and let $\alpha \in (0, c]$. Then

$${\mathcal{F}_f}^\alpha = f^{c-\alpha}.$$

PROOF.

Let $F \in (\mathcal{F}_f)^\alpha$. Then there exists $\nu \in \mathcal{F}_f,\ \beta < \alpha$ such that $F = \nu^\beta$. Since $\nu \in \mathcal{F}_f$, we have $\nu^\beta = F \in f^{c-\alpha}$.

Conversely, if $F \in f^{c-\alpha}$ then $f(F) \overset{\text{def}}{=} t > c - \alpha$. Let $\nu = (c-t)1_X \vee 1_F$. We intend to invoke Lemma 7.3.3 to show that $\nu \in \mathcal{F}_f$. To this end, let $0 \le \gamma < c$.

If $\gamma \in [c-t, c)$ then $\nu^\gamma = F$ and so $f(\nu^\gamma) = f(F) = t \ge c - \gamma$.

If $\gamma \in [0, c-t)$ then $\nu^\gamma = X$ and so $f(\nu^\gamma) = f(X) = c \ge c - \gamma$.

We therefore have $\nu^\gamma \in f_{c-\gamma}$ for all $\gamma \in [0, c)$ and so $\nu \in \mathcal{F}_f$ and $F = \nu^{c-t}$ with $c - t < \alpha$. Thus $F \in \mathcal{F}_f^\alpha$.

□

7.3.7 Lemma
If $\mathcal{F}$ is a prefilter on X with $c(\mathcal{F}) = c > 0$ then, for $\alpha \in [0, c)$:

$$(f_\mathcal{F})^\alpha = \mathcal{F}^{c-\alpha}.$$

PROOF.

$$\begin{aligned}
A \in (f_\mathcal{F})^\alpha &\iff f_\mathcal{F}(A) = c - \inf S_\mathcal{F}(A) > \alpha \\
&\iff \inf S_\mathcal{F}(A) < c - \alpha \\
&\iff c - \alpha \in S_\mathcal{F}(A) \\
&\iff A \in \mathcal{F}^{c-\alpha}
\end{aligned}$$

□

We now establish the g-filter analogue of Theorem 5.3.2(5)

7.3.8 Lemma
If f is a g-filter with $c(f) = c$ and $0 \leq \alpha < c$ then:

$$f^\alpha = \bigcup_{c>\beta>\alpha} f^\beta.$$

Proof.

If $F \in f^\alpha$ choose β such that $\alpha < \beta < f(F)$. Then $F \in f^\beta$ and so $F \in \bigcup_{\beta>\alpha} f^\beta$.

Conversely, let $F \in \bigcup_{c>\beta>\alpha} f^\beta$. Then $F \in f^\beta$ for some $\beta > \alpha$. Thus $f(F) > \beta > \alpha$ and hence $F \in f^\alpha$.

□

7.3.9 Corollary
If f is a g-filter with $c(f) = c$ and $0 < \alpha \leq c$ then:

$$f^{c-\alpha} = \bigcup_{0<\beta<\alpha} f^{c-\beta}.$$

In [9] we saw that saturated prefilters are specified by their α-level filters. We show that a similar situation pertains for g-filters

7.3.10 Lemma
If f and g are g-filters with $c(f) \neq c(g)$ then $f \neq g$.

7.3.11 Lemma
Let f and g be g-filters with $c(f) = c(g) = c$. Then:

$$\forall \alpha < c,\ f^\alpha = g^\alpha \iff f = g.$$

Proof.

$$\begin{aligned}
\forall \alpha < c,\ f^\alpha = g^\alpha \quad &\iff \forall \alpha < c,\ \forall A \subseteq X,\ A \in f^\alpha \iff A \in g^\alpha \\
&\iff \forall A \subseteq X,\ \forall \alpha \leq c,\ (f(A) > \alpha \iff g(A) > \alpha) \\
&\iff \forall A \subseteq X,\ f(A) = g(A) \\
&\iff f = g.
\end{aligned}$$

□

We have seen that to each g-filter there corresponds a saturated prefilter and, conversely, to each prefilter there corresponds a g-filter. This inspires the following theorem.

7.3.12 Theorem
Let

$$\mathcal{S}(X) \stackrel{\text{def}}{=} \{\mathcal{F} \in 2^{I^X} : \mathcal{F} \text{ is a saturated prefilter on } X\ \},$$

$$\mathcal{G}(X) \stackrel{\text{def}}{=} \{f \in I^{2^X} : f \text{ is a g-filter on } X\ \}.$$

Then

$$\psi : \mathcal{S}(X) \to \mathcal{G}(X),\ \mathcal{F} \mapsto f_\mathcal{F}$$

is a bijection.

PROOF.

We first show that ψ is injective. To this end, let $\mathcal{F}, \mathcal{G} \in \mathcal{S}(X)$ with $\mathcal{F} \neq \mathcal{G}$.

If $c(\mathcal{F}) \neq c(\mathcal{G})$ then:

$$\begin{aligned} f_{\mathcal{F}}(X) &= c(\mathcal{F}) - \inf\{\alpha \leq c(\mathcal{F}) : X \in \mathcal{F}^\alpha\} \\ &= c(\mathcal{F}) - 0 \\ &\neq c(\mathcal{G}) \\ &= f_{\mathcal{G}}(X) \end{aligned}$$

and so $f_{\mathcal{F}} \neq f_{\mathcal{G}}$.

If $c(\mathcal{F}) = c(\mathcal{G}) = c$ then:

$$\exists \alpha \leq c,\ \mathcal{F}^\alpha \neq \mathcal{G}^\alpha.$$

This follows from the fact that *saturated* prefilters are completely determined by their α-level filters [9]: Theorem 2 and [**?**]: Theorem 11). Suppose that $F \in \mathcal{F}^\alpha \setminus \mathcal{G}^\alpha$. Then $\alpha \in S_{\mathcal{F}}(F) \setminus S_{\mathcal{G}}(F)$. Thus inf $S_{\mathcal{F}}(F) < \alpha$ and $\alpha \leq$ inf $S_{\mathcal{G}}(F)$. Thus

$$f_{\mathcal{F}}(F) = c - \inf\ S_{\mathcal{F}}(F) > c - \alpha \geq c - \inf\ S_{\mathcal{G}}(F) = f_{\mathcal{G}}(F).$$

So, once again, $f_{\mathcal{F}} \neq f_{\mathcal{G}}$.

In order to show that ψ is surjective, let $f \in \mathcal{G}(X)$ and let $c(\mathcal{F}) = c$. Then:

$$\mathcal{F}_f = \{\nu \in I^X : \forall \alpha \leq c,\ \forall \beta < \alpha,\ \nu^\beta \in f^{c-\alpha}\}.$$

Now, appealing to Lemmas 7.3.6 and 7.3.7, we have

$$\forall \alpha \in [0, c),\ (f_{\mathcal{F}_f})^\alpha = (\mathcal{F}_f)^{c-\alpha} = f^{c-(c-\alpha))} = f^\alpha.$$

It therefore follows from Lemma 7.3.11 that

$$\psi(\mathcal{F}_f) = f_{\mathcal{F}_f} = f.$$

□

We extract the following corollaries.

7.3.13 Corollary
If f is a g-filter on X then:

$$f_{\mathcal{F}_f} = f.$$

7.3.14 Corollary
If $\mathcal{F}$ is a saturated prefilter on X then:

$$\mathcal{F}_{f_{\mathcal{F}}} = \mathcal{F}.$$

PROOF.

Let $\psi : \mathcal{S} \to \mathcal{G}$ as in the theorem. Then:

$$\psi(\mathcal{F}) = f_{\mathcal{F}}$$

and

$$\psi(\mathcal{F}_{f_{\mathcal{F}}}) = f_{(\mathcal{F}_{f_{\mathcal{F}}})} = f_{\mathcal{F}}.$$

Thus it follows from the injectivity of ψ that $\mathcal{F} = \mathcal{F}_{f_{\mathcal{F}}}$.

□

We have developed the g-filter analogues of various prefilter notions and it is natural therefore to seek a g-filter analogue of the saturation operator. In other words, if f is a g-filter, we seek a definition of $\widehat{f}$, the *saturation* of f, which is consistent with the theory which we have developed thus far. We would require, among other things, that the saturation of the g-filter associated with a prefilter is the g-filter associated with the saturation of the prefilter. In symbols:

$$\widehat{f_{\mathcal{F}}} = f_{\widehat{\mathcal{F}}}.$$

However, we have the following lemma.

7.3.15 Lemma
If $\mathcal{F}$ is a prefilter then:

$$f_{\mathcal{F}} = f_{\widehat{\mathcal{F}}}.$$

PROOF.

For $F \subseteq X$:

$$\begin{aligned} f_{\widehat{\mathcal{F}}}(F) &= c(\widehat{\mathcal{F}}) - \inf\{\alpha : F \in (\widehat{\mathcal{F}})^\alpha\} \\ &= c(\mathcal{F}) - \inf\{\alpha : F \in (\mathcal{F})^\alpha\} \\ &= f_{\mathcal{F}}(F). \end{aligned}$$

□

Thus, for a prefilter $\mathcal{F}$

$$\widehat{f_{\mathcal{F}}} = f_{\mathcal{F}}.$$

The most natural definition of $\widehat{f}$ which accomplishes this is the simple

$$\widehat{f} \stackrel{\text{def}}{=} f.$$

In this sense, g-filters are already saturated. This explains why, in [15], the definition of a generalised uniformity did not include a saturation condition. The situation is also illustrated by the following theorem which extends Theorem 7.3.4.

7.3.16 Theorem
If $\mathcal{F}$ is a prefilter then:

$$\mathcal{F}_{f_{\mathcal{F}}} = \widehat{\mathcal{F}}.$$

PROOF.

From Theorem 7.3.4 we know that $\mathcal{F}_{f_{\mathcal{F}}}$ is a saturated prefilter and so, according to [29], Theorem 11, we must show that

$$\forall \alpha \leq c(\widehat{\mathcal{F}}) = c(\mathcal{F}),\ (\mathcal{F}_{f_{\mathcal{F}}})^\alpha = (\widehat{\mathcal{F}})^\alpha.$$

Now

$$(\mathcal{F}_{f_{\mathcal{F}}})^\alpha = (f_{\mathcal{F}})^{c-\alpha} = \mathcal{F}^\alpha = \widehat{\mathcal{F}}^\alpha.$$

□

From this last result we obtain the following characterisation of the saturation of a prefilter.

7.3.17 Corollary
If $\mathcal{F}$ is a prefilter with $c(\mathcal{F}) = c > 0$ then:

$$\widehat{\mathcal{F}} = \{\nu \in I^X : \forall 0 < \alpha \leq c,\ \forall \beta < \alpha,\ \nu^\beta \in \mathcal{F}^\alpha\}.$$

Proof.

$$\begin{aligned} \nu \in \widehat{\mathcal{F}} &\iff \nu \in \mathcal{F}_{f_\mathcal{F}} \\ &\iff \forall 0 < \alpha \leq c, \forall \beta < \alpha,\ \nu^\beta \in (f_\mathcal{F})^{c-\alpha} = \mathcal{F}^\alpha. \end{aligned}$$

□

7.4 Prime G-Filters

We seek a suitable definition of a prime g-filter which ties in with the theory of prime prefilters.

7.4.1 Definition
We call a g-filter f on X *prime* if

$$\forall A, B \subseteq X,\ f(A \cup B) = f(A) \vee f(B).$$

For example:
for $a \in X, \alpha \in I_0$, it is straightforward to check that $f = f_{\alpha,a}$ defined earlier is a g-filter and that f is prime.

The Theorem 5.3.10, which characterises the minimal prime prefilters finer than a given prefilter, has found a number of applications. With this in mind, we attempt to construct a similar theory of prime g-filters.

We first find the connection between prime g-filters and ultrafilters.

7.4.2 Lemma
Let $\mathbb{F}$ be a filter on X and let $0 < \alpha \leq 1$. Then

$$\mathbb{F} \text{ is an ultrafilter } \iff \alpha 1_\mathbb{F} \text{ is a prime g-filter.}$$

Proof.

($\Rightarrow$)

Let $\mathbb{F}$ be an ultrafilter. If $A \cup B \in \mathbb{F}$ then $\alpha 1_\mathbb{F}(A \cup B) = \alpha$. Furthermore, since $\mathbb{F}$ is an ultrafilter, $A \in \mathbb{F}$ or $B \in \mathbb{F}$. Thus

$$\alpha 1_\mathbb{F}(A) \vee \alpha 1_\mathbb{F}(B) = \alpha = \alpha 1_\mathbb{F}(A \cup B).$$

If $A \cup B \notin \mathbb{F}$ then $1_\mathbb{F}(A \cup B) = 0$. Since $\mathbb{F}$ is a filter, $A \notin \mathbb{F}$ and $B \notin \mathbb{F}$ and hence

$$\alpha 1_\mathbb{F}(A) \vee \alpha 1_\mathbb{F}(B) = 0 = \alpha 1_\mathbb{F}(A \cup B).$$

($\Leftarrow$)

Let $\alpha 1_\mathbb{F}$ be prime and let $A \cup B \in \mathbb{F}$. Then

$$\alpha 1_\mathbb{F}(A \cup B) = \alpha = \alpha 1_\mathbb{F}(A) \vee \alpha 1_\mathbb{F}(B).$$

Therefore

$$\alpha 1_\mathbb{F}(A) = \alpha \text{ or } \alpha 1_\mathbb{F}(B) = \alpha$$

and so $A \in \mathbb{F}$ or $B \in \mathbb{F}$. Thus $\mathbb{F}$ is an ultrafilter.

□

7.4.3 Theorem
Let f be a g-filter with $c(f) = c$. Then

$$f \text{ is prime} \iff f_c \text{ is an ultrafilter.}$$

PROOF.

($\Rightarrow$)

$$\begin{aligned} A \cup B \in f_c &\iff f(A \cup B) = f(A) \vee f(B) = c \\ &\iff f(A) = c \text{ or } f(B) = c \\ &\iff A \in f_c \text{ or } B \in f_c \end{aligned}$$

($\Leftarrow$)

If $\alpha < c$ then

$$\begin{aligned} \alpha < f(A \cup B) &\Rightarrow A \cup B \in f^\alpha = f_c \\ &\Rightarrow f(A \cup B) = c \text{ and } A \in f_c \text{ or } B \in f_c \\ &\Rightarrow f(A) \vee f(B) = c = f(A \cup B) \end{aligned}$$

□

7.4.4 Corollary
If f is a prime g-filter with $c(f) = c$ then $f_c = f^0$.

PROOF.

We have $f_c \subseteq f^0$ and f_c is an ultrafilter.

□

The reader can check that if $A \subseteq X$, $\alpha > 0$ and $\mathbb{F} = \langle \{A\} \rangle$ then

$$\alpha 1_{\mathbb{F}} \text{ is prime} \iff A \text{ is a singleton.}$$

If $\mathbb{F}$ is a filter then we define

$$\mathbb{P}(\mathbb{F}) \stackrel{\text{def}}{=} \{\mathbb{K} : \mathbb{F} \subseteq \mathbb{K},\ \mathbb{K} \text{ is an ultrafilter}\}$$

We now investigate the situation with regard to prime g-filters finer than a given g-filter.

7.4.5 Lemma
If f is a g-filter then

$$\mathbb{F} \in \mathbb{P}(f^0) \Rightarrow 1_{\mathbb{F}} \text{ is a prime g-filter and } f \le 1_{\mathbb{F}}.$$

PROOF.

Let $g = 1_{\mathbb{F}}$. Then, by Lemma 7.4.2, g is prime.
If $A \subseteq X$ and $f(A) > 0$ then $A \in f^0 \subseteq \mathbb{F}$. Thus $g(A) = 1 \ge f(A)$ and so $f \le g$.

□

7.4.6 Corollary
If f is a g-filter, $\alpha \ge c = c(f)$ and $\mathbb{F} \in \mathbb{P}(f^0)$ then $\alpha 1_{\mathbb{F}}$ is a prime g-filter with $f \le \alpha 1_{\mathbb{F}}$.

PROOF.

It follows from lemma 7.4.2 that $\alpha 1_{\mathbb{F}}$ is prime. Furthermore, if $A \subseteq X$ then

$$\begin{aligned} f(A) > 0 &\Rightarrow A \in f^0 \subseteq \mathbb{F} \\ &\Rightarrow \alpha 1_{\mathbb{F}}(A) = \alpha \ge c = f(X) \ge f(A) \end{aligned}$$

□

7.4.7 Theorem
If f is a prime g-filter with $c(f) = c$ and $\mathbb{F} = f_c$ then

$$f = c1_{\mathbb{F}}.$$

PROOF.

Let $A \subseteq X$. If $f(A) > 0$ then $A \in f^0 = f_c = \mathbb{F}$ and hence $f(A) = c = c1_{\mathbb{F}}(A)$.
If $f(A) = 0$ then $A \notin \mathbb{F}$ and so $f(A) = 0 = c1_{\mathbb{F}}(A)$.

□

Thus prime g-filters are precisely those g-filters of the form $\alpha 1_{\mathbb{F}}$ with $\mathbb{F}$ an ultrafilter.
If f is a g-filter on X, let

$$\mathcal{P}(f) \stackrel{\text{def}}{=} \{g : g \text{ is a prime g-filter and } f \leq g.\}$$

We now aim for the g-filter equivalent of Lowen's Theorem 5.3.10.

7.4.8 Theorem
If f is a g-filter with $c(f) = c$ then

$$\mathcal{P}(f) = \{\alpha 1_{\mathbb{F}} : \mathbb{F} \in \mathbb{P}(f^0),\ \alpha \geq c\}.$$

PROOF.

