MODULI SPACES OF MEROMORPHIC GSp_{2n}-CONNECTIONS

RESEARCH STATEMENT — *NEAL LIVESAY*

1. Introduction

1.1. **Motivation.** Representation theory is a branch of mathematics that involves studying abstract algebraic structures by representing their elements as matrices. For example, a representation of a group is a concrete realization of the elements of the group as invertible matrices, with the group operation corresponding to matrix multiplication. Representation theory has a pervasive influence throughout mathematics. It also plays an important role in physics, chemistry, and other sciences as it provides a precise language to study the effects of symmetry in a physical system.

My research is based on the application of representation theory to the study of differential equations (or DE's). A fundamental problem in the theory of DE's is the classification of first-order singular linear differential operators up to certain "symmetries" described by a group. For example, consider a first-order system of linear DE's defined in some connected open set U in the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. It is well-known that a set of linearly independent solutions to the DE in U can be analytically continued along loops in $\mathbb{P}^1$ to get a new set of linearly independent solutions. These new solutions are potentially distinct when the loop is not contractible; i.e., the loop "runs around a singularity" $y \in \mathbb{P}^1$. The datum encoding this change is known as the monodromy group M^y (in Greek, *mono* = "single" and *dromos* = "running"). Hence, there is a map from singular first-order systems of DE's with singularities $\{y_i\}_{i=1}^k \subset \mathbb{P}^1$ to sets of monodromy groups $\{M^{y_i}\}_{i=1}^k$.

Consider an inverse problem: when can a given set of groups $\{M^{y_i}\}_{i=1}^k$ be realized as the set of monodromy groups for a differential equation? This is (roughly) the Riemann–Hilbert Problem.

$$\underline{\text{Global object}} \qquad\qquad \underline{\text{Local data}}$$

$$\left\{ \begin{array}{c} \text{systems of} \\ \text{DE's on } \mathbb{P}^1 \\ \text{with singularities} \\ \{y_1, \dots, y_k\} \end{array} \right\} \underset{\text{inverse problem}}{\overset{\text{monodromy}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{sets of groups} \\ \{M^{y_1}, \dots, M^{y_k}\} \end{array} \right\}$$

A modern, algebro-geometric variant of this problem involves the study of meromorphic G-connections (or, equivalently, flat G-bundles, for G a complex reductive algebraic group) on $\mathbb{P}^1$ with specified local isomorphism classes. Much research has concentrated on GL_n-connections with regular singularities; for example, Deligne [10] proved a Riemann–Hilbert correspondence in this case. Meromorphic G-connections with *irregular* singularities are less understood, but have been studied extensively in recent years due to their role in a collection of influential conjectures known as the geometric Langlands program (see, e.g., §1 in [6] for details). My research focuses on the construction of *moduli spaces* (i.e., geometric objects encoding a classification) of meromorphic GSp_{2n}-connections on $\mathbb{P}^1$ with irregular singularities and specified local isomorphism classes.

1.2. **Outline of statement.** My main contributions are as follows:

- I make concrete the abstract theory of formal G-connections [4] for the case that G is the general symplectic group GSp_{2n}. I conjecture that much of this concrete theory should translate to other classical groups.

- I construct symplectic moduli spaces of both framed and framable GSp_{2n}-connections with specified formal isomorphism classes, and I construct Poisson moduli spaces of GSp_{2n}-connections with specified fixed combinatorics.

The outline of this research statement is as follows. In Section 2, I give some background on the classical *leading term* analysis of formal connections, and I discuss limitations of this classical theory. In Section 3, I describe how these limitations are overcome by a more general, Lie-theoretic analysis of *regular strata* [4]. Moreover, I discuss my concrete realization of this theory for formal GSp_{2n}-connections. Section 4 contains statements of my **main results** — the explicit constructions of moduli spaces of meromorphic GSp_{2n}-connections on $\mathbb{P}^1$ with specified local isomorphism classes — generalizing the construction in [6] for GL_n-connections. To conclude, a list of potential future projects is given in Section 5, followed by a list of citations.

