THE ABSOLUTE ORDERS ON THE COXETER GROUPS A_n AND B_n ARE SPERNER

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Abstract. Over 50 years ago, Rota posted the following celebrated "Research Problem": prove or disprove that the partial order of partitions on an n-set (i.e., the refinement order) is Sperner for all n. A counterexample was eventually discovered by Canfield in 1978. However, Harper and Kim recently proved that a closely related order — i.e., the refinement order on the symmetric group — is not only Sperner, but strong Sperner. Equivalently, the well-known absolute order on the symmetric group is strong Sperner. In this paper, we extend these results by giving a concise, elegant proof that the absolute orders on the Coxeter groups A_n and B_n are strong Sperner.

1. Introduction

In 1928, Sperner [8] proved that the poset of subsets of $[n] = \{1, 2, \ldots, n\}$ has the property that none of its antichains (i.e., a collection of pairwise incomparable vertices in the poset) has cardinality larger than the largest rank. In 1967, Rota [7] famously conjectured that the refinement order Π_n (i.e., the poset of partitions of $[n]$) has this same property (which became known as the *Sperner property*) for all n. In 1978, Canfield [2] discovered a counterexample to Rota's conjecture for n larger than Avogadro's number. Although the refinement order Π_n is not Sperner for n sufficiently large, there is a closely related poset on the symmetric group S_n (also called the refinement order) which Harper and Kim [5] recently proved is not only Sperner for all n, but strong Sperner. The refinement order on S_n is anti-isomorphic to a well-known (see, e.g., [1]) order on S_n called the *absolute order*; i.e., $x \leq y$ in the refinement order if and only if $y \leq x$ in the absolute order. Hence an immediate corollary to [5] is that the absolute orders S_n are strong Sperner.

The main result in this paper is Theorem 5.1, which states that the absolute orders on the Coxeter groups A_n and B_n are strong Sperner. The key to the proof lies in showing that each of these absolute orders contain a product of "claws" as a spanning subposet, which is strong Sperner by Harper's Product Theorem [3].

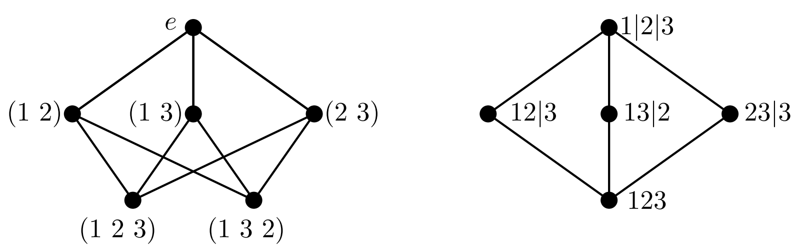

Figure 1. The refinement orders on S_3 and Π_3 respectively.

2. The regular n-simplex and n-cube and their symmetries

A partial order, called an *absolute order*, can be defined on the symmetry group of a regular polytope. The absolute orders of interest in this paper are associated to the n-simplex and n-cube. We recall some basic facts about these polytopes and their symmetries. The regular n-simplex Δ_n is the convex hull of the standard basis $\{e_1, e_2, \ldots, e_{n+1}\}$ for $\mathbb{R}^{n+1}$. Each i-dimensional face (or *i-face*) of Δ_n corresponds with a subset of $[n+1] = \{1, 2, \ldots, n+1\}$ of size $i+1$. Hence the vertices are singletons and the facets (i.e., the $(n-1)$-faces) are n-sets. The symmetry group A_n of Δ_n is the group of permutations of $[n+1]$ (i.e., the symmetric group S_{n+1}). The set of reflections in A_n consists of all transpositions $(i\ j)$, $i \neq j$.