Let $g \in \mathcal{P}(f)$ with $c(g) = \alpha$ and $\mathbb{F} = g_\alpha$. Then, by Theorem 7.4.7, $g = \alpha 1_{\mathbb{F}}$ with $\mathbb{F}$ an ultrafilter. Furthermore, since $f \leq g$, we have $c(f) \leq \alpha = c(g)$ and $\mathbb{F} \supseteq f^0$.
Conversely, if $g = \alpha 1_{\mathbb{F}}$ then, by Corollary 7.4.8, $g \in \mathcal{P}(f)$.

□

For a g-filter f let us define

$$\mathcal{P}_m(f) \stackrel{\text{def}}{=} \{g : g \text{ is a minimal prime g-filter and } f \leq g\}.$$

It is now an easy matter to obtain a characterisation of the minimal prime g-filters which are finer than a given g-filter.

7.4.9 Corollary
If f is a g-filter with $c(f) = c$ then

$$\mathcal{P}_m(f) = \{c1_{\mathbb{F}} : \mathbb{F} \in \mathbb{P}(f^0)\}$$

PROOF.

Let $g \in \mathcal{P}_m(f)$. Then $g = \alpha 1_{\mathbb{F}}$ for some $\alpha \geq c$ and some $\mathbb{F} \in \mathbb{P}(f^0)$. If $\alpha > c$ then we can choose β such that $c < \beta < \alpha$ and then $h = \beta 1_{\mathbb{F}} \in \mathcal{P}(f)$ with $h \leq g$ and $h \neq g$ which contradicts the minimality of g.

□

Our next task is to find the relationship between prime prefilters and prime g-filters. We first need the following lemma.

7.4.10 Lemma
Let $(I, \leq)$ be a totally ordered set and let $(X, \preceq)$ be a partially ordered set. Let

$$\varphi, \psi : (I, \leq) \to (X, \preceq)$$

be decreasing functions in the sense that

$$\forall \alpha, \beta \in I,\ (\alpha \leq \beta \Rightarrow \varphi(\beta) \preceq \varphi(\alpha),\ \psi(\beta) \preceq \psi(\alpha)).$$

Let $F \subseteq X$ have the property

$$\forall x,\ (x \in F,\ x \preceq y,\ \Rightarrow y \in F).$$

Then

$$\forall \alpha \in I,\ (\varphi(\alpha) \in F \text{ or } \psi(\alpha) \in F) \iff (\forall \alpha \in I,\ \varphi(\alpha) \in F) \text{ or } (\forall \alpha \in I,\ \psi(\alpha) \in F).$$

Proof.

We only have to show the forward implication so suppose that $\exists \alpha \in I$ such that $\varphi(\alpha) \notin F$. We must show that $\forall \beta \in I,\ \psi(\beta) \in F$. Now

$$\varphi(\alpha) \notin F \Rightarrow \psi(\alpha) \in F.$$

Thus if $\beta \leq \alpha$ then

$$\psi(\alpha) \preceq \psi(\beta) \Rightarrow \psi(\beta) \in F.$$

On the other hand, if $\alpha < \beta$ then

$$\begin{aligned} \varphi(\beta) \preceq \varphi(\alpha) \quad &\Rightarrow \varphi(\beta) \notin F(\text{ otherwise } \varphi(\alpha) \in F.\ ) \\ &\Rightarrow \psi(\beta) \in F. \end{aligned}$$

□

7.4.11 Corollary
Let $I \subseteq \mathbb{R}$ be an interval, X a set and let $\varphi, \psi : I \to \mathcal{P}(X)$ be functions with the property that

$$\forall \alpha, \beta \in I,\ (\alpha \leq \beta \Rightarrow \varphi(\beta) \subseteq \varphi(\alpha),\ \psi(\beta) \subseteq \psi(\alpha)).$$

and let $\mathbb{F}$ be a filter on X. Then

$$\forall \alpha \in I,\ (\varphi(\alpha) \in \mathbb{F} \text{ or } \psi(\alpha) \in \mathbb{F}) \iff (\forall \alpha \in I, \varphi(\alpha) \in \mathbb{F}) \text{ or } (\forall \in I,\ \psi(\alpha) \in \mathbb{F}).$$

7.4.12 Theorem
Let f be a prime g-filter on a set X with $c(f) = c$. Then $\mathcal{F}_f$ is also prime.

Proof.

Let $\mu \vee \nu \in \mathcal{F}_f$. Then, according to Lemma 7.3.3, Theorem **??** and Corollary **??**,

$$\forall \gamma \in [0, c),\ (\mu \vee \nu)^\gamma = \mu^\gamma \cup \nu^\gamma \in f_{c-\gamma} = f_c \stackrel{\text{def}}{=} \mathbb{F}$$

with $\mathbb{F}$ an ultrafilter on X. We therefore have

$$\forall \gamma \in [0, c),\ (\mu^\gamma \in \mathbb{F} \text{ or } \nu^\gamma \in \mathbb{F}).$$

We now invoke Corollary 7.4.11 and claim that

$$(\forall \gamma \in [0, c),\ \mu^\gamma \in \mathbb{F}) \text{ or } (\forall \gamma \in [0, c),\ \nu^\gamma \in \mathbb{F}).$$

This, together with Lemma 7.3.3, shows that $\mu \in \mathbb{F}$ or $\nu \in \mathbb{F}$.

□

7.4.13 Theorem
Let $\mathcal{F}$ be a prime prefilter on a set X with $c(\mathcal{F}) = c$. Then $f_{\mathcal{F}}$ is also prime.

PROOF.

We need to show that, for $A, B \subseteq X$

$$f_{\mathcal{F}}(A \cup B) \leq f_{\mathcal{F}}(A) \vee f_{\mathcal{F}}(B).$$

To this end let $0 < \alpha < f_{\mathcal{F}}(A \cup B)$. Then

$$\begin{aligned}
\alpha < c - \inf S_{\mathcal{F}}(A \cup B) \quad & \iff c - \alpha \in S_{\mathcal{F}}(A \cup B) \\
& \iff A \cup B \in \mathcal{F}^{c-\alpha} = \mathcal{F}_0 \\
& \iff A \in \mathcal{F}_0 \text{ or } B \in \mathcal{F}_0 \quad \text{(since } \mathcal{F}_0 \text{ is an ultrafilter)} \\
& \iff c - \alpha \in S_{\mathcal{F}}(A) \text{ or } c - \alpha \in S_{\mathcal{F}}(B) \\
& \Rightarrow \inf S_{\mathcal{F}}(A) \leq c - \alpha \text{ or } \inf S_{\mathcal{F}}(B) \leq c - \alpha \\
& \Rightarrow f_{\mathcal{F}}(A) \geq \alpha \text{ or } f_{\mathcal{F}}(B) \geq \alpha \\
& \Rightarrow f_{\mathcal{F}}(A) \vee f_{\mathcal{F}}(B) \geq \alpha.
\end{aligned}$$

Since α is arbitrary, we are done.

□

7.4.14 Corollary
If f is a g-filter and $\mathcal{F}$ is a prefilter then

$$\begin{aligned}
f \text{ is prime} \quad & \iff \mathcal{F}_f \text{ is prime} \\
\mathcal{F} \text{ is prime} \quad & \iff f_{\mathcal{F}} \text{ is prime.}
\end{aligned}$$

PROOF.

The proof follows immediately from **??**, **??**, 7.4.12 and 7.4.13.

□

Finally, we check that prime g-filters are preserved by functions.

7.4.15 Theorem
Let $h : X \to Y$ and let f be a prime g-filter on X. Then $h(f)$ is a prime g-filter on Y.

PROOF.

We saw in Theorem 3.6 that $h(f)$ is a g-filter on Y. Now let $A, B \in 2^Y$. Then

$$h(f)(A \cup B) = f(h^{-1}[A \cup B]) = f(h^{-1}[A] \cup h^{-1}[B]) = f(h^{-1}[A]) \vee f(h^{-1}[B]) = h(f)(A) \vee h(f)(B).$$

Thus $h(f)$ is prime.

□

Chapter 8

Generalised Uniform Spaces

8.1 Introduction

In [15] the notion of a generalised uniform space is introduced and studied. Here we study the generalised uniform space with the aid of generalised filters.

Let X be a set and $U \subseteq \mathcal{P}(X \times X) = 2^{X \times X}$ then we define

$$\Delta = \{(x,x) : x \in X\} \text{ and } U_s = \{(x,y) : (y,x) \in U\}.$$

We define

$$\overset{\circ}{U} = \{V \subseteq X \times X : V \circ V \subseteq U\}.$$

8.1.1 Definition
If X is a set then a function $d : 2^{X \times X} \longrightarrow I$ is *generalised uniformitiy* on X iff

1. d is a g-filter and $c(d) = 1$;
2. $\forall\, U \subseteq X \times X,\ d(U) > 0 \Rightarrow \Delta \subseteq U$;
3. $\forall\, U \subseteq X \times X,\ d(U_s) \geq d(U)$;
4. $\forall\, U \subseteq X \times X,\ d(U) \leq \sup_{V \in \overset{\circ}{U}} d(V)$.

we call (X, d) a *generalised uniform space (or g-uniform space).*

The following lemma establish a generalised uniform space from a uniform space.

8.1.2 Lemma
If $(X, \mathbb{D})$ is a uniform space then $1_{\mathbb{D}} : 2^{X \times X} \longrightarrow I$ is a generalised uniform space.

PROOF.

$1_{\mathbb{D}}$ is a g-filter and $1_{\mathbb{D}}(X \times X) = 1$.
If $U \subseteq X \times X$ and $1_{\mathbb{D}}(U) > 0$ then $\Delta \subseteq U$.
Let $U \subseteq X \times X$. Then if $1_{\mathbb{D}}(U) > 0$ then $U \in \mathbb{D}$ and so $U_s \in \mathbb{D}$. Therefore $1_{\mathbb{D}}(U_s) \geq 1_{\mathbb{D}}(U)$.

Let $U \subseteq X \times X$. If $1_{\mathbb{D}}(U) > 0$ then $U \in \mathbb{D}$ and so $\exists V \in \mathbb{D}$ such that $V \in \overset{\circ}{U}$. Therefore $\sup_{V \in \overset{\circ}{U}} 1_{\mathbb{D}}(V) \geq 1_{\mathbb{D}}(U)$.

□

Next we see the α-level uniformities from a generalised uniformity.

8.1.3 Theorem

Let (X, d) be a generalised uniform space. Then for each $\alpha \in I_1$

$$d^\alpha = \{U \in X \times X : d(U) > \alpha\}$$

is a uniformity on X.

PROOF.

Let $\alpha \in I_1$. Since $c(d) = 1$, d^α is a filter.
Let $U \in d^\alpha$. Then $d(U) > \alpha \geq 0 \Rightarrow \Delta \subseteq U$.
Let $U \in d^\alpha$. Then $d(U_s) \geq d(U) > \alpha \Rightarrow U_s \in d^\alpha$.
Let $U \in d^\alpha$. Then $\sup_{V \in \mathring{U}} d(V) \geq d(U) > \alpha$. Therefore $\exists V \in \mathring{U}$ such that $d(V) > \alpha$. Thus $\exists V \in d^\alpha$ such that $V \circ V \subseteq U$.
Hence the result.
The uniformity d^α will be referred to as the *α-level uniformity of d.*
Our next task to build a fuzzy uniformity from a generalised uniformity. This can be done using previous theorem with theorem 6.6.2.

□

8.1.4 Theorem

Let (X, d) be a generalised uniform space. Then

$$\mathcal{D}_d \stackrel{\text{def}}{=} \{\sigma \in I^{X\times X} : \forall\ \alpha \in (0,1),\ \forall\ \beta < \alpha,\ \sigma^\beta \in d^{1-\alpha}\}.$$

is a fuzzy uniformity on X.

PROOF.

We have d^α is a uniformity for each $\alpha \in I_1$.
Let $d(\alpha) = d^{1-\alpha}$ for each $\alpha \in I_0$. Then we have

$$0 < \beta \leq \alpha \leq 1 \Rightarrow d(\beta) \subseteq d(\alpha).$$

and for each $\alpha \in I_0$,

$$d(\alpha) = d^{1-\alpha} = \bigcup_{0<\beta<\alpha} d^{1-\beta} = \bigcup_{0<\beta<\alpha} d(\beta) \quad \text{by 7.3.10.}$$

Therefore

$$\mathcal{D}_d = \{\sigma \in I^{X\times X} : \forall\ \alpha \in (0,1),\ \forall\ \beta < \alpha,\ \sigma^\beta \in d(\alpha)\}.$$

is a fuzzy uniformity on X.

□

Now we try to establish a generalised uniformity from a fuzzy uniformity.

Let $(X, \mathcal{D})$ be a fuzzy uniform space. Then $\mathcal{D}$ is a saturated prefilter with $c(\mathcal{D}) = 1$. For $U \subseteq X \times X$ we have

$$S_\mathcal{D}(U) = \{\alpha \in (0,1] : U \in \mathcal{D}^\alpha\}.$$

and

$$d_\mathcal{D} = \begin{cases} 1 - \inf S_\mathcal{D}(U) & \text{if } S_\mathcal{D}(U) \neq \emptyset \\ 0 & \text{if } S_\mathcal{D}(U) = \emptyset. \end{cases}$$

8.1.5 Theorem

Let $(X, \mathcal{D})$ be a fuzzy uniform space. Then $d_\mathcal{D} : 2^{X\times X} \longrightarrow I$ is a generalised uniformity.

PROOF.

(i) We have $d_{\mathcal{D}}$ is a g-filter and $d_{\mathcal{D}}(X \times X) = 1$.

(ii) Let $U \subseteq X \times X$ and $d_{\mathcal{D}}(U) > 0$. Then $S_{\mathcal{D}}(U) \neq \emptyset$. Therefore $U \in \mathcal{D}^1 \Rightarrow \Delta \subseteq U$.

(iii) Let $U \subseteq X \times X$ and $d_{\mathcal{D}}(U) > 0$. Then $\exists \beta < \alpha$ such that $S_{\mathcal{D}}(U) = (\beta, 1]$. Therefore $\forall\ \alpha \in (\beta, 1]$, $U \in \mathcal{D}^\alpha$ and so $U_s \in \mathcal{D}^\alpha$ and hence $S_{\mathcal{D}}(U_s) = (\beta, 1]$. Therefore $d_{\mathcal{D}}(U) = d_{\mathcal{D}}(U_s)$.

(iv) let $U \subseteq X \times X$ and $\alpha < d_{\mathcal{D}}(U)$. Then $\inf\ S_{\mathcal{D}}(U) < 1-\alpha$. Therefore $1-\alpha \in S_{\mathcal{D}}(U) \Rightarrow U \in \mathcal{D}^{1-\alpha} \Rightarrow \exists V \in \mathcal{D}^{1-\alpha}$ such that $V \circ V \subseteq U$. Thus $V \in \mathring{U}$ and $1-\alpha \in S_{\mathcal{D}}(V)$. Thereofore $d_{\mathcal{D}}(V) > 1 - (1 - \alpha) = \alpha$. So $\sup\limits_{V \in \mathring{U}} d_{\mathcal{D}}(V) > \alpha$. Since α is arbitrary, $\sup\limits_{V \in \mathring{U}} d_{\mathcal{D}}(V) \geq d_{\mathcal{D}}(U)$.

Hence the result.

□

8.1.6 Theorem

The collection of fuzzy uniform spaces is in a one-to-one correspondence with the collection of generalised uniform spaces.

PROOF.

Let $|FUS|$ be the collection of fuzzy uniform spaces and $|GUS|$ be the collection of generalised uniform spaces. Then define

$$\psi : |FUS| \longrightarrow |GUS|,\ (X, \mathcal{D}) \mapsto (X, d_{\mathcal{D}}).$$

Let $(X_1, \mathcal{D}_1) \neq (X_2, \mathcal{D}_2)$.
If $X_1 \neq X_2$ then clearly $(X_1, d_{\mathcal{D}_1}) \neq (X_2, d_{\mathcal{D}_2})$.
If $\mathcal{D}_1 \neq \mathcal{D}_2$ then we have $d_{\mathcal{D}_1} \neq d_{\mathcal{D}_2}$.
Consequently ψ is injective.
To see ψ is surjective, let $(X, D) \in |GUS|$. Then d is generalised filter on X.
By 7.3.13, $\psi(X, \mathcal{D}_d) = (X, d_{\mathcal{D}_d}) = (X, d)$.
Therefore ψ is bijective.

□

8.1.7 Corollary

Let $(X, \mathcal{D})$ be a fuzzy uniform space and (X, d) be a generalised uniform space. Then

$$\mathcal{D}_{d_{\mathcal{D}}} = \mathcal{D} \text{ and } d_{\mathcal{D}_d} = d.$$

PROOF.

Since $\mathcal{D}$ is saturated prefiter and by (7.3.14) and (7.3.15) the result follows.

□

8.1.8 Proposition

Let $(X, \mathbb{D})$ be a uniform space. Then

$$\forall\ \alpha \in I_1,\ (1_{\mathbb{D}})^\alpha = \mathbb{D}.$$

PROOF.

Let $\alpha \in I_1$. Then

$$U \in (1_{\mathbb{D}})^\alpha \Longleftrightarrow 1_{\mathbb{D}}(U) > \alpha \Longleftrightarrow 1_{\mathbb{D}}(U) > 0 \Longleftrightarrow U \in \mathbb{D}.$$

□

8.2 Uniformly Continuous Functions

8.2.1 Definition
Let (X, d) and (Y, e) be generalised uniform spaces and $f : X \longrightarrow Y$ be a mapping. Then f is said to be uniformly continuous if

$$\forall\ V \subseteq Y \times Y,\ d((f \times f)^{\leftarrow}(V)) \geq e(V).$$

We obtain α-level theorem for uniformly continuous.

8.2.2 Theorem
Let (X, d) and (Y, e) be generalised uniform spaces. Then

(a) $f : (X, d) \longrightarrow (Y, e)$ is uniformly continuous $\Rightarrow \forall\ \alpha \in I_1, f : (X, d^\alpha) \longrightarrow (Y, e^\alpha)$ is uniformly continuous.