2. Background: Meromorphic connections and their localizations

2.1. **Global objects: meromorphic connections.** Let $\mathcal{O}$ be the structure sheaf on $\mathbb{P}^1$, and let $\mathcal{K}$ be its function field (i.e., meromorphic functions). A **meromorphic** (GL_n)**-connection** (V, ∇) **on** $\mathbb{P}^1$ is a rank n trivializable vector bundle V equipped with a $\mathbb{C}$-derivation $\nabla : V \to V \otimes_{\mathcal{O}} \Omega^1_{\mathcal{K}/\mathbb{C}}$. After fixing a global trivialization $\phi : V \xrightarrow{\sim} V^{\mathrm{triv}}$, a connection can be expressed in matrix form $\nabla = d + [\nabla]_\phi$, where $[\nabla]_\phi \in \mathfrak{gl}_n(\Omega^1_{\mathcal{K}/\mathbb{C}})$ is the **connection matrix of ∇ with respect to ϕ.** This is analogous to expressing a linear map as a matrix after fixing an ordered basis. Alternatively, a connection may be expressed in terms of an ordinary $\mathcal{K}$-entried matrix by contracting with the Euler vector field $\tau = z\frac{d}{dz}$; i.e., by taking $\nabla_\tau := \iota_\tau(\nabla)$. The resulting contracted matrix form is

$$\nabla_\tau = \tau + [\nabla_\tau]_\phi$$

with $[\nabla_\tau]_\phi \in \mathfrak{gl}_n(\mathcal{K})$. It is easily seen that a horizontal section of ∇_τ corresponds to a solution of a first-order system of linear algebraic differential equations, symbolically denoted as:

$$\begin{array}{ccc} \{\text{Meromorphic connections}\} & \longleftrightarrow & \{\text{1st-order system of linear DE's}\} \\ \text{horizontal section } \nabla_{z\frac{d}{dz}}(v) = 0 & \longleftrightarrow & \text{solution } z\dfrac{d}{dz}(v) = -[\nabla_\tau]_\phi(v) \end{array}$$

More generally, a **meromorphic G-connection,** for G a reductive group, is a flat structure ∇ on a principal G-bundle (see §2.4 in [3] for more details). In this case, the connection matrices are elements of $\mathfrak{g}(\mathcal{K})$.

2.2. **Local objects: formal connections.** A global meromorphic G-connection induces a formal connection at each singularity $y_i \in \mathbb{P}^1$ by taking Laurent series expansions. Let $\mathfrak{o} := \mathbb{C}[\![z]\!]$ be the ring of formal power series and let $F := \mathbb{C}(\!(z)\!)$ be the fraction field of Laurent series. A **formal** GL_n**-connection** $(V, \widehat{\nabla})$ is an n-dimensional F-vector space V equipped with a $\mathbb{C}$-derivation $\widehat{\nabla} : V \to V \otimes_F \Omega^1_{F/\mathbb{C}}$. Similar to the global case, contracting with $\tau = z\frac{d}{dz}$ and fixing a local trivialization $\phi : V \xrightarrow{\sim} F^n$ produces a local matrix form

$$\widehat{\nabla}_\tau = \tau + [\widehat{\nabla}_\tau]_\phi$$

with $[\widehat{\nabla}_\tau]_\phi \in \mathfrak{gl}_n(F)$. More generally, the localizations of meromorphic G-connections, known as **formal G-connections,** have connection matrices in $\mathfrak{g}(F)$.