The n-cube $\square_n$ is the convex hull in $\mathbb{R}^n$ of the Cartesian product $\{-1, 1\}^n \subset \mathbb{R}^n$. The dual polytope to the n-cube is the n-cross-polytope $\diamondsuit_n$, which is the convex hull of $\{\pm e_1, \pm e_2, \ldots, \pm e_n\} \subset \mathbb{R}^n$. Each i-face of $\diamondsuit_n$ corresponds to a subset $S \subset \{\pm j\}_{j=1}^n$ of size $i+1$ with the property that $k \in S$ implies $-k \notin S$. The symmetry group B_n for each of the dual polytopes $\square_n$ and $\diamondsuit_n$ is the group of *signed permutations*; i.e., the permutations w of the set $\{\pm j\}_{j=1}^n$ with the property that $w(-i) = -w(i)$ for all i. Following [6], we denote the signed permutation with cycle form $(a_1\ a_2\ \cdots\ a_k)(-a_1\ -a_2\ \cdots\ -a_k)$ by $(\!(a_1, a_2, \ldots, a_k)\!)$, and $(a_1\ a_2\ \cdots\ a_k\ -a_1\ -a_2\ \cdots\ -a_k)$ by $[a_1, a_2, \ldots, a_k]$. The set of reflections in B_n corresponds to the union of $\{[i]\}_{i=1}^n$ and $\{(\!(i,j)\!), (\!(i,-j)\!)\}_{1 \leq i < j \leq n}$.

Lemma 2.1. *For any pair (C, C') of distinct facets in Δ_n (resp. $\square_n$), there is a unique reflection in A_n (resp. B_n) mapping C to C'.*

Proof. Let $C \neq C'$ be facets in Δ_n. Since $C \neq C'$ correspond to subsets of $[n+1]$ of size n, it follows that $C - C' = \{i\}$ and $C' - C = \{j\}$ for some $i \neq j$. The unique reflection mapping C to C' is $(i\ j)$.

Now let $C \neq C'$ be facets in $\square_n$. The facets of $\square_n$ correspond to the vertices of $\diamondsuit_n$, which in turn correspond to elements of $\{\pm j\}_{j=1}^n$. Suppose without loss of generality that C corresponds to 1. Either C' corresponds to -1, j for some $j \neq 1$, or $-j$ for some $j \neq 1$. In any case, there is a unique reflection in B_n mapping C to C' (specifically, the reflections $[1]$, $(\!(1,j)\!)$, and $(\!(1,-j)\!)$, respectively). $\square$

Define a *(complete) flag* $\mathscr{F} = (\mathcal{P}_i)_{i=0}^n$ in an n-dimensional regular polytope $\mathcal{P}$ to be a sequence of faces in $\mathcal{P}$, ordered by containment, with $\dim(\mathcal{P}_i) = i$. The action of A_n (resp. B_n) on Δ_n (resp. $\square_n$) induces a simply transitive action on the associated set of flags. Hence if we designate some flag in Δ_n or $\square_n$ — call it the *standard flag* $\mathscr{F}^{\text{std}} = (\mathcal{P}_i^{\text{std}})_{i=0}^n$ — then a correspondence between elements in the polytope's symmetry group and its set of flags can be defined via $w \mapsto w \cdot \mathscr{F}^{\text{std}}$. Note that, for all $i \in [0, n]$, the i-faces for the n-simplex (resp. the n-cube) are i-simplices (resp. i-cubes).

3. Posets, the Sperner property, and the absolute orders

Let P be a (finite graded) poset with rank decomposition $P = \bigsqcup_{i=0}^r P_i$. A *$k$-family* in P is a subset of P containing no chain of size $k+1$. The poset P is defined to be *k-Sperner* if the union of the k largest rank levels P_i is a k-family of maximal size; *strong Sperner* if P is k-Sperner for all $k \in [1, r+1]$; and *rank unimodal* if $|P_0| \leq |P_1| \leq \cdots \leq |P_{j-1}| \leq |P_j| \geq |P_{j+1}| \geq \cdots \geq |P_r|$ for some j. Note that the

1-Sperner property is otherwise known as the Sperner property, and a 1-family is otherwise known as an antichain.

Lemma 3.1. *Suppose that P is a spanning subposet of P'; i.e., suppose P has the same vertex set and rank function as P'. If P is rank unimodal and strong Sperner, then so is P'.*

Proof. Since P is rank unimodal, its largest k rank levels can be chosen so that their ranks are consecutive. Their union is a k-family in both P and P'. Since P is k-Sperner, this union is a k-family in P of maximal size, and therefore a k-family in P' of maximal size. □

Define a *k-claw* $C_k = \bigsqcup_{l=0}^{1}(C_k)_l$ to be the graded poset with $|(C_k)_0| = 1$, $|C_k| = k-1$, and whose underlying graph is complete bipartite. It is not the case that a product of Sperner (or even strong Sperner) posets is necessarily Sperner. However, there is a strengthening of the strong Sperner property called the *normalized flow property* (abbreviated NFP) which is well-behaved under taking products by Harper's Product Theorem [3].