(b) $\forall\ \alpha \in (0, 1),\ f : (X, d^\alpha) \longrightarrow (Y, e^\alpha)$ is uniformly continuous $\Rightarrow f : (X, d) \longrightarrow (Y, e)$ is uniformly continuous.

Proof.

$$\begin{aligned}
&f \text{ is } d - e \text{ uniformly continuous}\\
&\Longleftrightarrow \forall\ V \subseteq Y \times Y,\ d((f \times f)^{\leftarrow}(V)) \geq e(V)\\
&\Longleftrightarrow \forall\ V \subseteq Y \times Y,\ \forall\ \alpha \in (0,1),\ (e(V) > \alpha \Rightarrow d((f \times f)^{\leftarrow}(V)) > \alpha)\\
&\Longleftrightarrow \forall\ V \subseteq Y \times Y,\ \forall\ \alpha \in (0,1),\ (V \in e^\alpha \Rightarrow (f \times f)^{\leftarrow}(V) \in d^\alpha)\\
&\Longleftrightarrow \forall\ \alpha \in (0,1),\ f : (X, d^\alpha) \longrightarrow (Y, e^\alpha) \text{ is uniformly continuous.}
\end{aligned}$$

To complete the proof we only have to show, if f is $d - e$ uniformly continuous then $f : (X, d^0) \longrightarrow (Y, e^0)$ is uniformly continuous. This can be proved by taking $\alpha = 0$ in the above result.

□

The following three theorems simpilifies some of the the work later on when we map a morphism of a category into a morphim of another category.

8.2.3 Theorem
Let $(X, \mathbb{D})$ and $(Y, \mathbb{E})$ be uniform spaces. Then

$f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ is uniformly continuous $\Rightarrow f : (X, 1_{\mathbb{D}}) \longrightarrow (Y, 1_{\mathbb{E}})$ is uniformly continuous.

Proof.

Let $V \subseteq Y \times Y$. Then

$$1_{\mathbb{E}}(V) = 1 \Rightarrow V \in \mathbb{E} \Rightarrow (f \times f)^{\leftarrow}(V) \in \mathbb{D} \Rightarrow 1_{\mathbb{D}}((f \times f)^{\leftarrow}(V)) = 1.$$

It follows that $\forall\ V \subseteq Y \times Y,\ 1_{\mathbb{D}}((f \times f)^{\leftarrow}(V)) \geq 1_{\mathbb{E}}(V)$ and hence that f is $1_{\mathbb{D}} - 1_{\mathbb{E}}$ uniformly continuous.

□

8.2.4 Theorem
Let (X, d) and (Y, e) be generalised uniform spaces. Then

$f : (X, d) \longrightarrow (Y, e)$ is uniformly continuous $\Longleftrightarrow f : (X, \mathcal{D}_d) \longrightarrow (Y, \mathcal{D}_e)$ is uniformly continuous.

PROOF.

$$\begin{aligned}
& f \text{ is } d-e \text{ uniformly continuous} \\
\iff & \forall\ \alpha \in (0,1),\ f:(X,d^{\alpha}) \longrightarrow (Y,e^{\alpha}) \text{ is uniformly continuous} \\
\iff & f:(X,\mathcal{D}_d) \longrightarrow (Y,\mathcal{D}_e \text{ is uniformly continuous (by 6.6.7).}
\end{aligned}$$

□

8.2.5 Theorem

Let $(X,\mathcal{D})$ and $(Y,\mathcal{E})$ be fuzzy uniform spaces. Then

$$f:(X,\mathcal{D}) \longrightarrow (Y,\mathcal{E}) \text{ is uniformly continuous } \iff f:(X,d_{\mathcal{D}}) \longrightarrow (Y,d_{\mathcal{E}}) \text{ is uniformly continuous .}$$

PROOF.

$$\begin{aligned}
& f \text{ is } \mathcal{D}-\mathcal{E} \text{ uniformly continuous} \\
\iff & \forall\ \alpha \in (0,1),\ f \text{ is } \mathcal{D}^{\alpha}-\mathcal{E}^{\alpha} \text{ uniformly continuous} \\
\iff & \forall\ \alpha \in (0,1),\ f \text{ is } d_{\mathcal{D}}^{1-\alpha}-d_{\mathcal{E}}^{1-\alpha} \text{ uniformly continuous} \\
\iff & \forall\ \alpha \in (0,1),\ f \text{ is } d_{\mathcal{D}}-d_{\mathcal{E}} \text{ uniformly continuous.}
\end{aligned}$$

□

Chapter 9

Fuzzy Filters

9.1 Inroduction

In [22], the notion of a fuzzy filter is introduced and studied. We adopt a slightly different definition of a fuzzy filter. In [28] some special propeties of fuzzy filters can be found. Some results are from J.Gutiérrez through private communication.

9.1.1 Definition
We call a non-zero function $\varphi : I^X \to I$ a *fuzzy filter* or a *f-filter* on X if φ satifies the following conditions.

1. $\varphi(0) = 0$;
2. $\forall \mu, \nu \in I^X,\ \varphi(\mu \wedge \nu) \geq \varphi(\mu) \wedge \varphi(\nu)$;
3. $\forall \mu, \nu \in I^X,\ \mu \leq \nu \Rightarrow \varphi(\mu) \leq \varphi(\nu)$.

Of course, the condition that φ is non-zero is equivalent to the condition $\varphi(1) > 0$. This definition of a fuzzy filter differs from the definition in [22], where it is required that $\varphi(1) = 1$.

If $\varphi : I^X \longrightarrow I$ is a function and $\mu \in I^X$, we define

$$\langle\varphi\rangle(\mu) \stackrel{\text{def}}{=} \sup_{\nu \leq \mu} \varphi(\nu).$$

We call a non-zero function $\varphi : I^X \to I$ a *fuzzy filter base* or a *f-filter base* on X if φ satisfies the following conditions.

1. $\varphi(0) = 0$;
2. $\forall \nu_1, \nu_2 \in I^X,\ \varphi(\nu_1) \wedge \varphi(\nu_2) \leq \langle\varphi\rangle(\nu_1 \wedge \nu_2)$.

It follows immediately that a fuzzy filter is a fuzzy filter base.

9.1.2 Theorem
If X is a set and φ is a fuzzy filter base on X then $\langle\varphi\rangle$ is a fuzzy filter on X.

Proof.

(i) $\langle\varphi\rangle(0) = \sup_{\nu \leq 0} \varphi(\nu) = \varphi(0) = 0$.

(ii) Let $\mu, \nu \in I^X$. If $\langle\varphi\rangle(\mu) \wedge \langle\varphi\rangle(\nu) = 0$ then $\langle\varphi\rangle(\mu) \wedge \langle\varphi\rangle(\nu) \leq \langle\varphi\rangle(\mu \wedge \nu)$. So let

$$0 < \alpha < \langle\varphi\rangle(\mu) \wedge \langle\varphi\rangle(\nu).$$

Then

$$0 < \alpha < \sup_{\mu_1 \le \mu} \varphi(\mu_1) \wedge \sup_{\nu_1 \le \nu} \varphi(\nu_1).$$

So there exists $\mu_1 \le \mu$ and $\nu_1 \le \nu$ such that

$$0 < \alpha < \varphi(\mu_1) \wedge \varphi(\nu_1) \le \langle\varphi\rangle(\mu_1 \wedge \nu_1).$$

Thus there exists $\psi \le \mu_1 \wedge \nu_1 \le \mu \wedge \nu$ such that $0 < \alpha < \varphi(\psi)$. Therefore $\alpha < \langle\varphi\rangle(\mu \wedge \nu)$ and hence, since α is arbitrary, it follows that

$$\langle\varphi\rangle(\mu) \wedge \langle\varphi\rangle(\nu) \le \langle\varphi\rangle(\mu \wedge \nu).$$

(iii) Let $\mu \le \nu$. Then

$$\langle\varphi\rangle(\mu) = \sup_{\mu_1 \le \mu} \varphi(\mu_1) \le \sup_{\mu_1 \le \nu} \varphi(\mu_1) = \langle\varphi\rangle(\nu).$$

□

9.1.3 Definition

We define the *characteristic*, $c(\varphi)$, of a fuzzy filter φ by

$$c(\varphi) \overset{\text{def}}{=} \sup_{\nu \in I^X} \varphi(\nu).$$

9.1.4 Theorem

Let φ be a fuzzy filter base on a set X . Then

$$c(\varphi) = c(\langle\varphi\rangle).$$

Proof.

$$c(\langle\varphi\rangle) = \sup_{\nu \in I^X} \langle\varphi\rangle(\nu) = \sup_{\nu \in I^X} \sup_{\mu \le \nu} \varphi(\mu) = \sup_{\mu \in I^X} \varphi(\mu) = c(\varphi).$$

□

9.1.5 Lemma

Let φ be fuzzy filter on a set X and let $\mu, \nu \in I^X$. Then

1. $c(\varphi) = \varphi(1)$;
2. $\varphi(\mu \wedge \nu) = \varphi(\mu) \wedge \varphi(\nu)$.

Proof.

(1) Let φ be fuzzy filter. Then we have

$$\mu \le \nu \Rightarrow \varphi(\mu) \le \varphi(\nu).$$

Therefore $c(\varphi) = \sup_{\nu \in I^X} \varphi(\nu) = \varphi(1)$.

(2) We have $\varphi(\mu \wedge \nu) \ge \varphi(\mu) \wedge \varphi(\nu)$. But $\mu \wedge \nu \le \mu$ and $\mu \wedge \nu \le \nu$. Consequently,

$$\varphi(\mu \wedge \nu) \le \varphi(\mu) \wedge \varphi(\nu).$$

□

Remark When we try to show $\varphi : I^X \longrightarrow I$ is a fuzzy filter it is enough to show that φ satisfies the following conditions.

(1) $\varphi(0) = 0$;

(2) $\forall\ \mu, \nu \in I^X,\ \varphi(\mu \wedge \nu) = \varphi(\mu) \wedge \varphi(\nu)$.

9.1.6 Examples

(1) Let $0 \neq \mu \in I^X$. For each $\nu \in I^X$ we define,

$$\varphi_\mu(\nu) = \begin{cases} \alpha & \text{if } \nu \geq \mu \\ 0 & \text{if } \nu \ngeq \mu. \end{cases}$$

then φ_μ is a fuzzy filter on X, with $c(\varphi) = \alpha$.

(2) For each $\mu \in I^X$ we define

$$\varphi(\mu) = \inf\ \mu.$$

then φ is a fuzzy filter on X with $c(\varphi) = 1$.
Since:

$$\varphi(0) = 0 \text{ and } \varphi(1) = 1.$$

$$\begin{aligned} \inf\ \mu \wedge \nu &= \inf_{x \in X} (\mu \wedge \nu)(x) = \bigwedge_{x \in X} (\mu(x) \wedge \nu(x) \\ &= (\bigwedge_{x \in X} \mu(x)) \wedge (\bigwedge_{x \in X} \nu(x)) = \inf\ \mu \wedge \inf\ \nu. \end{aligned}$$

Therefore $\varphi(\mu \wedge \nu) = \varphi(\mu) \wedge \varphi(\nu)$.

(3) Let $x_0 \in X$ be fixed and for each $\nu \in I^X$ we define

$$\varphi_{x_0}(\mu) = \mu(x_0)$$

Then φ_{x_0} is a fuzzy filter on X with $c(\varphi) = 1$.

(4) Let $\mathcal{F}$ be a prefiter on X with $c(\mathcal{F}) = 1$. For each $\mu \in I^X$ we define

$$\varphi(\mu) = 1 - c^\mu(\mathcal{F}).$$

Then φ is a fuzzy filter on X with $c(\varphi) = 1$.
Since:

$$\varphi(0) = 1 - c(\mathcal{F}) = 1 - 1 = 0.$$

$$\varphi(1) = 1 - c^1(\mathcal{F}) = 1 - 0 = 1.$$

Let $A = \{\alpha \in I : \mu + \alpha \in \mathcal{F}\},\ B = \{\alpha \in I : \nu + \alpha \in \mathcal{F}\}$ and $C = \{\alpha \in I : \mu \wedge \nu + \alpha \in \mathcal{F}\}$. Now we have to show

$$\inf C = (\inf A) \wedge (\inf B).$$

$$\begin{aligned} \alpha \in C &\Rightarrow \mu \wedge \nu + \alpha \in \mathcal{F} \\ &\Rightarrow \mu + \alpha \in \mathcal{F} \text{ and } \nu + \alpha \in \mathcal{F} \\ &\Rightarrow \alpha \in A \text{ and } \alpha \in B. \end{aligned}$$

Therefore $C \subseteq A$ and $C \subseteq B$. Hence $\inf C \geq \inf A$ and $\inf C \geq \inf B$.
If $\inf C > \inf A$ then $\exists \gamma : \inf C > \gamma > \inf A$. Now we have to show $\inf C = \inf B$.
Assume that $\inf C > \inf B$ then $\exists \beta : \inf C > \beta > \inf B$. Therefore

$$\inf C > \beta \vee \gamma > \inf A \text{ and } \inf C > \beta \vee \gamma > \inf B$$

and hence $\beta \vee \gamma \in A$ and $\beta \vee \gamma \in B$. Thus $\mu + \beta \vee \gamma \in \mathcal{F}$ and $\nu + \beta \vee \gamma \in \mathcal{F}$. So $(\mu + \beta \vee \gamma) \wedge (\nu + \beta \vee \gamma) = \mu \wedge \nu + \beta \vee \gamma$. Therefore $\beta \vee \gamma \in C$ and so $\beta \vee \gamma > \inf C$. This is a contradiction. Therefore $\inf C = (\inf A) \wedge (\inf B)$. So

$$\begin{aligned} 1 - \inf C &= 1 - (\inf A) \wedge (\inf B) \\ &= (1 - \inf A) \wedge (1 - \inf B). \end{aligned}$$

Therefore

$$\varphi(\mu \wedge \nu) = \varphi(\mu) \wedge \varphi(\nu).$$

9.1.7 Definition
A fuzzy filter φ is said to be *prime* if it satisfies,
$\forall\ \mu,\nu \in I^X,\ \varphi(\mu \vee \nu) = \varphi(\mu) \vee \varphi(\nu)$.

9.1.8 Definition
A fuzzy filter φ is said to be *stratified* if
$\forall\ \alpha \in [0,1],\ \varphi(\alpha 1_X) = \alpha \wedge \varphi(1)$.

9.1.9 Proposition
Given a fuzzy stratified filter φ, if for any $\mu \in I^X$ we define,

$$\tilde{\varphi}(\mu) = \inf_{\varepsilon\in I_0} \varphi(\mu+\varepsilon)$$

then $\tilde{\varphi}$ is also a fuzzy stratified filter finer than that φ which will be called the *saturated hull* of φ.

Proof.

Let φ be a fuzzy stratified filter and

$$\tilde{\varphi}(\mu) = \inf_{\varepsilon\in I_0} \varphi(\mu+\varepsilon).$$

Then

$$\tilde{\varphi}(0) = \inf_{\varepsilon\in I_0}\ \varepsilon \wedge \varphi(1) = 0.$$

$$\begin{aligned}\tilde{\varphi}(\mu\wedge\nu) &= \inf_{\varepsilon\in I_0}\ \varphi(\mu\wedge\nu+\varepsilon)\\ &= \inf_{\varepsilon\in I_o}\ \varphi(\mu+\varepsilon\wedge\nu+\varepsilon)\\ &= \inf_{\varepsilon\in I_0}\ \varphi(\mu+\varepsilon)\ \wedge\ \inf_{\varepsilon\in I_0}\ \varphi(\nu+\varepsilon)\\ &= \tilde{\varphi}(\mu)\wedge\tilde{\varphi}(\nu).\end{aligned}$$

$$\tilde{\varphi}(\alpha 1_X) = \inf_{\varepsilon\in I_0}\ \varphi(\alpha 1_X+\varepsilon) = \inf_{\varepsilon\in I_0}(\alpha+\varepsilon)\wedge\varphi(1) = \alpha\wedge\varphi(1) = \alpha\wedge\tilde{\varphi}(1).$$

Finally, since φ is non-decreasing, it follows that $\tilde{\varphi}$ is finer than φ.

9.1.10 Definition
A fuzzy filter φ is said to be *saturated* if it is stratified and $\tilde{\varphi} = \varphi$.

9.1.11 Proposition
A fuzzy stratified filter φ on X is saturated iff for each family $(\mu_\varepsilon)_{\varepsilon\in I_0} \subseteq I^X$ we have

$$\varphi(\sup_{\varepsilon\in I_0}(\mu_\varepsilon-\varepsilon)) \geq \inf_{\varepsilon\in I_0}\varphi(\mu_\varepsilon).$$

Proof.

($\Rightarrow$)
Let φ be saturated, $\{\mu_\varepsilon\}_{\varepsilon\in I_0} \subseteq I^X$ and $\mu = \sup_{\varepsilon\in I_0}(\mu_\varepsilon-\varepsilon)$. Then

$$\varphi(\mu) = \tilde{\varphi}(\mu) = \inf_{\varepsilon\in I_0}\ \varphi(\mu+\varepsilon).$$

But we have $\forall\ \varepsilon\in I_0,\ \mu+\varepsilon \geq \mu_\varepsilon$.
Therefore $\forall\ \varepsilon\in I_0,\ \varphi(\mu+\varepsilon)\geq\varphi(\mu_\varepsilon)$ and hence

$$\begin{aligned}\varphi(\mu) &= \inf_{\varepsilon\in I_0}\ \varphi(\mu+\varepsilon)\\ &\geq \inf_{\varepsilon\in I_0}\ \varphi(\mu_\varepsilon).\end{aligned}$$

(⇐)

Conversly, if for each family $\{\mu_\varepsilon\}_{\varepsilon\in I_0} \subseteq I^X$,

$$\varphi(\sup_{\varepsilon\in I_0}(\mu_\varepsilon - \varepsilon)) \geq \inf_{\varepsilon\in I_0} \varphi(\mu_\varepsilon).$$

Then for any $\mu \in I^X$, we can consider the family $\{\mu + \varepsilon\}_{\varepsilon\in I_0} \subseteq I^X$ and we have

$$\varphi(\sup_{\varepsilon\in I_0}(\mu + \varepsilon - \varepsilon)) \geq \inf_{\varepsilon\in I_0} \varphi(\mu + \varepsilon).$$

Thus $\varphi(\mu) \geq \tilde{\varphi}(\mu)$. Therefore $\varphi(\mu) = \tilde{\varphi}(\mu)$ and hence $\varphi = \tilde{\varphi}$.