It is well-known that there is a simply transitive action of $\mathrm{GL}_n(\mathbb{C})$ on the space of ordered bases for the vector space $\mathbb{C}^n$, and that this action corresponds to the conjugation action on matrices. Analogously, there is a simply transitive action of $\mathrm{G}(F)$ on the space of trivializations of V. The corresponding action of $\mathrm{G}(F)$ on connection matrices is the **local gauge change action** $g \cdot [\widehat{\nabla}_\tau]_\phi := [\widehat{\nabla}_\tau]_{g\cdot\phi}$, given by the formula

$$g \cdot [\widehat{\nabla}_\tau]_\phi = \mathrm{Ad}(g)([\widehat{\nabla}_\tau]_\phi) - \tau(g)g^{-1}.$$

2.3. **Formal types.** For classification theorems involving spaces of linear maps, it is often desirable to have explicit normal forms for similarity classes (e.g., the Jordan canonical form). In [2], Boalch constructed moduli spaces of meromorphic GL_n-connections with formal gauge classes having diagonal normal forms referred to as *formal types*. The existence of a diagonal formal type for a formal gauge class is determined by an analysis of *leading terms*.

Suppose a connection matrix is expanded with respect to the naïve degree filtration on $\mathfrak{gl}_n(F)$:

$$[\widehat{\nabla}_\tau] = \left(M_{-r}z^{-r} + M_{-r+1}z^{-r+1} + \ \dots\ \right)$$

for $M_i \in \mathfrak{gl}_n(\mathbb{C})$. If the **leading term** M_{-r} is regular semisimple, then there exists a local gauge change $g \in \mathrm{GL}_n(\mathfrak{o})$ that simultaneously diagonalizes terms of all degrees in the connection matrix; i.e.,

$$g \cdot [\widehat{\nabla}_\tau] = \left(D_{-r}z^{-r} + \ \dots\ + D_0\right)$$

for D_i diagonal (see, e.g., [19]). A diagonalized connection matrix of this form is referred to as a **formal type** for $\widehat{\nabla}$, and it is unique up to an action by the affine Weyl group $\widehat{W}$ for GL_n.

2.4. **Limitations of the classical theory.** Many connections of interest do not have regular semisimple leading terms. For example, in [20], Witten considered generalized Airy connections, which have connection matrices

$$\begin{pmatrix} 0 & z^{-(s+1)} \\ z^{-s} & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^{-(s+1)} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z^{-s}\right].$$

When $s = 0$, this is the GL_2-version of the Frenkel–Gross rigid flat G-bundle on $\mathbb{P}^1$ with the roles of 0 and ∞ reversed [11]. For these connections, not only does the leading term fail to be regular semisimple, but it is nilpotent; hence, leading term analysis fails to provide an explicit normal form for this connection. Recently, Bremer and Sage [3, 4, 5, 6] have developed a powerful theory of fundamental strata which generalizes the classical theory of leading terms. Moreover, they extended the notion of formal types to a much larger class of gauge classes, including the gauge classes of generalized Airy connections.

3. Strata

Fundamental strata were originally developed by Bushnell and Frölich [7, 8] to study supercuspidal representations of GL_n over p-adic fields. As mentioned in Section 2.4, Bremer and Sage have pioneered a geometric theory of fundamental strata [3, 4, 5, 6] which generalizes the classical leading term theory for formal GL_n-connections. The key idea underlying this approach is to consider connection matrices in terms of a class of Lie-theoretically defined filtrations on $\mathfrak{gl}_n(F)$, rather than solely considering the naïve degree filtration. This allows for the definition of formal types for more general connections. Moreover, since this approach is purely Lie-theoretic, it can be adapted to study meromorphic G-connections for G a reductive group [4]. One of my main contributions involves the concrete realization of this theory of fundamental strata for GSp_{2n}-connections; see Proposition 1 in Section 3.4.

The general symplectic group $\mathrm{GSp}_{2n}(\mathbb{C})$ is a central extension of $\mathrm{Sp}_{2n}(\mathbb{C})$ consisting of linear transformations of $\mathbb{C}^{2n}$ preserving a symplectic form $\langle , \rangle$ up to an invertible scalar. It is convenient to express vectors in $\mathbb{C}^{2n}$ and elements of GSp_{2n} with respect to the ordered symplectic basis $(e_1, e_2, \dots, e_n, f_n, f_{n-1}, \dots, f_1)$ where $\langle e_i, f_j \rangle = \delta_{i,j}$, so that the standard Borel subalgebra $\mathfrak{b}$ and Borel subgroup B are upper triangular.