Lemma 3.2. *Let $\{k_i\}_{i=1}^n \subset \mathbb{Z}_+$. The product poset $\prod_{i=1}^n C_{k_i}$ is strong Sperner.*

Proof. Any k-claw C_k has the NFP by [4, note on p. 162]. If the capacity of each vertex in each of the claws C_{k_i} and C_{k_j} is defined to be 1, then it is clear that C_{k_i} and C_{k_j} satisfy the hypotheses of Harper's Product Theorem [3]. Thus $C_{k_i} \times C_{k_j}$ has NFP. By induction, $\prod_{i=1}^n C_{k_i}$ has the NFP, and is therefore strong Sperner. □

We briefly recall some generalities about absolute orders; see, e.g., [1] for details. Let W be a finite Coxeter group with set of reflections T. The *absolute length* l_T on W is the word length with respect to T. The *absolute order* on W is defined by

$$\pi \leq \mu \text{ if and only if } l_T(\mu) = l_T(\pi) + l_T(\pi^{-1}\mu)$$

for all $\pi, \mu \in W$. Equivalently, the absolute order is the partial order on W generated by the covering relations $w \to tw$, where $w \in W$, $t \in T$, and $l_T(w) < l_T(tw)$. This order is graded with rank function l_T. The absolute length generating function $\mathrm{P}_W(q) = \sum_{w \in W} q^{l_T(w)}$ satifies $\mathrm{P}_W(q) = \prod_{i=1}^n (1 + (d_i - 1)q)$, where $(d_i)_{i=1}^n$ is the degree sequence for W (and $n = \mathrm{rank}(W)$) [1, p. 35]. It follows that $|T| = |l_T^{-1}(1)| = \sum_{i=1}^n (d_i - 1)$. Moreover, the rank sequence $\left(|l_T^{-1}(i)|\right)_{i=0}^n$ for any absolute order is strictly log-concave by [9, Theorem 4.5.2], and thus all of the absolute orders are rank unimodal.

4. Factoring elements of A_n and B_n

In order to show that the absolute orders A_n and B_n contain a product of claws as a spanning subposet, we first prove that any element of A_n or B_n can be factored with respect to symmetries of a flag in the associated regular polytope. For all that follows, $\mathcal{P}$ denotes the regular n-simplex or n-cube, and W denotes the corresponding symmetry group. Note that if $\mathcal{P}$ equals Δ_n or $\square_n$, each reflective symmetry of an i-face $\mathcal{P}_i$ of $\mathcal{P}$ uniquely extends to a reflective symmetry of $\mathcal{P}$. Define $T_{\mathcal{P}_i}$ to be the embedding of the set of reflections of $\mathcal{P}_i$ into W.

Lemma 4.1. *Let $\mathcal{P}$ be the n-simplex or n-cube, and let W be the corresponding group of symmetries with degree sequence $(d_i)_{i=1}^n$. Fix a standard flag $(\mathcal{P}_i^{std})_{i=0}^n$ in $\mathcal{P}$, and set $T_i = T_{\mathcal{P}_i^{std}}$. It follows that, for all $i \in [1, n]$, $|T_i - T_{i-1}| = d_i - 1$.*

Proof. The n-simplex (resp. the n-cube) has the property that, for each i, each of its i-faces is an i-simplex (resp. i-cube). Hence the symmetry group for any of its i-faces is A_i (resp. B_i). If the degree sequence for the n-simplex (resp. n-cube) is $(d_j)_{j=1}^n$, then the degree sequence associated to an i-face is $(d_j)_{j=1}^i$. It follows that $|T_i - T_{i-1}| = |T_i| - |T_{i-1}| = \sum_{j=1}^i (d_j - 1) - \sum_{j=1}^{i-1}(d_j - 1) = d_i - 1$. □

It is easily verified that the relation between a regular polytope and the degree sequence of its symmetry group described in Lemma 4.1 is satisfied by *precisely* the n-simplices, n-cubes, and m-gons (and none of the other regular polytopes). For ease of reference, we note here that the degree sequence $(d_i)_{i=1}^n$ for A_n is defined by $d_i = i+1$, for B_n by $d_i = 2i$, and for $I_2(m)$ by $d_1 = 2$ and $d_2 = m$.