□

9.2 Fuzzy Filters with Characteristic 1

We will see in chapter 10 that a super uniformity is a fuzzy filter with characteristic 1 plus other conditions. So let's investigate some properties of fuzzy filters with characteristic 1.

9.2.1 Proposition

Let φ be a fuzzy filter on X with $c(\varphi) = 1$. For each $\alpha \in (0, 1]$ the collection

$$\varphi_\alpha = \{\mu \in I^X : \varphi(\mu) \geq \alpha\}$$

is a prefilter. Furthermore the family $\{\varphi_\alpha\}_{\alpha\in I_0}$ is non-increasing and for $\alpha \in I_0$, $\varphi_\alpha = \bigcap_{\beta<\alpha} \varphi_\beta$.

Proof.

Let $\alpha \in I_0$. Then

$$1 \in \varphi_\alpha \text{ and } 0 \notin \varphi_\alpha \text{ and so } \varphi_\alpha \neq \emptyset.$$

Let $\mu,\ \nu \in \varphi_\alpha$. Then $\varphi(\mu) \geq \alpha$ and $\varphi(\nu) \geq \alpha$. Therefore $\varphi(\mu \wedge \nu) = \varphi(\mu) \wedge \varphi(\nu) \geq \alpha$ and hence $\mu \wedge \nu \in \varphi_\alpha$.

Let $\mu \in \varphi_\alpha$ and $\mu \leq \nu$. Then $\varphi(\mu) \geq \alpha$ and $\varphi(\mu) \leq \varphi(\nu)$. Therefore $\varphi(\mu) \geq \alpha$ and so $\nu \in \varphi_\alpha$.

Hence φ_α is a prefilter.

It is easy to see that

$$0 < \beta \leq \alpha \leq 1 \Rightarrow \varphi_\beta \supseteq \varphi_\alpha.$$

Let $\alpha \in I_0$. Then $\forall\ \beta < \alpha,\ \varphi_\alpha \subseteq \varphi_\beta$. Therefore $\varphi_\alpha \subseteq \bigcap_{\beta<\alpha} \varphi_\beta$.

Let $\mu \in \bigcap_{\beta<\alpha} \varphi_\beta$. Then $\forall\ \beta < \alpha,\ \mu \in \varphi_\beta$. Therefore $\forall\ \beta < a$, $\varphi(\mu) \geq \beta$ and so $\varphi(\mu) \geq \sup_{\beta<\alpha}\beta = \alpha$. So $\mu \in \varphi_\alpha$ and hence $\varphi_\alpha = \bigcap_{\beta<\alpha} \varphi_\beta$.

□

We call φ_α the *α-level prefilter* of the given fuzzy filter φ and the construction

$$\varphi \longrightarrow \{\varphi_\alpha\}_{\alpha\in I_0}$$

will be called the *decomposition* of φ into the system of its α-level prefilters.

9.2.2 Lemma

Let $\{\mathcal{F}(\alpha)\}_{\alpha\in(0,1)}$ be a non-increasing collection of prefilters on a set X. Then for $\mu \in I^X$ let

$$A_\mu = \{\alpha \in (0, 1) : \mu \in \mathcal{F}(\alpha)\}.$$

Then $A_\mu = \emptyset$ or $\exists\ \beta \in (0, 1) : A_\mu = (0, \beta)$ or $A_\mu = (0, \beta]$.

Proof.

Let $A_\mu \neq \emptyset and \beta \in A_\mu$. Then $\mu \in \mathcal{F}(\beta)$. If $\alpha < \beta$ then $\mu \in \mathcal{F}(\beta) \subseteq \mathcal{F}(\alpha)$.

□

9.2.3 Proposition

Let $\{\mathcal{F}(\alpha)\}_{\alpha \in (0,1)}$ be a non-increasing collection of prefilters on a set X. If for any $\mu \in I^X$ we define

$$\varphi(\mu) = \sup\{\alpha \in (0,1) : \mu \in \mathcal{F}(\alpha)\}$$

then φ is a fuzzy filter with $c(\varphi) = 1$.

Furthermore $\mathcal{F}(\alpha)$ is exactly the α-level prefilter φ_α of φ iff the collection $\{\mathcal{F}_\alpha\}_{\alpha \in (0,1)}$ satisfies the condition that any $\alpha \in (0,1)$, $\mathcal{F}(\alpha) = \bigcap_{\beta < \alpha} \mathcal{F}(\beta)$.

Proof.

Let $\{\mathcal{F}(\alpha)\}_{\alpha \in (0,1)}$ be a non-increasing collection of prefilters. Then

$$0 < \alpha \leq \beta < 1 \Rightarrow \mathcal{F}(\alpha) \supseteq \mathcal{F}(\beta),$$

$$\varphi(0) = \sup \emptyset = 0 \text{ and } \varphi(1) = \sup(0,1) = 1.$$

We have to show

$$\varphi(\mu \wedge \nu) = \varphi(\mu) \wedge \varphi(\nu). \text{ That is } \sup A_{\mu \wedge \nu} = \sup A_\mu \wedge \sup A_\nu.$$

Now

$$\begin{aligned} \alpha \in A_{\mu \wedge \nu} &\Rightarrow \mu \wedge \nu \in \mathcal{F}(\alpha) \\ &\Rightarrow \mu \in \mathcal{F}(\alpha) \text{ and } \nu \in \mathcal{F}(\alpha) \\ &\Rightarrow \alpha \in A_\mu \text{ and } \alpha \in A_\nu. \end{aligned}$$

Therefore $A_{\mu \wedge \nu} \subseteq A_\mu$ and $A_{\mu \wedge \nu} \subseteq A_\nu$. Hence $\sup A_{\mu \wedge \nu} \leq \sup A_\nu$ and $\sup A_{\mu \wedge \nu} \leq \sup A_\nu$.

If $\sup A_{\mu \wedge \nu} < \sup A_\mu$ then $\exists \beta : \sup A_{\mu \wedge \nu} < \beta < \sup A_\mu$. Now we have to show $\sup A_{\mu \wedge \nu} = \sup A_\nu$.

Assume that $\sup A_{\mu \wedge \nu} < \sup A_\nu$ then $\exists\, \gamma : \sup A_{\mu \wedge \nu} < \gamma < \sup A_\nu$. Therefore

$$\sup A_{\mu \wedge \nu} < \beta \wedge \gamma < \sup A_\mu \text{ and } \sup A_{\mu \wedge \nu} < \beta \wedge \gamma < \sup A_\nu$$

and hence $\beta \wedge \gamma \in A_\mu$ and $\beta \wedge \gamma \in A_\nu$. Therefore $\mu \in \mathcal{F}(\beta \wedge \gamma)$ and $\nu \in \mathcal{F}(\beta \wedge \gamma)$ and so $\mu \wedge \nu \in \mathcal{F}(\beta \wedge \gamma)$. Hence $\beta \wedge \gamma < \sup A_{\mu \wedge \nu}$. This is a contradiction. Therefore $\sup A_{\mu \wedge \nu} = \sup A_\mu \wedge A_\nu$. Thus

$$\varphi(\mu \wedge \nu) = \varphi(\mu) \wedge \varphi(\nu).$$

Therefore φ is a fuzzy filter.

($\Rightarrow$)

If $\mathcal{F}(\alpha)$ is exactly the α-level prefilter φ_α of φ then by above proposition for any $\alpha \in (0,1)$

$$\mathcal{F}(\alpha) = \bigcap_{\beta < \alpha} \mathcal{F}(\beta).$$

($\Leftarrow$)

Let the collection $\{\mathcal{F}(\alpha)\}_{\alpha \in (0,1)}$ satisfy the condition that any $\alpha \in I_0$, $\mathcal{F}(\alpha) = \bigcap_{\beta < \alpha} \mathcal{F}(\beta)$.

Let $\gamma \in (0,1)$. Then

$$\mathcal{F}(\gamma) = \bigcap_{\beta < \gamma} \mathcal{F}(\beta) \text{ and } \varphi_\gamma = \{\mu \in I^X : \varphi(\mu) \geq \gamma\}.$$

Now we have to show $\mathcal{F}(\gamma) = \varphi_\gamma$.

Let $\mu \in \mathcal{F}(\gamma)$. Then $\forall\ \beta < \gamma,\ \mu \in \mathcal{F}(\beta)$. Therefore

$$\varphi(\mu) = \sup\{\alpha \in I_0 : \mu \in \mathcal{F}(\alpha)\} \geq \gamma.$$

Therefore $\mu \in \varphi_\gamma$.
Let $\mu \in \varphi_\gamma$. Then $\varphi(\mu) \geq \gamma$. Thus

$$\sup\{\alpha \in I_0 : \mu \in \mathcal{F}(\alpha)\} \geq \gamma.$$

Therefore $\forall\ \alpha < \gamma,\ \mu \in \mathcal{F}(\alpha)$ and hence $\mu \in \mathcal{F}(\gamma)$.

□

9.2.4 Proposition
Let φ be a fuzzy filter on X with $c(\varphi) = 1$. If φ is prime then for all $\alpha \in I_0,\ \varphi_\alpha$ is prime.

PROOF.

Let $\alpha \in I_0,\ \mu \vee \nu \in \varphi_\alpha$. Then $\varphi(\mu \vee \nu) \geq \alpha$.
Therefore $\varphi(\mu \vee \nu) = \varphi(\mu) \vee \varphi(\nu) \geq \alpha$ and hence

$$\varphi(\mu) \geq \alpha \text{ or } \varphi(\nu) \geq \alpha.$$

Thus

$$\mu \in \varphi_\alpha \text{ or } \nu \in \varphi_\alpha.$$

Hence φ_α is prime prefilter.

□

9.2.5 Proposition
Let $\{\mathcal{F}(\alpha)\}_{\alpha \in (0,1)}$ be a non-increasing collection of prefilters on X. If for all $\alpha \in (0,1),\ \mathcal{F}(\alpha)$ is prime, then the fuzzy filter generated by this collection is prime.

PROOF.

We have

$$\varphi(\mu) = \sup\{\alpha \in (0,1) : \mu \in \mathcal{F}(\alpha)\} = \sup A_\mu$$

is a fuzzy filter. Then

$$\begin{aligned} \alpha \in A_{\mu\vee\nu} \quad &\Rightarrow \mu \vee \nu \in \mathcal{F}(\alpha) \\ &\Rightarrow \mu \in \mathcal{F}(\alpha) \text{ or } \nu \in \mathcal{F}(\alpha) \text{ [Since } \mathcal{F}(\alpha) \text{ is prime]} \\ &\Rightarrow \mu \in \mathbb{F}_\alpha \text{ or } \nu \in \varphi_\alpha. \end{aligned}$$

Therefore

$$A_{\mu\vee\nu} \subseteq A_\mu \text{ or } A_{\mu\vee\nu} \subseteq A_\nu$$

and so

$$\sup A_{\mu\vee\nu} \leq \sup A_\nu \text{ or } \sup A_{\mu\vee\nu} \leq \sup A_\nu.$$

Thus

$$\sup A_{\mu\vee\nu} \leq \sup A_\mu \ \vee \ \sup A_\nu.$$

We have

$$\alpha \in A_\mu \Rightarrow \alpha \in A_{\mu\vee\nu}.$$

Therefore $A_\mu \subseteq A_{\mu\vee\nu}$ and so $\sup A_\mu \leq \sup A_{\mu\vee\nu}$. Similarly $\sup A_\nu \leq \sup A_{\mu\vee\nu}$. Therefore $\sup A_\mu \ \vee \ \sup A_\nu \leq \sup A_{\mu\vee\nu}$. Hence $\sup A_\mu \ \vee \ \sup A_\nu = \sup A_{\mu\vee\nu}$. Thus $\varphi(\mu \vee \nu) = \varphi(\mu) \vee \varphi(\nu)$.

□

The proof of the following lemma is straightforward.

9.2.6 Lemma

Let φ be fuzzy filter with $c(\varphi) = 1$. then

$$\forall\ \alpha \in [0,1],\ \varphi(\alpha 1_X) = \alpha.$$

9.2.7 Proposition

Let φ be a fuzzy filter on X with $c(\varphi) = 1$. If φ is stratified then for all $\alpha \in I_0,\ C(\varphi_\alpha) = [0,\alpha)$.

Proof.

Let $\alpha \in I_0$. Then

$$\varphi_\alpha = \{\mu \in I^X : \varphi(\mu) \geq \alpha\}$$

is a prefilter. we have

$$C(\varphi_\alpha) = \{\beta \in I : \beta 1_X \in \varphi_\alpha\}.$$

Let $\gamma < \alpha$. Then $\varphi(\gamma 1_X) = \gamma < \alpha$. Therefore $\gamma 1_X \notin \varphi_\alpha$. Thus $\gamma \in C(\varphi_\alpha)$.
Let $\gamma \geq \alpha$. Then $\varphi(\gamma 1_X) = \gamma \geq \alpha$. Therefore $\gamma 1_X \in \varphi_\alpha$. Thus $\gamma \notin C(\varphi_\alpha)$.
Hence $C(\varphi_\alpha) = [0,\alpha)$.

□

9.2.8 Proposition

Let $\{\mathcal{F}(\alpha)\}_{\alpha\in(0,1)}$ be a non-increasing collection of prefilters on X. If for all $\alpha \in (0,1),\ C(\mathcal{F}(\alpha)) = [0,\alpha)$ then the fuzzy filter generated by this collection is stratified.

Proof.

Let $\{\mathcal{F}(\alpha)\}_{\alpha\in(0,1)}$ be a non-increasing collection of prefilters on X and for any $\alpha \in (0,1)$,

$$C(\mathcal{F}(\alpha)) = [0,\alpha).$$

Thus

$$\{\beta \in I : \beta 1_X \notin \mathcal{F}(\alpha)\} = [0,\alpha).$$

Let

$$\varphi(\mu) = \sup\{\alpha \in (0,1) : \mu \in \mathcal{F}(\alpha)\}.$$

Then φ is a fuzzy filter with $c(\varphi) = 1$. Now we have to show that $\forall\ \gamma \in I,\ \varphi(\gamma 1_X) = \gamma$.
Let $\gamma \in I$. Then

$$\gamma 1_X \notin \mathcal{F}(\alpha) \text{ for } \gamma < \alpha$$

and

$$\gamma 1_X \in \mathcal{F}(\alpha) \text{ for } \gamma \geq \alpha.$$

Therefore

$$\varphi(\gamma 1_X) = \sup(0,\gamma] = \gamma.$$

Hence f is stratified.

□

9.2.9 Proposition

Let φ be a fuzzy filter on X with $c(\varphi) = 1$. If φ is saturated, then for all $\alpha \in I_0,\ \varphi_\alpha$ is saturated.

PROOF.

Let $\alpha \in I_0$ and $\forall\, \varepsilon \in I_0,\ \mu + \varepsilon \in \varphi_\alpha$. Then

$$\forall\, \varepsilon \in I_0,\ \varphi(\mu + \varepsilon) \geq \alpha.$$

Since φ is a saturated fuzzy filter and so

$$\varphi(\mu) = \inf_{\varepsilon \in I_0} \varphi(\mu + \varepsilon) \geq \alpha.$$

Therefore $\mu \in \varphi_\alpha$. Hence φ_α is a saturated prefilter.

□

9.2.10 Proposition

Let $\{\mathcal{F}(\alpha)\}_{\alpha \in (0,1)}$ be a non-increasing collection of prefilters on X such that for each $\alpha \in (0,1),\ C(\mathcal{F}(\alpha)) = [0, \alpha)$. If for all $\alpha \in (0,1),\ \mathcal{F}(\alpha)$ is saturated, then the fuzzy filter generated by the collection is saturated.

PROOF.

Let $\{\mathcal{F}(\alpha)\}_{\alpha \in (0,1)}$ be a non-increasing collection of saturated prefilters on X and for each $\alpha \in (0,1),\ C(\mathcal{F}(\alpha)) = [0, \alpha)$. Then we have

$$\varphi(\mu) = \sup\{\alpha \in (0,1) : \mu \in \mathcal{F}(\alpha)\}$$

is a stratified fuzzy filter.

Let $\gamma < \inf_{\varepsilon \in I_0} \varphi(\mu + \varepsilon)$. Then $\forall\, \varepsilon \in I_0,\ \varphi(\mu + \varepsilon) > \gamma$ and hence

$$\forall\, \varepsilon \in I_0,\ \mu + \varepsilon \in \mathcal{F}(\alpha).$$

But $\mathcal{F}(\gamma)$ is saturated. So $\mu \in \mathcal{F}(\gamma)$ and hence $\varphi(\mu) \geq \gamma$. Consequently,

$$\varphi(\mu) \geq \inf_{\varepsilon \in I_0} \varphi(\mu + \varepsilon) = \tilde{\varphi}(\mu).$$

Hence $\varphi(\mu) = \tilde{\varphi}(\mu)$. Thus φ is saturated.

□

9.2.11 Proposition

Let φ be fuzzy filter with $c(\varphi) = 1$. Then for all $\alpha \in I_0$, we have $\widetilde{\varphi_\alpha} = \tilde{\varphi}_\alpha$.

PROOF.