3.1. **Moy–Prasad filtrations.** A Bruhat–Tits building is a polysimplicial structure defined for reductive groups over fields with discrete valuations (such as $\mathrm{GSp}_{2n}(F)$). There is a correspondence between facets in the Bruhat–Tits building $\mathscr{B}(\mathrm{G})$ and parahoric subgroups in $\mathrm{G}(F)$ that is analogous to the correspondence between simplices in the spherical building (associated to the complex group $\mathrm{G}(\mathbb{C})$) and parabolic subgroups in $\mathrm{G}(\mathbb{C})$. Given any point x in $\mathscr{B}(\mathrm{G})$, Moy and Prasad [14, 15] have defined a decreasing $\mathbb{R}$-filtration $(\widehat{\mathfrak{g}}_{x,r})_r$ on $\widehat{\mathfrak{g}} := \mathfrak{g}(F)$ with a discrete collection of steps, referred to as the **critical numbers for** x**.** For example, the filtration $(\widehat{\mathfrak{gsp}_2})_{x,r}$, for x the origin in $\mathscr{B}(\mathrm{GSp}_2)$, is the usual degree filtration on $\widehat{\mathfrak{gsp}_2}$ with critical numbers $\mathbb{Z}$ (note that $\mathrm{GSp}_2 \cong \mathrm{GL}_2$). On the other hand, the **Iwahori filtration** $(\mathfrak{i}^r)_r := (\widehat{\mathfrak{gsp}_2})_{x,r}$, for x the barycenter of the fundamental alcove in $\mathscr{B}(\mathrm{GSp}_2)$, has critical numbers $\frac{1}{2}\mathbb{Z}$:

$$\cdots \supsetneq \overset{\mathfrak{i}^{-\frac{3}{2}}}{\begin{pmatrix} z^{-1}\mathfrak{o} & z^{-2}\mathfrak{o} \\ z^{-1}\mathfrak{o} & z^{-1}\mathfrak{o} \end{pmatrix}} \supsetneq \overset{\mathfrak{i}^{-1}}{\begin{pmatrix} z^{-1}\mathfrak{o} & z^{-1}\mathfrak{o} \\ \mathfrak{o} & z^{-1}\mathfrak{o} \end{pmatrix}} \supsetneq \overset{\mathfrak{i}^{-\frac{1}{2}}}{\begin{pmatrix} \mathfrak{o} & z^{-1}\mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{pmatrix}} \supsetneq \overset{\mathfrak{i}^{0}}{\begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & \mathfrak{o} \end{pmatrix}} \supsetneq \overset{\mathfrak{i}^{\frac{1}{2}}}{\begin{pmatrix} z\mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} \end{pmatrix}} \supsetneq \cdots$$

There are corresponding $\mathbb{R}_{\geq 0}$-filtrations $(\widehat{\mathrm{GSp}_{2n}})_{x,r}$ on the parahoric subgroups $(\widehat{\mathrm{GSp}_{2n}})_x$. In particular, $(\widehat{\mathrm{GSp}_{2n}})_{x+}$ is the pro-unipotent radical of $(\widehat{\mathrm{GSp}_{2n}})_x$ (see §2.1 in [4] for general definitions).

3.2. **Fundamental strata.** Given $x \in \mathscr{B}(\mathrm{GSp}_{2n})$ and r a nonnegative integer, a GSp_{2n}**-stratum of depth** r is a triple (x, r, β) with β a functional on the successive quotient $(\widehat{\mathfrak{gsp}_{2n}})_{x,r}/(\widehat{\mathfrak{gsp}_{2n}})_{x,r+}$ (see [3] for a general definition). The stratum is **fundamental** if β satisfies a certain degeneracy condition. A formal GSp_{2n}-connection $\widehat{\nabla}$ **contains** (x, r, β) if $[\widehat{\nabla}_\tau] \in (\widehat{\mathfrak{gsp}_{2n}})_{x,-r}$ and β is induced by $[\widehat{\nabla}_\tau]$. The basic idea is that β roughly plays the role of a nonnilpotent "leading term" of the connection matrix with respect to the Moy–Prasad filtration at x. For example, the generalized Airy connection in Section 2.4 (viewed as a GSp_2-connection) contains the fundamental stratum $(x, 2s+1, \beta)$ with x the barycenter of the fundamental alcove and β the functional induced by the nonnilpotent matrix $\begin{pmatrix} 0 & z^{-s+1} \\ z^{-s} & 0 \end{pmatrix}$.