Proposition 4.2. *Let $\mathcal{P}$ be the n-simplex or n-cube, and let W be the associated symmetry group. Fix a standard flag $\mathscr{F}^{std} = (\mathcal{P}_i^{std})_{i=0}^n$ in $\mathcal{P}$, and set $T_i = T_{\mathcal{P}_i^{std}}$.*

(1) *Any element $w \in W$ has a unique factorization of the form*
$$w = r_n r_{n-1} \cdots r_2 r_1$$
with $r_i \in (T_i - T_{i-1}) \sqcup \{e\}$ for each i, where e is the identity in W.

(2) *Given such a factorization, the length can be computed via*
$$l_T\left(\prod_{i=0}^{n-1} r_{n-i}\right) = |\{i : r_i \neq e\}|.$$

(3) *Finally, $\prod_{i=0}^{n-1} r_{n-i}$ covers $\prod_{i=0}^{n-1} r'_{n-i}$ if there exists k such that $r_k \neq r'_k = e$ and $r_j = r'_j$ for all $j \neq k$.*

Proof. We begin by proving (1). The claim is clearly true for $n = 1$. Now let $n > 1$ be arbitrary, and suppose the claim is true for $n-1$. Let $w \in W$, with corresponding flag $\mathscr{F} = (\mathcal{P}_i)_{i=0}^n$. If $(\mathcal{P}_i)_{i=0}^{n-1}$ is a flag in the "standard facet" $\mathcal{P}_{n-1}^{\text{std}}$, then the claim follows by the inductive hypothesis. Suppose instead that $(\mathcal{P}_i)_{i=0}^{n-1}$ is a flag in some other facet C. Lemma 2.1 implies that there is a unique reflection $r_n \in (T_n - T_{n-1}) - \{e\}$ mapping C to $\mathcal{P}_{n-1}^{\text{std}}$. By the inductive hypothesis, it follows that $r_n \cdot \mathscr{F} = (r_{n-1} \cdots r_2 r_1) \cdot \mathscr{F}^{\text{std}}$ with $r_i \in (T_i - T_{i-1}) \sqcup \{e\}$ for all $i \in [1, n-1]$. Therefore $w \cdot \mathscr{F}^{\text{std}} = \mathscr{F} = r_n(r_{n-1} \cdots r_2 r_1) \cdot \mathscr{F}^{\text{std}}$, and the claim follows.

To prove (2), we first let $\mathcal{P}$ be the n-simplex and let W be its symmetry group. Assume without loss of generality that $\mathscr{F}^{\text{std}} = ([i+1])_{i=0}^n$. Then $T_i - T_{i-1}$ consists of all transpositions $(j\ (i+1))$ with $j \in [1, i]$. If w is a product of elements in $T_{i-1} \sqcup \{e\}$, then w is a permutation of $[i]$. Hence $l_T(r_i w) > l_T(w)$, which implies that $l_T(r_i w) = l_T(w) + 1$. The claim follows from a straight-forward induction on n. Now let $\mathcal{P}$ be the n-cube and W its symmetry group. Assume without loss of generality $\mathscr{F}^{\text{std}} = (\mathcal{P}_i^{\text{std}})_{i=0}^n$ is chosen so that the symmetries of $\mathcal{P}_i^{\text{std}}$ correspond to symmetries of $\{\pm 1, \pm 2, \ldots, \pm i\}$. Then T_i consists of all reflections of the form $[j]$, $(\!(j,k)\!)$, and $(\!(-j,k)\!)$ with $j, k \in [1, i]$ and $j \neq k$, and $T_i - T_{i-1}$ consists of all reflections of the form $[i]$, $(\!(j,i)\!)$, and $(\!(-j,i)\!)$ with $j \in [1, i-1]$. Similar to the case above, the product $r_i w$ of r_i in $T_i - T_{i-1}$ with a product w of reflections in T_{i-1} has $l_T(r_i w) > l_T(w)$. Hence $l_T(r_i w) = l_T(w) + 1$, and the claim follows by induction.