Let $\alpha \in I_0$. Then

$$\begin{aligned}
\mu \in \tilde{\varphi_\alpha} \quad &\Longleftrightarrow \forall\, \varepsilon \in I_0,\ \mu + \varepsilon \in \varphi_\alpha \\
&\Longleftrightarrow \forall\, \varepsilon \in I_o,\ \varphi(\mu + \varepsilon) \geq \alpha \\
&\Longleftrightarrow \tilde{\varphi}(\mu) = \inf_{\varepsilon \in I_0} \varphi(\mu + \varepsilon) \geq \alpha \\
&\Longleftrightarrow \mu \in \tilde{\varphi}_\alpha.
\end{aligned}$$

□

9.2.12 Proposition

Let $\{\mathcal{F}(\alpha)\}_{\alpha \in (0,1)}$ be a non-increasing collection of prefilters on X such that for each $\alpha \in (0,1),\ C(\mathcal{F}(\alpha)) = [0, \alpha)$. The saturated hull of the fuzzy filter generated $\{\mathcal{F}(\alpha)\}_{\alpha \in (0,1)}$ is just the fuzzy filter generated by $\{\widetilde{\mathcal{F}(\alpha)}\}_{\alpha \in (0,1)}$.

Proof.

For $\mu \in I^X$, let
$$\varphi(\mu) = \sup\{\alpha \in (0,1) : \mu \in \mathcal{F}(\alpha)\}$$
and
$$\overset{\star}{\varphi}(\mu) = \sup\{\alpha \in (0,1) : \mu \in \widetilde{\mathcal{F}(\alpha)}\}.$$
Now we have to show $\tilde{\varphi}(\mu) = \overset{\star}{\varphi}(\mu)$.

Let $\gamma < \overset{\star}{\varphi}(\mu)$. Then $\mu \in \widetilde{\mathcal{F}(\gamma)}$. Therefore
$$\forall\, \varepsilon \in I_0,\ \mu + \varepsilon \in \mathcal{F}(\gamma).$$
So
$$\forall\, \varepsilon \in I_0,\ \varphi(\mu + \varepsilon) \geq \alpha.$$
Therefore
$$\tilde{\varphi}(\mu) = \inf_{\varepsilon \in I_0} \varphi(\mu + \varepsilon) \geq \gamma.$$
Hence $\tilde{\varphi}(\mu) \geq \overset{\star}{\varphi}(\mu)$.

On the other hand let $\gamma < \tilde{\varphi}(\mu)$. Then
$$\forall\, \varepsilon \in I_0,\ \varphi(\mu + \varepsilon) > \gamma.$$
So
$$\forall\, \varepsilon \in I_0,\ \mu + \varepsilon \in \mathcal{F}(\gamma).$$
Therefore $\mu \in \widetilde{\mathcal{F}(\gamma)}$ and hence $\overset{\star}{\varphi}(\mu) \geq \gamma$. Consequently, $\overset{\star}{\varphi}(\mu) \geq \tilde{\varphi}(\mu)$.

Hence $\overset{\star}{\varphi}(\mu) = \tilde{\varphi}(\mu)$.

□

9.3 Fuzzy Filters From G-Filters and G-Filters From Fuzzy Filters

If $f : 2^X \to I$ is a g-filter we define $\varphi_f : I^X \to I$ by
$$\varphi_f(\mu) \overset{\text{def}}{=} \sup_{\alpha \in (0,1)} f(\mu^\alpha) \wedge \alpha.$$
If $\varphi : I^X \to I$ is a f-filter we define $f_\varphi : 2^X \to I$ by
$$f_\varphi(A) \overset{\text{def}}{=} \varphi(1_A).$$

9.3.1 Theorem

Let X be a set, f a g-filter on X and φ a f-filter on X. Then

1. φ_f is a stratified f-filter and $c(\varphi_f) = c(f)$;
2. If f is prime then so is φ_f;
3. f_φ is a g-filter and $c(f_\varphi) = c(\varphi)$;
4. If φ is prime then so is f_φ;
5. $f_{\varphi_f} = f$;
6. $\varphi_{f_\varphi} \leq \varphi$.

PROOF.

(1)

Firstly:

$$c(\varphi_f) = \varphi_f(1) = \sup_{\alpha\in(0,1)} f(1^\alpha) \wedge \alpha = \sup_{\alpha\in(0,1)} f(X) \wedge \alpha = f(X) = c(f),$$

$$\varphi_f(0) = \sup_{\beta\in(0,1)} f(0^\beta) \wedge \beta = \sup_{\beta\in(0,1)} f(\emptyset) \wedge \beta = 0$$

and for $\alpha \in (0, 1]$ we have:

$$\varphi_f(\alpha 1_X) = \sup_{\beta\in(0,1)} f(\alpha 1_X^\beta) \wedge \beta = \sup_{\beta\in(0,\alpha)} f(X) \wedge \beta = f(X) \wedge \alpha = c(f) \wedge \alpha = c(\varphi_f) \wedge \alpha.$$

If $\varphi_f(\mu) \wedge \varphi_f(\nu) > t$ then there exists $\alpha_1, \alpha_2 > t$ such that $f(\mu^{\alpha_1}) > t$ and $f(\nu^{\alpha_2}) > t$. Let $\alpha = \alpha_1 \wedge \alpha_2$. Then $\alpha > t$ and

$$f((\mu \wedge \nu)^\alpha) = f(\mu^\alpha \cap \nu^\alpha) = f(\mu^\alpha) \wedge f(\nu^\alpha) \geq f(\mu^{\alpha_1}) \wedge f(\nu^{\alpha_2}) > t.$$

Therefore

$$\varphi_f(\mu \wedge \nu) = \sup_{\beta\in(0,1)} f((\mu \wedge \nu)^\beta) \wedge \beta \geq f((\mu \wedge \nu)^\alpha) \wedge \alpha > t.$$

Consequently

$$\varphi_f(\mu \wedge \nu) \geq \varphi_f(\mu) \wedge \varphi_f(\nu).$$

Let $\mu \leq \nu$. Then for each $\alpha \in (0, 1)$ we have $\mu^\alpha \subseteq \nu^\alpha$. Hence

$$\varphi_f(\mu) = \sup_{\alpha\in(0,1)} f(\mu^\alpha) \wedge \alpha \leq \sup_{\alpha\in(0,1)} f(\nu^\alpha) \wedge \alpha = \varphi_f(\nu).$$

(2)

$$\begin{aligned}
\varphi_f(\mu \vee \nu) &= \sup_{\alpha\in(0,1)} f((\mu \vee \nu)^\alpha) \wedge \alpha \\
&= \sup_{\alpha\in(0,1)} f(\mu^\alpha \cup \nu^\alpha) \wedge \alpha \\
&= \sup_{\alpha\in(0,1)} (f(\mu^\alpha) \vee f(\nu^\alpha)) \wedge \alpha \\
&= \sup_{\alpha\in(0,1)} ((f(\mu^\alpha) \wedge \alpha) \vee (f(\nu^\alpha) \wedge \alpha)) \\
&= \sup_{\alpha\in(0,1)} (f(\mu^\alpha) \wedge \alpha) \vee \sup_{\alpha\in(0,1)} (f(\nu^\alpha) \wedge \alpha) \\
&= \varphi_f(\mu) \vee \varphi_f(\nu).
\end{aligned}$$

(3)

$$f_\varphi(\emptyset) = \varphi(1_\emptyset) = 0.$$

$$f_\varphi(A \cap B) = \varphi(1_{A\cap B}) = \varphi(1_A \wedge 1_B) = f_\varphi(A) \wedge f_\varphi(B).$$

If $A \subseteq B$ then

$$f_\varphi(A) = \varphi(1_A) \leq \varphi(1_B) = f_\varphi(B).$$

Finally,

$$c(f_\varphi) = f_\varphi(X) = \varphi(c_1) = c(\varphi).$$

(4)

$$f_\varphi(A \cup B) = \varphi(1_{A\cup B}) = \varphi(1_A \vee 1_B) = \varphi(1_A) \vee \varphi(1_B) = f_\varphi(A) \vee f_\varphi(B).$$

(5)

Let $A \subseteq X$ Then

$$f_{\varphi_f}(A) = \varphi_f(1_A) = \sup_{\alpha \in (0,1)} f(1_A^\alpha) \wedge \alpha = \sup_{\alpha \in (0,1)} f(A) \wedge \alpha = f(A).$$

(6)

Let $\mu \in I^X$. Then

$$\varphi_{f_\varphi}(\mu) = \sup_{\alpha \in (0,1)} f_\varphi(\mu^\alpha) \wedge \alpha = \sup_{\alpha \in (0,1)} \varphi(1_{\mu^\alpha}) \wedge \alpha = \sup_{\alpha \in (0,1)} \varphi(1_{\mu^\alpha} \wedge \alpha 1_X) \leq \varphi(\sup_{\alpha \in (0,1)} 1_{\mu^\alpha} \wedge \alpha 1_X) = \varphi(\mu).$$

□

We have seen how to obtain a fuzzy filter from a g-filter and vice-versa. We now show that

$$f \mapsto \varphi_f$$

is an injective function.

9.3.2 Theorem

If f and g are different g-filters then φ_f and φ_g are different.

PROOF.

If $f \neq g$ then there exists A such that $\alpha \stackrel{\text{def}}{=} f(A) > g(A) \stackrel{\text{def}}{=} \beta$. Let $\mu \stackrel{\text{def}}{=} 1_A$. Then

$$\begin{aligned} \varphi_f(\mu) &= \sup_{\gamma \in (0,1)} f(A) \wedge \gamma \\ &= \sup_{\gamma \in (0,1)} \alpha \wedge \gamma \\ &= \alpha > \beta \\ &= \varphi_g(\mu). \end{aligned}$$

□

So the collection of all g-filters on a set X embeds into the collection of all fuzzy filters on X.

More information regarding fuzzy filter can be found in [20, 21, 22, 40].

Chapter 10

Super Uniform Spaces

10.1 Introduction

In [28] the notion of a super uniformity introduced and studied.

10.1.1 Definition
A *fuzzy α-uniformity* with $\alpha \in I_0$ on X is a subset $\mathcal{U}^\alpha \subseteq I^{X\times X}$ which satisfies the following conditions:

1. $\mathcal{U}^\alpha$ is a saturated prefilter with characteristic set $[0, \alpha)$;
2. $\forall\ \sigma \in \mathcal{U}^\alpha,\ \forall\ x \in X,\ \sigma(x, x) \geq \alpha$;
3. $\forall\ \sigma \in \mathcal{U}^\alpha,\ \sigma_s \in \mathcal{U}^\alpha$;
4. $\forall\ \sigma \in \mathcal{U}^\alpha,\ \forall\ \varepsilon \in I_0,\ \exists \psi \in \mathcal{U}^\alpha : \psi \circ \psi \leq \sigma + \varepsilon$.

We call $(X,\mathcal{U}^\alpha)$ a *fuzzy α-uniform space* and the elements of $\mathcal{U}^\alpha$, *fuzzy α-entourages.*

Note: For $\alpha = 1$ they are the fuzzy uniformities on X.

10.1.2 Definition
A *fuzzy α-uniform base* with $\alpha \in I_0$ on X is a subset $\mathcal{B}^\alpha \subseteq I^{X\times X}$ which satifies the following conditions:

1. $\mathcal{B}^\alpha$ is a prefilter base with characteristic set $[0, \alpha)$;
2. $\forall\ \sigma \in \mathcal{B}^\alpha,\ \forall\ x \in X,\ \sigma(x, x) \geq \alpha$;
3. $\forall\ \sigma \in \mathcal{B}^\alpha,\ \forall\ \varepsilon \in I_0,\ \exists \psi \in \mathcal{B}^\alpha : \psi \leq \sigma_s + \varepsilon$;
4. $\forall\ \sigma \in \mathcal{B}^\alpha,\ \forall\ \varepsilon \in I_0,\ \exists \psi \in \mathcal{B}^\alpha : \psi \circ \psi \leq \sigma + \varepsilon$.

10.1.3 Definition
A *super uniformity* on X is a function $\delta : I^{X\times X} \longrightarrow I$ which satifies the following conditions:

1. δ is a saturated fuzzy filter;
2. $\forall\ \sigma \in I^{X\times X},\ \inf\limits_{x\in X} \sigma(x, x) \geq \delta(\sigma)$;
3. $\forall\ \sigma \in I^{X\times X},\ \delta(\sigma) = \delta(\sigma_s)$;
4. $\forall\ \sigma \in I^{X\times X},\ \forall\ \varepsilon \in I_0,\ \exists \psi \in I^{X\times X} : \psi \circ \psi - \varepsilon \leq \sigma$ and $\delta(\sigma) \leq \delta(\psi)$.

Note From the definition we can conclude that

$$\delta(\sigma) \geq \inf_{x,y \in X} \sigma(x,y) = \inf \sigma.$$

10.1.4 Proposition

If δ is a super uniformity on X, then for all $\alpha \in I_0$, $\delta_\alpha = \{\sigma \in I^{X\times X} : \delta(\sigma) \geq \alpha\}$ is a fuzzy α-uniformity. Moreover, $\delta_\alpha = \bigcap_{\beta<\alpha} \delta_\beta$.

PROOF.

Let $\delta : I^{X\times X} \longrightarrow I$ be a super uniformity and $\alpha \in I_0$. Then
(i) δ is a saturated fuzzy filter. Therefore

$$\delta_\alpha = \{\sigma \in I^{X\times X} : \delta(\sigma) \geq \alpha\}.$$

is a saturated prefilter with $C(\delta_\alpha) = [0, \alpha)$.
(ii) Let $\sigma \in \delta_\alpha$ and $x \in X$. Then

$$\sigma(x,x) \geq \inf_{x \in X} \sigma(x,x) \geq \delta(\sigma) \geq \alpha.$$

(iii) Let $\sigma \in \delta_\alpha$. Then

$$\delta(\sigma_s) = \delta(\sigma) \geq \alpha \Rightarrow \sigma_s \in \delta_\alpha.$$

(iv) Let $\sigma \in \delta_\alpha$ and $\varepsilon \in I_0$. Then $\exists \psi \in I^{X\times X}$ such that

$$\psi \circ \psi - \varepsilon \leq \sigma \text{ and } \delta(\sigma) \leq \delta(\psi).$$

Thus $\psi \in \delta_\alpha$ such that $\psi \circ \psi \leq \sigma + \varepsilon$.
Since δ is a fuzzy filter. So we have for $\alpha \in I_0$,

$$\delta_\alpha = \bigcap_{\beta<\alpha} \delta_\beta.$$

Hence the result.

□

10.1.5 Proposition

Let $\{\mathcal{U}^\alpha\}_{\alpha \in (0,1)}$ be a non-increasing collection of fuzzy α-uniformity on a set$X$ such that for each $\alpha \in (0,1), \mathcal{U}^\alpha = \bigcap_{\beta<\alpha} \mathcal{U}^\beta$. If for each $\sigma \in I^{X\times X}$ we define

$$\delta(\sigma) = \sup\{\alpha \in (0,1) : \sigma \in \mathcal{U}^\alpha\}.$$

Then δ is a super uniformity.

PROOF.

Let $\{\mathcal{U}^\alpha\}_{\alpha \in (0,1)}$ be a non-increasing collection of fuzzy α-uniformities on a set$X$ such that for each $\alpha \in (0,1), \mathcal{U}^\alpha = \bigcap_{\beta<\alpha} \mathcal{U}^\beta$. Then for each $\sigma \in I^{X\times X}$

$$\delta(\sigma) = \sup\{\alpha \in (0,1) : \sigma \in \mathcal{U}^\alpha\}$$

is a saturated fuzzy fiter.

Let $\sigma \in I^{X\times X}$ and $\gamma < \delta(\sigma)$. Then $\sigma \in U^\gamma$ and hence $\forall\ x \in X,\ \sigma(x,x) \geq \gamma$. So

$$\inf_{x \in X} \sigma(x,x) \geq \gamma.$$

Therefore

$$\inf_{x\in X} \sigma(x,x) \geq \delta(\sigma).$$

We have

$$\begin{aligned}\delta(\sigma) &= \sup\{\alpha \in (0,1) : \sigma \in U^\alpha\} \\ &= \sup\{\alpha \in (0,1) : \sigma_s \in U^\alpha\} \\ &= \delta(\sigma_s).\end{aligned}$$

Let $\sigma \in I^{X\times X}$ and $\varepsilon \in I_0$. Let $\delta(\sigma) = \alpha > 0$. Then $\forall\ \beta < \alpha,\ \sigma \in U^\beta$. So

$$\sigma \in \bigcap_{\beta<\alpha} U^\beta = U^\alpha.$$

Therefore $\exists \psi \in U^\alpha$ such that $\psi \circ \psi \leq \sigma + \varepsilon$ and we have $\delta(\sigma) = \alpha \leq \delta(\psi)$.

□

10.1.6 Theorem

Let $\{\mathbb{D}(\alpha)\}_{\alpha\in(0,1)}$ be a non-decreasing collection of uniformities on a set X. For each $\alpha \in (0,1)$ let

$$\mathcal{D}(\alpha) = \{\sigma \in I^{X\times X} : \forall \beta < \alpha, \forall \gamma < \beta, \sigma^\gamma \in \mathbb{D}(\beta)\} = \bigcap_{\beta<\alpha} (\mathbb{D}(\beta))^\beta.$$

Then:

1. $\mathcal{D}(\alpha)$ is a fuzzy α-uniformity, which we call the fuzzy α-uniformity associated with the collection $\{\mathbb{D}(\alpha)\}_{\alpha\in(0,1)}$,
2. The family $\{\mathcal{D}(\alpha)\}_{\alpha\in(0,1)}$ is non-increasing,
3. $\forall\ \alpha \in (0,1),\ \mathcal{D}(\alpha) = \bigcap_{\alpha'<\alpha} \mathcal{D}(\alpha')$.