While it is not the case that every formal GSp_{2n}-connection has a nonnilpotent leading term, it is the case that every GSp_{2n}-connection contains a fundamental stratum (x, r, β). The depth of a fundamental stratum detects the "irregularity" of a singularity; for example, $\widehat{\nabla}$ is irregular singular if and only if $r > 0$, and is regular otherwise.

3.3. **Regular strata and formal types.** The analogues of nonnilpotent leading terms are fundamental strata (as discussed in Section 3.2), and the analogues of regular semisimple leading terms are S**-regular strata.** These are fundamental strata that are centralized in a graded sense by a (possibly nonsplit) maximal torus $S \subset \mathrm{GSp}_{2n}(F)$. The following GSp_{2n}-variant of the classical result in Section 2.3 is a consequence of [4, Theorem 5.1]: If $\widehat{\nabla}$ contains an S-regular stratum (x, r, β), then $[\widehat{\nabla}_\tau]$ is $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-gauge equivalent to a regular semisimple element in $\mathfrak{s}^{-r}/\mathfrak{s}^1$. The functional A corresponding to this "diagonalized" matrix, referred to as an S**-formal type,** is unique up to an action by the relative affine Weyl group $\widehat{W}_S$. Hence the $\widehat{W}_S$-orbit space of S-formal types of depth r is isomorphic to the moduli space for the category $\mathscr{C}(S, r)$ of formal connections containing S-regular strata of depth r.

3.4. **Regular maximal tori and points supporting regular strata.** It is not the case that every torus in $\mathrm{GSp}_{2n}(F)$ centralizes a regular stratum. To elaborate, it is well-known that there is a correspondence between conjugacy classes in the Weyl group W for a reductive group G and conjugacy classes of maximal tori in $\widehat{\mathrm{G}}$ (see, e.g., [12, Lemma 2]). Bremer and Sage [4, Corollary 4.10] proved that S-regular strata exist if and only if S is a **regular maximal torus;** i.e., S corresponds to a regular conjugacy class in W. These regular Weyl group classes were classified by Springer [18].

It is also not the case that every point $x \in \mathscr{B}$ supports an S-regular stratum; for example, there is a certain compatibility necessary between the natural filtration on the Cartan subalgebra $\mathfrak{s}$ and the Moy–Prasad filtration at x. On the other hand, for the computations involved in constructing moduli spaces, it is preferable to choose points x giving a "best possible" filtration to support a given regular stratum. One of my contributions has been the statement of an explicit correspondence between regular maximal tori S and well-behaved points in the Bruhat–Tits building that support S-regular strata, as described in Proposition 1 below.

Proposition 1. *(L., in preparation) There is an explicit correspondence between regular maximal tori S in $\mathrm{GSp}_{2n}(F)$ and certain barycenters of facets in $\mathscr{B}(\mathrm{GSp}_{2n})$ that support S-regular strata. Furthermore, there are explicit normal forms for formal GSp_{2n}-connections containing regular strata that are suitable for the construction of moduli spaces.*

4. Global theory and main results

Boalch [2] constructed moduli spaces of "framable" and "framed" GL_n-connections with a specified set of irregular singularities and corresponding formal isomorphism classes determined by S-formal types $\mathbf{A} = \{A^1, \dots, A^k\}$ for S the split diagonal torus (see §5.1 in [6] for details on framable and framed connections). Bremer and Sage [6] generalized these constructions for GL_n-connections with specified formal types each corresponding to arbitrary regular maximal tori in $\mathrm{GL}_n(F)$. In each of these cases, the moduli spaces are realized as symplectic reductions of products of symplectic manifolds — which Boalch [2] referred to as "extended orbits" — encoding the local data (see §5.1 [6] for more on extended orbits).