Finally, to prove (3), let w and w' be elements of W with the property that their expansions $w = \prod_{i=0}^{n-1} r_i$ and $w' = \prod_{i=0}^{n-1} r'_i$ satisfy $r_k \neq r'_k = e$ for some k and $r_j = r'_j$ for all $j \neq k$. By Proposition 4.2.2, it follows that $l_T(w') + 1 = l_T(w)$. Set $\sigma = \prod_{i=0}^{k-1} r_i$ and $\tau = \prod_{i=k+1}^{n-1} r_i$, so that $w = \sigma r_k \tau$ and $w' = \sigma\tau$. Then

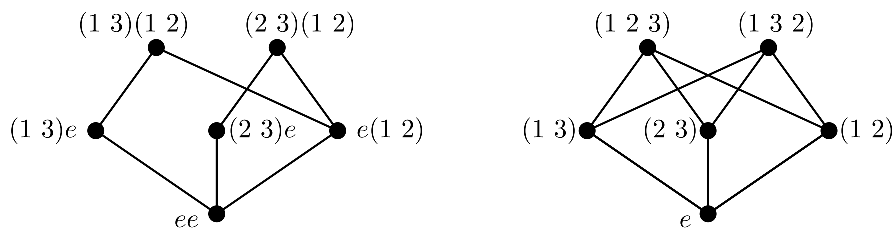

FIGURE 2. The product of claws $C_3 \times C_2$ (left) can be viewed as a spanning subposet of the absolute order on A_3 (right).

$l_T((w')^{-1}w) = l_T(\tau^{-1}\sigma^{-1}\sigma r_k \tau) = l_T(\tau^{-1} r_k \tau) = 1$. Since $l_T(w') + l_T((w')^{-1}w) = l_T(w)$, it follows that w covers w'. □

5. Main result

Theorem 5.1. *The absolute orders on A_n and B_n are strong Sperner.*

Proof. Let $\mathcal{P}$ be the n-simplex or n-cube, and let W be the associated symmetry group. Fix a standard flag $\mathscr{F}^{\mathrm{std}} = (\mathcal{P}_i^{\mathrm{std}})_{i=0}^n$ in $\mathcal{P}$, and set $T_i = T_{\mathcal{P}_i^{\mathrm{std}}}$. Let $(d_i)_{i=1}^n$ be the degree sequence for W. Consider the product poset

$$\prod_{i=0}^{n-1} C_{d_{n-i}} = C_{d_n} \times \cdots \times C_{d_2} \times C_{d_1}$$

of claws C_{d_i}. For each i, define a bijective correspondence between the vertices of the claw C_{d_i} and the elements of $(T_i - T_{i-1}) \sqcup \{e\}$ by mapping the $d_i - 1$ vertices in $(C_{d_i})_1$ bijectively onto $T_i - T_{i-1}$ (such a bijection exists by Lemma 4.1) and the rank 0 vertex in C_{d_i} to e. These bijective correspondences between claws and sets of reflections induce a bijective correspondence $\phi(r_n, \ldots, r_2, r_1) = r_n \cdots r_2 r_1$ between the vertices of the product poset $\prod_{i=0}^{n-1} C_{d_{n-i}}$ and the vertices of the absolute order W by Proposition 4.2(1).

We claim that $\prod_{i=0}^{n-1} C_{d_{n-i}}$ can be viewed as a spanning subposet of W via the above bijection between of the vertex sets. It suffices to prove that if y covers x in $\prod_{i=0}^{n-1} C_{d_{n-i}}$, then $\phi(y)$ covers $\phi(x)$ in W. Suppose that $(r_n, \ldots, r_2, r_1)$ covers $(r'_n, \ldots, r'_2, r'_1)$ in the product of claws. Then there exists k for which $r_k \neq r'_k = e$ and $r_j = r'_j$ for all $j \neq k$. By Proposition 4.2(3), the claim immediately follows. By Lemma 3.2, $\prod_{i=0}^{n-1} C_{d_{n-i}}$ is strong Sperner. Since $\prod_{i=0}^{n-1} C_{d_{n-i}}$ is a spanning subposet of W, it follows by Lemma 3.1 that W is strong Sperner. □

Remark 5.2. It is straight-forward to verify that Lemma 2.1, Lemma 4.1, and Proposition 4.2 extend to the regular n-gons. Moreover, Theorem 5.1 extends to the dihedral groups $I_2(m)$ for all m; i.e., the absolute order $I_2(m)$ contains $C_m \times C_2$ as a spanning subposet. Therefore, the dihedral groups are strong Sperner.

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