Furthermore, if for each $\sigma \in I^{X\times X}$ we define

$$\delta(\sigma) = \sup\{\alpha \in (0,1) : \sigma \in \mathcal{D}(\alpha)\},$$

we obtain a super uniformity δ such that for each $\alpha \in (0,1)$ the corresponding α-level uniformities are

$$\delta_\alpha = \{\sigma \in I^{X\times X} : \delta(\sigma) \geq \alpha\} = \mathcal{D}(\alpha) \text{ and } \delta_1 = \bigcap_{\alpha<1} \mathcal{D}(\alpha).$$

We call δ the super uniformity generated by the collection $\{\mathbb{D}(\alpha)\}_{\alpha\in(0,1)}$.

Proof.

(a) We shall first prove that for each $\alpha \in (0,1)$ the collection $\mathcal{D}(\alpha) = \{\sigma \in I^{X\times X} : \forall\beta < \alpha, \forall\gamma < \beta, \sigma^\gamma \in \mathbb{D}(\beta)\}$ is a fuzzy α-uniformity.

$$\begin{aligned}C(\mathcal{D}(\alpha)) &= \{t \in I : t1_X \notin \mathcal{D}(\alpha)\} = \{t \in I : \exists\beta < \alpha, \exists\gamma < \beta,\ (t1_X)^\gamma \notin \mathbb{D}(\beta)\} \\ &= \{t \in I : \exists\beta < \alpha, \exists\gamma < \beta, t \leq \gamma\} = \{t \in I : \exists\beta < \alpha, t < \beta\} = \{t \in I : t < \alpha\} \\ &= [0,\alpha).\end{aligned}$$

$\mathcal{D}(\alpha) = \bigcap_{\beta<\alpha} (\mathbb{D}(\beta))^\beta$ is a saturated prefilter because it is an intersection of saturated prefilters.

Let $\sigma \in \mathcal{D}(\alpha)$ and $x \in X$. For each $\beta < \alpha$ and each $\gamma < \beta$ we have $\sigma^\gamma \in \mathbb{D}(\beta)$ and hence $(x,x) \in \sigma^\gamma$. Thus for each $\beta < \alpha$ and each $\gamma < \beta$ we have $\sigma(x,x) > \gamma$. So for each $\beta < \alpha$ we have $\sigma(x,x) \geq \beta$ and hence $\sigma(x,x) \geq \alpha$.

Let $\sigma \in \mathcal{D}(\alpha)$. Since $(\sigma^\gamma)_s = (\sigma_s)^\gamma$. Therefore $\sigma_s \in \mathcal{D}(\alpha)$.

Let $\sigma \in \mathcal{D}(\alpha)$, $\epsilon \in I_0$ and choose $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n = \alpha$ and $\alpha_i - \alpha_{i-1} < \frac{1}{2}\epsilon$ for each $i \in \{1, \ldots, n\}$.

For $i \in \{0, 1, \ldots, n-1\}$ we have $\sigma^{\alpha_i} \in \mathbb{D}(\alpha_{i+1})$ and so there exists $U_{\alpha_{i+1}} \in \mathbb{D}(\alpha_{i+1})$ such that $U_{\alpha_{i+1}} \circ U_{\alpha_{i+1}} \subseteq \sigma^{\alpha_i}$.

Let $U'_{\alpha_1} = U_{\alpha_1}$ and $U'_{\alpha_i} = \bigcap_{j \leq i} U_{\alpha_j}$ for each $i \in \{2, \ldots, n\}$. Then, since for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, i\}$, $U_{\alpha_j} \in \mathbb{D}(\alpha_j) \subseteq \mathbb{D}(\alpha_i)$, we have $U'_{\alpha_i} \in \mathbb{D}(\alpha_i)$ and $U'_{\alpha_1} \supseteq U'_{\alpha_2} \supseteq \cdots \supseteq U'_{\alpha_n}$. So we can state that:

for any $i \in \{1, \ldots, n\}$, there exists $U_{\alpha_i} \in \mathbb{D}(\alpha_i)$ such that $U_{\alpha_i} \circ U_{\alpha_i} \subseteq \sigma^{\alpha_{i-1}}$ and $U_{\alpha_1} \supseteq U_{\alpha_2} \supseteq \cdots \supseteq U_{\alpha_n}$. Let

$$U_{\alpha_0} := X \times X$$

and let

$$\psi := \sup_{i \in \{1, \ldots, n\}} \alpha_i 1_X \wedge 1_{U_{\alpha_{i-1}}}.$$

Thus $\psi \in \mathcal{D}(\alpha)$ since if $0 \leq \gamma < \beta < \alpha$ then for some $i \in \{0, \ldots, n-1\}$ we have $\alpha_i \leq \beta < \alpha_{i+1}$. Thus $\gamma < \alpha_{i+1}$ and hence $\psi^\gamma \supseteq \psi_{\alpha_{i+1}} = U_{\alpha_i} \in \mathbb{D}(\alpha_i) \subseteq \mathbb{D}(\beta)$. It follows that $\psi^\gamma \in \mathbb{D}(\beta)$ and so $\psi \in \mathcal{D}(\alpha)$.

If $\sigma(x, y) > \alpha_{n-2}$ then $\sigma(x, y) + \varepsilon > \alpha_{n-2} + (\alpha_n - \alpha_{n-2}) = \alpha_n = \alpha$ and hence we have $(\psi \circ \psi)(x, y) \leq \alpha \leq \sigma(x, y) + \varepsilon$.

If $\sigma(x, y) \leq \alpha_{n-2}$ there exists some $i \leq n-2$ such that $\alpha_{i-1} \leq \sigma(x, y) \leq \alpha_i$. Since $(x, y) \notin \sigma^{\alpha_i}$ we have $(x, y) \notin U_{\alpha_{i+1}} \circ U_{\alpha_{i+1}}$ and so for no $z \in X$ do we have $(x, z) \in U_{\alpha_{i+1}}$ and $(z, y) \in U_{\alpha_{i+1}}$. In other words, for each $z \in X$ either $(x, z) \notin U_{\alpha_{i+1}}$ or $(z, y) \notin U_{\alpha_{i+1}}$. Thus for each $z \in X$ either $\psi(x, z) \leq \alpha_{i+1}$ or $\psi(z, y) \leq \alpha_{i+1}$. Consequently,

$$\psi \circ \psi(x, y) = \sup_{z \in X} \psi(x, z) \wedge \psi(z, y) \leq \alpha_{i+1} < \alpha_{i-1} + \varepsilon \leq \sigma(x, y) + \varepsilon$$

Therefore there exists $\psi \in \mathcal{D}(\alpha)$ such that $\psi \circ \psi \leq \sigma + \varepsilon$.

For each $\alpha \in (0, 1)$ we have

$$\begin{aligned} \bigcap_{\alpha' < \alpha} \mathcal{D}(\alpha') &= \{\sigma \in I^{X \times X} : \forall \alpha' < \alpha, \forall \beta < \alpha', \forall \gamma < \beta, \sigma^\gamma \in \mathbb{D}(\beta)\} \\ &= \{\sigma \in I^{X \times X} : \forall \beta < \alpha, \forall \gamma < \beta, \sigma^\gamma \in \mathbb{D}(\beta)\} = \mathcal{D}(\alpha). \end{aligned}$$

We now appeal to (10.1.5) and claim that the mapping defined for each $\sigma \in I^{X \times X}$ by

$$\delta(\sigma) = \sup\{\alpha \in (0, 1) : \sigma \in \mathcal{D}(\alpha)\}$$

is a super uniformity such that for each $\alpha \in (0, 1)$ the corresponding α-level uniformities are

$$\delta_\alpha = \{\sigma \in I^{X \times X} : \delta(\sigma) \geq \alpha\} = \mathcal{D}(\alpha) \text{ and } \delta_1 = \bigcap_{\alpha < 1} \delta_\alpha = \bigcap_{\alpha < 1} \mathcal{D}(\alpha).$$

In particular, for a fuzzy uniformity $\mathcal{D}$, the collection $\{\mathbb{D}(\alpha) = \mathcal{D}^\alpha\}_{\alpha \in (0,1)}$, is a non-decreasing collection of uniformities and in that case, the fuzzy α-uniformity associated with this collection is $\mathcal{D}(\alpha) = \bigcap_{\beta < \alpha} (\mathcal{D}^\beta)^\beta$.

□

The proof of the following corollary is straightforward.

10.1.7 Corollary

Let $(X, \mathbb{D})$ be uniform space. Then the super uniformity $\delta_{\mathbb{D}}$ generated by the collection $\{\mathbb{D}(\alpha) = \mathbb{D}\}_{\alpha \in (0,1)}$ is

$$\delta_{\mathbb{D}}(\sigma) = \sup\{\alpha \in (0, 1) : \sigma \in \bigcap_{\beta < \alpha} \mathbb{D}^\beta\} = \sup\{\alpha : \alpha \in \mathbb{D}^\alpha\} = \sup\{\alpha : \sigma^\alpha \in \mathbb{D}\}.$$

10.1.8 Definition

A non-decreasing collection of uniformities $\{\mathbb{D}(\alpha)\}_{\alpha\in(0,1)}$ is said to be *generated from below* if for each $\alpha \in (0,1)$ we have $\mathbb{D}(\alpha) = \bigcup_{\alpha'<\alpha} \mathbb{D}(\alpha')$.

In the case $\{\mathbb{D}(\alpha) = \mathcal{D}^\alpha\}_{\alpha\in(0,1)}$, where $\mathcal{D}$ is a fuzzy uniformity, it is easy to check that it is generated from below.

10.1.9 Theorem

The super uniformities generated by two different, non-decreasing collections of uniformities, which are generated from below, are different.

Proof.

Let $\{\mathbb{D}(\alpha)\}_{\alpha\in(0,1)}$ and $\{\mathbb{D}^(\alpha')\}_{\alpha\in(0,1)}$ two different non-decreasing collections of uniformities which are generated from below. Then there exists $\alpha \in (0,1)$ and $U \subseteq X\times X$ such that $U \in \mathbb{D}(\alpha) - \mathbb{D}'(\alpha)$. Since $U \in \mathbb{D}(\alpha) = \bigcup_{\alpha'<\alpha} \mathbb{D}(\alpha')$, there exists $\alpha' < \alpha$ such that $U \in \mathbb{D}(\alpha')$. We now consider $\sigma_U = (\alpha 1_X \wedge 1_U) \vee \alpha' 1_X \in I^{X\times X}$.
⋆ If $\beta \in [\alpha', \alpha)$ and $\gamma < \beta$ then

$$\sigma_U^\gamma = \begin{cases} U & \text{if } \gamma \geq \alpha' \\ X\times X & \text{if } \gamma < \alpha'. \end{cases}$$

Thus in any case ${\sigma_U}^\gamma \in \mathbb{D}(\alpha') \subseteq \mathbb{D}(\beta)$.
⋆ If $\beta < \alpha'$ and $\gamma < \beta$ then ${\sigma_U}^\gamma = X\times X \in \mathbb{D}(\beta)$.

Therefore, for each $\beta < \alpha$ and each $\gamma < \beta$ we have ${\sigma_U}^\gamma \in \mathbb{D}(\beta)$, hence $\sigma_U \in \mathcal{D}(\alpha)$ and so $\delta(\sigma_U) \geq \alpha$.

On the other hand we know $U \notin \mathbb{D}'(\alpha)$. Thus for each $\alpha'' \in (\alpha', \alpha)$ there exists γ and β such that $\alpha' < \gamma < \beta < \alpha'' < \alpha$ and ${\sigma_U}^\gamma = U \notin \mathbb{D}'(\beta)$. Thus $\sigma_U \notin \mathcal{D}'(\alpha'')$ and therefore $\delta'(\sigma_U) \leq \alpha'$. Consequently δ and δ' are different.

10.1.10 Theorem

Let (X, δ) be super uniform space. Then

$$\mathbb{D}_\delta = \{\sigma^\alpha : \alpha < 1,\ \delta(\sigma) = 1\}$$

is a uniformity on X.

Proof.

If $\sigma^\alpha = \emptyset \in \mathbb{D}_\delta$ then $\sigma \leq \alpha 1_X$. Therefore $\delta(\sigma) = 1 \leq \delta(\alpha 1_X) = \alpha$ and this contradiction shows $\emptyset \notin \mathbb{D}_\delta$. $\delta(1) = 1$ and $\sigma^\alpha = X\times X$ for any $\alpha < 1$. Therefore $X \times X \in \mathbb{D}_\delta$.

Let $\sigma^\alpha \in \mathbb{D}_\delta$ and $\sigma^\alpha \subseteq U$. Then $\sigma \leq \alpha 1_X \vee 1_{\sigma^\alpha} \leq \alpha 1_X \vee 1_U$. Therefore $\delta(\sigma) = 1 \leq \delta(\alpha 1_X \vee 1_U)$ and for $\beta \in [\alpha, 1)$ we have $(\alpha 1_X \vee 1_U)^\beta = U \in \mathbb{D}_\delta$.

Let $\sigma^\alpha, \psi^\beta \in \mathbb{D}_\delta$. Then $\delta(\sigma \wedge \psi) = 1$ and $(\sigma\wedge\psi)^{\alpha\vee\beta} \in \mathbb{D}_\delta$. But $(\sigma\wedge\psi)^{\alpha\vee\beta} \subseteq \sigma^\alpha \cap \psi^\beta$. Therefore $\sigma^\alpha \cap \psi^\beta \in \mathbb{D}_\delta$.

Let $\sigma^\alpha \in \mathbb{D}_\delta$. For each $x \in X$ we know that $\sigma(x,x) \geq \delta(\sigma) = 1 > \alpha$ and so $(x,x) \in \sigma^\alpha$. Therefore $\Delta \subseteq \sigma^\alpha$.

Let $\sigma^\alpha \in \mathbb{D}_\delta$. Since $(\sigma^\alpha)_s = (\sigma_s)^\alpha$, we have $(\sigma^\alpha)_s \in \mathbb{D}_\delta$.

Let $\sigma^\alpha \in \mathbb{D}_\delta$. We therefore have $\sigma \in I^{X\times X}$ and $\varepsilon = \frac{1-\alpha}{2} > 0$. Thus $\exists \psi \in I^{X\times X}$ such that $\psi\circ\psi \leq \sigma + \varepsilon$ and $\delta(\psi) \geq \delta(\sigma) = 1$. Hence for $\beta = \frac{1+\alpha}{2} \in (\alpha, 1)$ we have $\psi^\beta \circ \psi^\beta \subseteq (\psi\circ\psi)^\beta \subseteq (\sigma+\varepsilon)^\beta = \sigma^{\beta-\varepsilon} = \sigma^\alpha$. Thus $\exists \psi^\beta \in \mathbb{D}_\delta$ such that $\psi^\beta \circ \psi^\beta \subseteq \sigma^\alpha$.

□

10.1.11 Theorem
Let (X, δ) be a super uniform space. Then

$$\mathcal{D}_\delta = \{\sigma \in I^{X\times X} : \delta(\sigma) = 1\}$$

is a fuzzy uniformity on X.

PROOF.

We have $0 \notin \mathcal{D}_\delta$ since $\delta(0) = 0$. $\delta(1) = 1$ and so $1 \in \mathcal{D}_\delta$.
Let $\sigma, \psi \in \mathcal{D}_\delta$. Then $\delta(\sigma) = 1$ and $\delta(\psi) = 1$. So

$$\delta(\sigma \wedge \psi) = \delta(\sigma) \wedge \delta(\psi) = 1.$$

Therefore $\sigma \wedge \psi \in \mathcal{D}_\delta$.
Let $\sigma \in \mathcal{D}_\delta$ and $\sigma \leq \psi$. Then $1 = \delta(\sigma) \leq \delta(\psi)$. Therefore $\psi \in \mathcal{D}_\delta$. Hence $\mathcal{D}_\delta$ is a prefilter.
Let $\forall\ \varepsilon \in I_0,\ \sigma + \varepsilon \in \mathcal{D}_\delta$. Then $\forall\ \varepsilon \in I_0,\ \delta(\sigma + \varepsilon) = 1$. So

$$\tilde{\delta}(\sigma) = \inf_{\varepsilon \in I_0} \delta(\sigma + \varepsilon) = 1.$$

Therefore $\sigma \in \mathcal{D}_\delta$ and hence $\mathcal{D}_\sigma$ is a saturated prefiter.

Let $\sigma \in \mathcal{D}_\delta$ and $x \in X$. Then

$$\sigma(x, x) \geq \delta(\sigma) = 1.$$

Let $\sigma \in \mathcal{D}_\delta$. Then $\delta(\sigma_s) = \delta(\sigma) = 1$ and so $\sigma_s \in \mathcal{D}_\delta$.

Let $\sigma \in \mathcal{D}_\delta,\ \varepsilon \in I_0$. Then $\exists \psi \in I^{X\times X}$ such that $\psi \circ \psi - \varepsilon \leq \sigma$ and $1 = \delta(\sigma) \leq \delta(\psi)$. Therefore $\psi \in \mathcal{D}_\delta$ such that $\psi \circ \psi \leq \sigma + \varepsilon$. Therefore $\mathcal{D}_\delta$ is a fuzzy uniform space.

□

10.2 Uniformly Continuous Functions

10.2.1 Definition
Let $(X, \mathcal{U}^\alpha)$ and $(Y, \mathcal{V}^\alpha)$ be two fuzzy α-uniform spaces. Then a mapping $f : (X, \mathcal{U}^\alpha) \longrightarrow (Y, \mathcal{V}^\alpha)$ is said to be a *uniformly continuous* if

$$\forall\ \psi \in \mathcal{V}^\alpha,\ (f \times f)^{-1}[\psi] \in \mathcal{U}^\alpha.$$

That is, $\forall\ \psi \in \mathcal{V}^\alpha,\ \exists \sigma \in \mathcal{U}^\alpha$ such that $(f \times f)[\sigma] \leq \psi$.

10.2.2 Definition
Let (X, δ_X) and (Y, δ_Y) be super uniform spaces. Then a mapping $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is said to be *uniformly continuous* if

$$\forall\ \psi \in I^{Y\times Y},\ \delta_X((f \times f)^{-1}[\psi]) \geq \delta_Y(\psi).$$

10.2.3 Proposition
Let (X, δ_X) and (Y, δ_Y) be super uniform spaces. Then

1. $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is uniformly continuous $\Rightarrow \forall\ \alpha \in I_0,\ f : (X, (\delta_X)_\alpha) \longrightarrow (Y, (\delta_Y)_\alpha)$ is uniformly continuous.
2. $\forall\ \alpha \in (0, 1),\ f : (X, (\delta_X)_\alpha) \longrightarrow (Y, (\delta_Y)_\alpha)$ is uniformly continuous $\Rightarrow f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is uniformly continuous.

PROOF.

(1) Let $\psi \in (\delta_Y)_\alpha$. Then $\delta_Y(\psi) \geq \alpha$. Therefore

$$\delta_X((f \times f)^{-1}[\psi]) \geq \delta_Y[\psi] \geq \alpha.$$

Thus $(f \times f)^{-1}[\psi] \in (\delta_X)_\alpha$.

(2) Let $\psi \in I^{Y \times Y}$ and $\alpha \in (0,1)$ such that $\delta_Y(\psi) \geq \alpha$. Then $\psi \in (\delta_Y)_\alpha$ and hence $(f \times f)^{-1}[\psi] \in (\delta_X)_\alpha$. Thus $\delta_X((f \times f)^{-1}[\psi]) \geq \alpha$. Therefore $\delta_X((f \times f)^{-1}[\psi]) \geq \delta_Y(\psi)$. □

10.2.4 Theorem
Let $(X, \mathbb{D})$ and $(Y, \mathbb{E})$ be uniform spaces. If $f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ is uniformly continuous then $f : (X, \delta_{\mathbb{D}}) \longrightarrow (Y, \delta_{\mathbb{E}})$ is uniformly continuous.

PROOF.

Let $f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ be uniformly continuous and $\psi \in I^{Y \times Y}$.
If $\delta_{\mathbb{E}}(\psi) = \sup\{\alpha : \psi^\alpha \in \mathbb{E}\} > t$ then $\psi^t \in \mathbb{E}$ and so $(f \times f)^{\leftarrow}(\psi^t) \in \mathbb{D}$. Therefore

$$((f \times f)^{-1}[\psi])^t = (f \times f)^{\leftarrow}(\psi^t) \in \mathbb{D}.$$

So $\delta_{\mathbb{D}}((f \times f)^{-1}[\psi]) \geq t$. Consequently,

$$\delta_{\mathbb{D}}((f \times f)^{-1}[\psi]) \geq 1_{\mathbb{E}}(\psi).$$

□

10.2.5 Theorem
Let (X, δ_X) and (Y, δ_Y) be super uniform spaces. If $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is uniformly continuous then $f : (X, \mathbb{D}_{\delta_X}) \longrightarrow (Y, \mathbb{D}_{\delta_Y})$ is uniformly continuous.

PROOF.

Let $\psi \in \mathbb{D}_{\delta_Y}$. Then $\delta_Y(\psi) = 1$. Therefore

$$\delta_X((f \times f)^{-1}[\psi]) \geq \delta_Y[\psi] = 1.$$

and

$$(f \times f)^{\leftarrow}(\psi^\alpha) = ((f \times f)^{-1}[\psi])^\alpha \in \mathbb{D}_{\delta_X}.$$

□

10.2.6 Theorem
Given a fuzzy uniformity $\mathcal{D}$ on a set X, $\delta_{\mathcal{D}}$ denote the super uniformity generated by the collection $\{\mathcal{D}^\alpha\}_{\alpha \in (0,1)}$. Suppose $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ are fuzzy uniform spaces and $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is uniformly continuous then $f : (X, \delta_{\mathcal{D}}) \longrightarrow (Y, \delta_{\mathcal{E}})$.

PROOF.

Let $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is uniformly continuous and $\psi \in I^{Y \times Y}$.
If $\delta_{\mathcal{E}}(\psi) > \alpha$ then $\psi \in \mathcal{E}(\alpha)$. That is $\forall\ \beta < \alpha$, $\forall\ \gamma < \beta$, $\psi^\gamma \in \mathcal{E}^\beta$ and so $\exists\ \psi' \in \mathcal{E}$ and $\gamma' < \beta$ such that $\psi^\gamma = {\psi'}^{\gamma'}$. Now $(f \times f)^{-1}[\psi'] \in \mathcal{D}$ and so

$$((f \times f)^{-1}[\psi])^\gamma = (f \times f)^{\leftarrow}(\psi^\gamma) = (f \times f)^{\leftarrow}({\psi'}^{\gamma'}) = (f \times f)^{-1}[\psi'])^{\gamma'} \in \mathcal{D}^\beta.$$

Therefore $\delta_{\mathcal{D}}((f \times f)^{-1}[\psi]) \geq \alpha$. Consequently,

$$\delta_{\mathcal{E}}(\psi) \leq \delta_{\mathcal{D}}((f \times f)^{-1}[\psi]).$$

□

10.2.7 Theorem
Let (X, δ_X) and (Y, δ_Y) be super uniform spaces. If $f : (X, \mathbb{D}_X) \longrightarrow (Y, \delta_Y)$ is uniformly continuous then $f : (X, \mathcal{D}_{\delta_X}) \longrightarrow (Y, \mathcal{D}_{\delta_Y})$ is uniformly continuous.

PROOF.

Let $\psi \in \mathcal{D}_{\delta_Y}$. Then $\delta_Y(\psi) = 1$. Therefore

$$\delta_X((f \times f)^{-1}[\psi]) \geq \delta_Y(\psi) = 1.$$

Hence $(f \times f)^{-1}[\psi] \in \mathcal{D}_{\delta_X}$.

□

Chapter 11

Categorical Embeddings

In this chapter we establish categorical embeddings from the category of uniform spaces into the categories of fuzzy uniform spaces, generalised uniform spaces and super uniform spaces. We also obtain categorical embeddings from the categories of fuzzy uniform spaces and generalised uniform spaces into the category of super uniform spaces. We show that the category of fuzzy uniform spaces and the category of generalised uniform spaces are isomorphic. These categorical relations are introduced and studied in [29].

11.1 Embedding into the Category of Fuzzy Uniform Spaces

Let US denote the category of uniform spaces with uniformly continuous maps and let FUS denote the category of fuzzy uniform spaces with uniformly continuous maps.

11.1.1 Theorem
Let

$$\omega_U : US \longrightarrow FUS, \qquad (X, \mathbb{D}) \mapsto (X, \mathbb{D}^1)$$

and let ω_U leave maps unchanged.
Let

$$i_U : FUS \longrightarrow US, \qquad (X, \mathcal{D}) \mapsto (X, \mathcal{D}^1)$$

and let i_U leave maps unchanged.
Then

(a) ω_U is a functor,

(b) i_U is a functor,

(c) $i_U \circ \omega_U = id_{US}$,

(d) ω_U is co-adjoint.

Proof.

(a) We have if $f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ is uniformly continuous in US then $f : (X, \mathbb{D}^1) \longrightarrow (Y, \mathbb{E}^1)$ is uniformly continuous in FUS.
Clearly ω_U preserves compositions and identites.
Therefore ω_U is a functor.

(b) We have if $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is a uniformly continuous in FUS then $f : (X, \mathcal{D}^1) \longrightarrow (Y, \mathcal{E}^1)$ is uniformly continuous in US.
Clearly I_U preserves compositions and identities.
Therefore I_U is a functor.

(c) Since $(\mathbb{D}^1)^1 = \mathbb{D}$, we have

$$(I_U \circ \omega_U)((X, \mathbb{D})) = I_U(X, \mathbb{D}^1) = (X, (\mathbb{D}^1)^1) = (X, \mathbb{D}).$$

Therefore $I_U \circ \omega_U = id_{US}$.
(d) We have $id_X : (X, (\mathcal{D}^1)^1) \longrightarrow (X, \mathcal{D})$ is uniformly continuous, since $\mathcal{D} \subseteq (\mathcal{D}^1)^1$.
We also have

$f : (Y, \mathbb{D}^1) \longrightarrow (X, \mathcal{D})$ is uniformly continuous
$\iff f : (Y, \mathcal{D}) \longrightarrow (X, \mathcal{D}^1)$ is uniformly continuous [since $(\mathbb{D}^1)^1 = \mathbb{D}$]
$\iff f : (Y, \mathbb{D}^1) \longrightarrow (X, (\mathcal{D}^1)^1)$ is uniformly continuous.

$$US \underset{i_U}{\overset{\omega_U}{\rightleftarrows}} FUS$$

$$(X, \mathcal{D}^1) \xleftarrow{!\, f} (Y, \mathbb{D}) \qquad \begin{array}{ccc} (X, \mathcal{D}) & & \\ id_X \uparrow & \nwarrow^{f} & \\ (X, (\mathcal{D}^1)^1) & \underset{f}{\leftarrow} & (Y, \mathbb{D}^1) \end{array}$$

For $(X, \mathcal{D}) \in Ob(FUS)$, $((X, \mathcal{D}^1), id_X)$ is a ω_U- co-universal arrow with domain $(X, \mathcal{D})$.
Therefore ω_U is a co-adjoint. ω_U embeds the category US as a coreflective subcategory of FUS.

□

11.2 Embedding into the Category of Generalised Uniform Spaces

Let GUS denote the category of generalised uniform spaces with uniformly continuous maps.

11.2.1 Theorem
Let

$$\varepsilon_U : US \longrightarrow GUS, \qquad (X, \mathbb{D}) \mapsto (X, 1_{\mathbb{D}})$$

and let ε_U leaves maps unchanged.
Let

$$\gamma_U : GUS \longrightarrow US, \qquad (X, d) \mapsto (X, d^0)$$

and let γ_U leaves maps unchanged.
Then

(a) ε_U is a functor,

(b) γ_U is a functor,

(c) $\gamma_U \circ \varepsilon_U = id_{US}$,

(d) ε_U is co-adjoint.

Proof.

(a) We have
(i) If $f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ is uniformly continuous then $f : (X, 1_{\mathbb{D}}) \longrightarrow (Y, 1_{\mathbb{E}})$ is uniformly continuous.
(ii) ε_U preserves compositions and identities.
Therefore ε_U is a functor.

(b) We have

(i) If $f : (X, d) \longrightarrow (Y, e)$ is uniformly continuous then $f : (X, d^0) \longrightarrow (Y, e^0)$ is uniformly continuous.

(ii) γ_U preserves compositions and identities.

Therefore γ_U is a functor.

(c) We have

$$(\gamma_U \circ \varepsilon_U)((X, \mathbb{D})) = \gamma_U((X, 1_{\mathbb{D}})) = (X, 1^0_{\mathcal{D}}) = (X, \mathbb{D}).$$

(d) First we have to show $d \leq 1_{d^0}$.

$$U \in d^0 \Longleftrightarrow d(U) > 0.$$

We have $d(U) > 0 \Rightarrow 1_{d^0}(U) = 1$ and hence

$$d \leq 1_{d^0}.$$

So we have $id_X : (X, 1_{d^0}) \longrightarrow (X, d)$ is uniformly continuous.

We also have

$$\begin{array}{l} f : (Y, 1_{\mathbb{D}}) \longrightarrow (X, d) \text{ is uniformly continuous} \\ \Rightarrow f : (Y, \mathbb{D}) \longrightarrow (X, d^0) \text{ is uniformly continuous [Since } 1^0_{\mathbb{D}} = \mathbb{D}] \\ \Rightarrow f : (Y, 1_{\mathbb{D}}) \longrightarrow (X, 1_{d^0}) \text{ is uniformly continuous.} \end{array}$$

$$US \underset{\gamma_U}{\overset{\varepsilon_U}{\rightleftarrows}} GUS$$

$$(X, d^0) \overset{!f}{\longleftarrow} (Y, \mathbb{D}) \qquad \begin{array}{ccc} (X, d) & & \\ id_X \uparrow & \nwarrow^{f} & \\ (X, 1_{d^0}) & \underset{f}{\leftarrow} & (Y, 1_{\mathbb{D}}) \end{array}$$

For $(X, d) \in Ob(GUS)$, $((X, d^0), id_X)$ is a ε_U- co-universal arrow with codomain (X, d).

Therefore ε_U is a co-adjoint. ε_U embeds the category US as a coreflective subcategory of FUS.

□

Next we establish an isomorphim between FUS and GUS.

11.2.2 Theorem

$$FUS \cong GUS$$

Proof.

Let

$$F : FUS \longrightarrow GUS, \qquad (X, \mathcal{D}) \mapsto (X, d_{\mathcal{D}})$$

and F leaves maps unchanged.

We have seen in 8.1.6 that F is bijective. We also have:

(a) if $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is uniformly continuous then $F(f) = f : (X, d_{\mathcal{D}}) \longrightarrow (Y, d_{\mathcal{E}})$ is uniformly continuous.

(b) F preserves composition and identities.

Therefore F is functor. Now we have to show there is functor $G : GUS \longrightarrow FUS$ such that $G \circ F = id_{FUS}$ and $F \circ G = id_{GUS}$. Let

$$G : GUS \longrightarrow FUS, \qquad (X, d) \mapsto (X, \mathcal{D}_d)$$

G leaves maps unchanged.

We have

(a) If $f : (X, d) \longrightarrow (Y, e)$ is uniformly continuous then $G(f) = f : (X, \mathcal{D}_d) \longrightarrow (Y, \mathcal{D}_e)$ is uniformly continuous.

(b) G preserves compositions and identities.

Therefore G is a functor. Now we have

$$(G \circ F)((X, \mathcal{D})) = G((X, d_{\mathcal{D}})) = (X, \mathcal{D}_{d_{\mathcal{D}}}) = (X, \mathcal{D}),$$
$$(F \circ G)((X, d)) = F((X, \mathcal{D}_d)) = (X, d_{\mathcal{D}_d}) = (X, d).$$

Hence the result.

□

11.3 Embeddings into the Category of Super Uniform Spaces

Let SUS denote the category of super uniform spaces with uniformly continuous maps.

11.3.1 Theorem

Let

$$\lambda_U : US \longrightarrow SUS, \qquad (X, \mathbb{D}) \mapsto (X, \delta_{\mathbb{D}})$$

and let λ_U leaves maps unchanged.

let

$$k_U : SUS \longrightarrow US, \qquad (X, \delta) \mapsto (X, \mathbb{D}_\delta)$$

and let k_U leaves maps unchanged.

Then

(a) λ_U is a functor,

(b) k_U is a functor,

(c) $k_U \circ \lambda_U = id_{US}$,

(d) λ_U embeds the category US as a coreflective subcategory of SUS,

(e) If $\sigma \in I^{X \times X}$ such that $\delta(\sigma) = 1$ then $\delta_{\mathbb{D}_\delta}(\sigma) = 1$.

Proof.

(a) We have

if $f : (X, \mathbb{D}) \longrightarrow (Y, \mathbb{E})$ is uniformly continuous then $f : (X, \delta_{\mathbb{D}}) \longrightarrow (Y, \delta_{\mathbb{E}})$ is uniformly continuous.

λ_U preserves compositions and identities.

Therefore λ_U is a functor.

(b) We have

if $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is uniformly continuous then $f : (X, \mathbb{D}_{\delta_X}) \longrightarrow (Y, \mathbb{D}_{\delta_Y}$ is uniformly continuous.

k_U preserves compositions and identities.

Therefore k_U is a functor.

(c) We have

$$(k_U \circ \lambda_U)((X, \mathbb{D})) = k_U((X, \delta_{\mathbb{D}})) = (X, \mathbb{D}_{\delta_{\mathbb{D}}}).$$

Now we have to show $\mathbb{D} = \mathbb{D}_{\delta_{\mathbb{D}}}$. We have

$$\begin{aligned} U \in \mathbb{D}_{\delta_{\mathbb{D}}} &\Longleftrightarrow \exists \alpha < 1,\ \exists \sigma \in I^{X \times X},\ : \delta_{\mathbb{D}}(\sigma) = 1 \text{ and } U = \sigma^\alpha \\ &\Longleftrightarrow \exists \alpha,\ \exists \sigma\ : U = \sigma^\alpha \text{ and } \forall\ \beta < 1,\ \sigma^\beta \in \mathbb{D} \\ &\Longrightarrow U = \sigma^\alpha \in \mathbb{D}. \end{aligned}$$

Conversly if $U \in \mathbb{D}$, let $\sigma = 1_U$. Then $\sigma^\alpha = U$ for all $\alpha \in (0,1)$ and so $\delta_{\mathbb{D}}(U) = 1$. It follows that $U = \sigma^\alpha \in \mathbb{D}_{\delta_{\mathbb{D}}}$.

(d) The injectivity of λ_U on morphisms follows from Theorem (10.1.9), together with the fact that λ_U leaves underlying maps unchanged.

(e) If $\delta(\sigma) = 1$ then for all $\alpha < 1$, $\sigma^\alpha \in \mathbb{D}_\delta$ and so $\sigma \in (\mathbb{D}_\delta)^1$. Therefore

$$\delta_{\mathbb{D}_\delta}(\sigma) = \sup\{\alpha : \sigma \in (\mathbb{D}_\delta)^\alpha\} = 1.$$

□

11.3.2 Theorem

Let

$$\varepsilon_F : FUS \longrightarrow SUS, \qquad (X, \mathcal{D}) \mapsto (X, \delta_{\mathcal{D}})$$

and let ε_F leaves maps unchanged.

Let

$$\gamma_F : SUS \longrightarrow FUS, \qquad (X, \delta) \mapsto (X, \mathcal{D}_\delta)$$

and let γ_F leaves maps unchanged.

Then

(a) ε_F is a functor,

(b) γ_F is a functor,

(c) $\gamma_F \circ \varepsilon_F = id_{FUS}$,

(d) ε_F embeds the category FUS as a coreflective subcategory of SUS,

(e) $\varepsilon_F \circ \omega_U = \lambda_U$,

(f) If $\sigma \in I^{X \times X}$ such that $\delta(\sigma) = 1$ then $\sigma_{\mathcal{D}_\delta}(\sigma) = 1$.

PROOF.

(a) We have

if $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is uniformly continuous then $f : (X, \delta_{\mathcal{D}}) \longrightarrow (Y, \delta_{\mathcal{E}})$ is uniformly continuous.

ε_F preserves compositions and identities.

Therefore ε_U is a functor.

(b) We have

if $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is uniformly continuous then $f : (X, \mathcal{D}_{\delta_X} \longrightarrow (Y, \mathcal{D}_{\delta_Y})$ is uniformly continuous.

γ_F preserves compositions and identities.

Therefore γ_F is a functor.

(c) We have

$$(\gamma_F \circ \varepsilon_F)((X, \mathcal{D})) = \gamma_F((X, \delta_{\mathcal{D}})) = (X, \mathcal{D}_{\delta_{\mathcal{D}}}).$$

Now we have to show that $\mathcal{D}_{\delta_{\mathcal{D}}} = \mathcal{D}$. We have

$$\begin{aligned}
\mathcal{D}_{\delta_{\mathcal{D}}} &= \{\sigma \in I^{X \times X} : \delta_{\mathcal{D}}(\sigma) = 1\} = \{\sigma \in I^{X \times X} : \forall\ \alpha < 1,\ \sigma \in \mathcal{D}(\alpha)\} \\
&= \{\sigma \in I^{X \times X} : \forall\ \alpha < 1,\ \forall\ \beta < \alpha,\ \forall\ \gamma < \beta, \sigma^\gamma \in \mathcal{D}^\beta\} \\
&= \{\sigma \in I^{X \times X} : \forall\ \beta < 1,\ \forall\ \gamma < \beta,\ \sigma^\gamma \in \mathcal{D}^\beta\}.
\end{aligned}$$

For any $\sigma \in \mathcal{D}$ and for any $\gamma < \beta < 1$ we have $\sigma^\gamma \in \mathcal{D}^\beta$ and so $\mathcal{D} \subseteq \mathcal{D}_{(\delta_{\mathcal{D}})}$. Thus for each $\alpha \in (0, 1]$ we have $\mathcal{D}^\alpha \subseteq (\mathcal{D}_{(\delta_{\mathcal{D}})})^\alpha$.

On the other hand, for each $\alpha \in (0,1]$,

$$(\mathcal{D}_{(\delta_{\mathcal{D}})})^{\alpha} = \{\sigma^{\alpha'} : \alpha' < \alpha, \sigma \in \mathcal{D}_{(\delta_{\mathcal{D}})}\} = \{\sigma^{\alpha'} : \alpha' < \alpha, \forall \beta < 1, \forall \gamma < \beta, \sigma^{\gamma} \in \mathcal{D}^{\beta}\} \subseteq \mathcal{D}^{\alpha}.$$

Therefore for each $\alpha \in (0,1]$, $(\mathcal{D}_{(\delta_{\mathcal{D}})})^{\alpha} = \mathcal{D}^{\alpha}$ and so $\mathcal{D}_{(\delta_{\mathcal{D}})} = \mathcal{D}$ [See Theorem 5.3.4].

(d) The injectivity of ε_F on morphisms follows from Theorem (10.1.9), together with the fact that ε_F leaves underlying maps unchanged.

(e) For each $(X, \mathbb{D}) \in |US|$ we have

$$(\varepsilon_F \circ \omega_U)((X, \mathbb{D})) = \varepsilon_F((X, \mathbb{D}^1)) = (X, \delta_{\mathbb{D}^1})$$

and $\delta_{\mathbb{D}^1}$ is the super uniformity generated by the collection $\{(\mathbb{D}^1)^{\alpha}\}_{\alpha \in (0,1)}$. But for each $\alpha \in (0,1)$ we have

$$(\mathbb{D}^1)^{\alpha} = \{\sigma^{\beta} : \beta < \alpha, \sigma \in \mathbb{D}^1\} = \mathbb{D}$$

and so $\delta_{\mathbb{D}^1}$ is the super uniformity generated by $\mathbb{D}$. That is,

$$(\varepsilon_F \circ \omega_U)((X, \mathbb{D})) = (X, \delta_{\mathbb{D}^1}) = (X, \delta_{\mathbb{D}}) = \lambda_U((X, \mathbb{D})).$$

(f) If $\delta(\sigma) = 1$ then $\sigma \in \mathcal{D}_{\delta} = \bigcap_{\alpha < 1} ((\mathcal{D}_{\delta})^{\alpha})^{\alpha}$. That is, $\delta_{\mathcal{D}_{\delta}}(\sigma) = 1$.

□

We have seen that there is a functor ε_F which embeds the category FUS into the category SUS, thereby making SUS an extension of FUS. The categorical isomorphism

$$F : FUS \to GUS, \quad G : GUS \to FUS$$

between FUS and GUS is established in [11.1.2]. We can therefore define

$$\varepsilon_G : GUS \to SUS$$

by

$$\varepsilon_G = \varepsilon_F \circ G$$

and it follows that ε_G is a functor which embeds the category GUS into SUS. If we define

$$\gamma_G = F \circ \gamma_F$$

then

$$\gamma_G \circ \varepsilon_G = id_{GUS}$$

and also

$$\varepsilon_G \circ \varepsilon_U = \lambda_U.$$

Thus SUS is also an extension of the category GUS.

Bibliography

[1] J. Adamek, H. Herrlich and G. E. Strecker, *Abstract and concrete categories*, 1990, Wiley-Interscience, New York.

[2] T. M. G. Ahsanullah, Some results on fuzzy neighbourhood spaces, Pure Math. Manuscript **4**(1985), 97- 106.

[3] T. M. G. Ahsanullah, Some properties of the level spaces of fuzzy neighbourhood spaces and fuzzy neighbouhood groups, J. Bangladesh Acad.Sci. **13(1)** (1989), 91-95.

[4] I. W. Alderton and F. Schwarz, Cartesian closed extensions of the category of uniform spaces, Topology & Appl. **57** (1994), 95-109.

[5] A. A. Allam , K. M. Abd El-Hakeim and N. N. Morsi, On fuzzy neighbourhood spaces, Fuzzy Sets and Systems **41** (1991), 201-212.

[6] G. Artico and R. Moresco, Fuzzy proximities compatible with Lowen fuzzy uniformities, Fuzzy Sets and Systems **21**, (1987) 85-98.

[7] G. Artico and R. Moresco, Fuzzy uniformities induced by fuzzy proximities, Fuzzy Sets and Systems **31** (1989), 111-121.

[8] N. Bourbaki, Element of Mathematics *General topology Chapter 1-4*, (1989), Springer-Verlag, Berlin Heidelberg, New York.

[9] M. H. Burton, The Relationship Between a Fuzzy Uniformity and its family of $\alpha-$level Uniformities, Fuzzy Sets and Systems **54(3)** (1993), 311-315.

[10] M. H. Burton, Cauchy Filters and Prefilters, Fuzzy Sets and Systems **54(3)** (1993), 317-331.

[11] M. H. Burton, Completeness in Fuzzy Uniform Spaces, Quaestiones Math. **16(1)** (1993), 13-36.

[12] M. H. Burton, Precompactness in Fuzzy Uniform Spaces, Quaestiones Math. **16(1)** (1993), 37-49.

[13] M. H. Burton, Boundedness in Uniform Spaces and Fuzzy Uniform Spaces, Fuzzy Sets and Systems **58(2)** (1993), 195-207.

[14] M. H. Burton, The Fuzzy Uniformisation of Function Spaces. To appear in Quaestiones Math.

[15] M. H. Burton, M. A. De Prada Vicente & J. Gutiérrez García, Generalised Uniform Spaces, J. Fuzzy Math. **4(2)**(1996), 363-380.

[16] M. H. Burton, M. Muraleetharan and J. Gutiérrez García, Generalised Filters, Submitted for publication.

[17] J. J. Chadwick, A Generalised Form of Compactness in Fuzzy Topological Spaces. J. Math. Anal. Appl. **162[1]** (1991), 92-110.

[18] J. J. Chadwick, Relative Compactness and Compactness of General Subsets of an I-Topological Space. Qaestiones Mathematicae **14(4)** (1991), 491-507.

[19] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. **24** (1968), 182-190.

[20] M. A. de Prada Vicente, Fuzzy filters, J. Math. Anal. Appl. **129** (1988), 560-568.

[21] M. A. de Prada Vicente and M. Macho Stadler, t-Prefilter theory, Fuzzy Sets and Systems **38** (1990), 115- 124.

[22] P. Eklund and W. Gähler, Fuzzy Filters, Functors and Convergence, In: *Applications of Category Theory to Fuzzy Sets, Chapter IV,* (1992), 109-136, Kluwer Academic Publishers.

[23] J. A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. **18** (1967), 145-174.

[24] J. A. Goguen, The Fuzzy Tychonoff theorem, J. Math. Anal. Appl. **43** (1973), 734-742.

[25] J. Goguen, Fuzzy sets and the social nature of truth, Advances in Fuzzy Set Theory and Applications **1979**.

[26] S. Gottwald, Set theory for fuzzy sets of higher level, Fuzzy Sets and Systems **2** (1979), 125-151.

[27] J. Gutiérrez García, M. A. de Prada Vicente & A. Šostak, Even Fuzzy Topologies and Related Structures, To appear.

[28] J. Gutiérrez, M. A. de Prada Vicente, Super Uniform Spaces, To appear.

[29] J. Gutiérrez García, M. A. De Prada Vicente & M. H. Burton, Embeddings into the Category of Super Uniform Spaces, To appear.

[30] H. Herrlich and G. E. Strecker, *Category Theory*, (1979), Heldermann Verlag, Berlin.

[31] U. Hohle, Probabalistic metrization of fuzzy uniformities, Fuzzy Sets and Systems **8** (1982), 63-69.

[32] U. Hohle, L. N. Stout, Foundations of fuzzy sets, Fuzzy Sets and Systems **40** (1991), 257-296.

[33] Hu Cheng-ming, Theory of convergence in fuzzy topological spaces, J. Fuzzy Math. **1(1)** (1993), 1-14.

[34] B. Hutton, Normality in fuzzy topological spaces, J. Math. Anal. Appl. **50** (1975), 74-79.

[35] B. Hutton, Uniformities on fuzzy topological spaces, J. Math. Anal Appl. **58** (1977), 559-571.

[36] B.Hutton, Products of fuzzy topological spaces, Topology & Appl. **11** (1980), 59-67.

[37] I. M. James, *Topological and uniform spaces*, 1987, Springer-Verlag, Berlin-New York.

[38] A. Kandil, G. Elsalamony and N. N. Morsi, On two theories of fuzzy syntopogeny for the framework FNS, Fuzzy Sets and Systems **57**, (1993), 227-238.

[39] A. Kandil, K. A. Hashem, N. N. Morsi , A Level-Topologies Criterion for Lowen Fuzzy Uniformizability, Fuzzy Sets and Systems **62** (1994), 211-226.

[40] A. K. Katsaras, Convergence of fuzzy filters in fuzzy topological spaces, Bull Math Roumanie, **27(75)** nr2 (1983), 131-137.

[41] A. K Katsaras, On fuzzy uniform spaces, J. Math. Anal.Appl. **101** (1984), 97-113.

[42] A. K. Katsaras, Fuzzy syntopogenous structures compatible with Lowen fuzzy uniformities and Artico-Moresco fuzzy proximities, Fuzzy Sets and Systems **36** (1990), 375-393.

[43] W. Kotzé, Quasi-coincidence and quasi-fuzzy Hausdorff, J. Math. Anal. Appl. **116(2)** (1986), 465-472.

[44] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. **56** (1976), 621-633.

[45] R.Lowen, Initial and final fuzzy topologies and the fuzzy Tychonoff theorem, J. Math. Anal. Appl. **58(1)** (1977), 11-21.

[46] R. Lowen, On fuzzy complements, Information Science, **14** (1978), 107-113.

[47] R.Lowen, A comparison of different compactness notions in fuzzy topological spaces, J. Math. Anal. Appl. **64(2)** (1978), 47-454.

[48] R. Lowen, Convergence in Fuzzy Topological Spaces, General Topology and its Applications **10** (1979), 147-160.

[49] R. Lowen, Convex fuzzy sets, Fuzzy Sets and Systems **3** (1980), 291-310.

[50] R. Lowen, Fuzzy Neighbourhood Spaces, Fuzzy Sets and Systems **7** (1980), 165-189.

[51] R. Lowen, Compact Hausdorff fuzzy topological spaces are topological, Topology & Appl. **12** (1981), 65-74.

[52] R. Lowen, Fuzzy Uniform Spaces, J. Math. Anal. Appl. **82** (1981), 370-385.

[53] R. Lowen and P. Wuyts, Completeness, Compactness and Precompactness in Fuzzy Uniform Spaces: Part 1, J. Math. Anal. Appl. **90(2)** (1982), 563-581.

[54] R.Lowen, Compactness notions in fuzzy neighbourhood spaces, Manuscripta Math. **38** (1982), 265-287.

[55] R. Lowen and P. Wuyts, Completeness, Compactness and Precompactness in Fuzzy Uniform Spaces: Part 2, J. Math. Anal. Appl. **92(2)** (1983), 342-371.

[56] R. Lowen, The relation between filter and net convergence in fuzzy topological spaces, Fuzzy Math. **12** (1983), 41-52.

[57] R. Lowen, On the existence of natural fuzzy topologies on spaces of probability measures, Math. Nachr. **115** (1984), 3357.

[58] R. Lowen, On the relation between fuzzy equalities pseudometrics and uniformities, Quaestiones Math. **7** (1984), 407-419.

[59] R. Lowen, On the fundamentals of fuzzy sets, Stochastica Ser2, (1984), 157-169.

[60] R. Lowen, Mathematics and fuzziness, some personal reflections, Information Sciences **36** (1985), 17-27.

[61] E. Lowen and R. Lowen, Characterisation of convergence in fuzzy topological spaces, Internat. J. Math. & Math. Sci. **8(3)** (1985), 497-511.

[62] R. Lowen, On the Existence of Natural Non-topological, Fuzzy Topological Spaces, (1985), Heldermann Verlag, Berlin.

[63] R. Lowen, P. Wuyts, and E. Lowen, On the reflectiveness and coreflectiveness of subcategories of FTS. Preprint (1986).

[64] E. Lowen and R. Lowen, The role of the diagonal oprtator in fuzzy topology, Fuzzy Sets and Systems **27** (1988), 53-61.

[65] R. Lowen and P. Wuyts, Concerning the constants in fuzzy topology, J. Math. Anal. Appl. **129** (1988), 256-268.

[66] R. Lowen, The shortest note on the smallest coreflective subcategory of TOP, Quaestiones Math. **12** (1989), 475-477.

[67] M. Macho Stadler, M. A. de Prada Vicente, Fuzzy Convergence Versus Convergence. Bull. Cal. Math. Soc. **85**(1993), 437-444.

[68] A. S. Mashour, N. N. Morsi and O. A. El-Tantaway, Fuzzy neighbourhood syntopogenous structures, Fuzzy Sets and Systems **43** (1991), 219-234.

[69] A. S. Mashour and N. N. Morsi, On regularity axioms in fuzzy neighbourhood spaces, Fuzzy Sets and Systems **44** (1991), 265-271.

[70] A. S. Mashour and N. N. Morsi, Fuzzy metric neighbourhood spaces, Fuzzy Sets and Systems **45** (1992), 367-388.

[71] K. Menger, Statistical metric spaces. Proc. Nat. Acad. Sci. USA **28** (1942), 535-537.

[72] N. N. Morsi, Dual fuzzy neighbourhood space I, Fuzzy Sets and Systems **44** (1991), 245-263.

[73] N. N. Morsi, Dual fuzzy neighbourhood spaces II, J. Fuzzy Math. **3(1)** (1995), 29-67.

[74] Pu Pan-Ming and Liu Ying-Ming, Fuzzy Topology.1.Neighborhood Structure of a Fuzzy Point and Moore-Smith Convergence, J. Math. Anal. Appl. **76** (1980), 571-599.

[75] R. D. Sarma and N. Ajmal, A new approach to fuzzy toplogical spaces and various fuzzy topological notions defined by nets, Fuzzy Ssts and Systems **73** (1995), 389-394.

[76] A. Sŏstak A, Basic structures of fuzzy topology, Preprint (1993) in Grahamstown.

[77] R. H. Warren, Neighbourhoods, bases and continuity in fuzzy topological spaces, Rocky Mountain J. Math. **8(3)** (1978), 459-470.

[78] R. H. Warren, Fuzzy topologies characterised by neighbourhood systems, Rocky Mountain J. Math. **9(4)** (1979), 761-764.

[79] R. H. Warren, Convergence in fuzzy topology, Rocky Mountain J. Math. **13(1)** (1983), 31-36.

[80] C. K. Wong, Fuzzy topology: product and quotient theorems, J. Math. Anal. Appl. **45** (1974), 512-521.

[81] S. Willard, *General Topology*, 1970, Addison-Wesley, Reading-Massachusetts.

[82] P. Wuyts, On the Determination of Fuzzy Topological Spaces and Fuzzy Neighbourhood Spaces by their Level-Topologies, Fuzzy Sets and Systems **12** (1984), 71-85.

[83] P. Wuyts, On Level-Topologies and Maximality of Fuzzy Topological Spaces, Fuzzy Sets and Systems **79**(1996), 337-339.

[84] L. A. Zadeh, Fuzzy Sets, Information and Control **8**(1965), 338-353.

[85] L. A. Zadeh, Fuzzy logic and approximate reasoning, 407-428.