I use a similar approach in Theorem 1 (my **first main result**) to construct explicit symplectic moduli spaces of both framable and framed GSp_{2n}-connections with specified formal types $\mathbf{A}$ that correspond to a collection of irregular singularities. I give an explicit construction of the extended orbits as symplectic manifolds in Proposition 2. Finally, I construct explicit Poisson moduli spaces of GSp_{2n}-connections with specified sets of "fixed combinatorics" that correspond to a collection of irregular singularities in Theorem 2 (my **second main result**).

4.1. **Extended orbits.** Roughly, a "framable extended orbit" $\mathscr{M}(A)$ contains information regarding both a formal isomorphism class and framing data. Extended orbits can be realized as symplectic reductions. To elaborate, let $\mathscr{M}(A)$ be the extended orbit of the S-formal type A. By Proposition 1, S corresponds to a point x in the fundamental alcove. Define $\mathscr{O}$ to be the $(\widehat{\mathrm{GSp}_{2n}})_x$-coadjoint orbit of A. Extended orbits can be constructed as explicit symplectic manifolds, as described in my result below.

Proposition 2. *(L., in preparation) The framable extended orbit $\mathscr{M}(A)$ is a symplectic manifold that is isomorphic to $(T^* \mathrm{GSp}_{2n}(\mathfrak{o}) \times \mathscr{O}) /\!\!/_0 (\widehat{\mathrm{GSp}_{2n}})_x$, and has a Hamiltonian action by the global gauge group $\mathrm{GSp}_{2n}(\mathbb{C})$. There is a similar construction for "framed extended orbits" $\widetilde{\mathscr{M}}(A)$.*

4.2. **Moduli spaces of framed and framable connections.** Let $\mathbf{A}$ be a collection of formal types with corresponding extended orbits $\{\mathscr{M}(A^i)\}_{i=1}^k$. There is a Hamiltonian action of the global gauge group $\mathrm{GSp}_{2n}(\mathbb{C})$ on the product $\prod_i \mathscr{M}(A^i)$ given by the diagonal action of $\mathrm{GSp}_{2n}(\mathbb{C})$ on each of the factors. The corresponding moment map $\mu : \prod_i \mathscr{M}_i \to (\mathfrak{gsp}_{2n}(\mathbb{C}))^\vee$ maps an element of the product of extended orbits to the sum of its residue terms. By the Residue Theorem (see, e.g., §II in [17]), the condition required for an element of the product to correspond to a meromorphic connection is satisfied precisely when it maps to 0 through the moment map. This fact allows for the construction of moduli spaces as symplectic reductions, as described in Theorem 1 below (my **first main result**).

Theorem 1. *(L., in preparation)*

(1) *The moduli space $\mathscr{M}^*(\mathbf{A})$ is a symplectic reduction of the product of local pieces:*

$$\mathscr{M}^*(\mathbf{A}) \cong \left(\prod_i \mathscr{M}(A^i) \right) /\!\!/_0 \mathrm{GSp}_{2n}(\mathbb{C}).$$

(2) *The moduli space* $\widetilde{\mathscr{M}}^*(\mathbf{A})$ *of framed* GSp_{2n}*-connections is constructed similarly. It is a smooth manifold. Moreover,* $\mathscr{M}^*(\mathbf{A})$ *is a symplectic reduction of* $\widetilde{\mathscr{M}}^*(\mathbf{A})$ *by a torus action.*

The constructions in (1) and (2) can each be extended to include connections with additional regular singularities.

4.3. **Moduli spaces of connections with fixed combinatorics.** Let $\mathbf{S} = \{S_i\}_{i=1}^k$ be a collection of regular maximal tori and let $\mathbf{r} = \{r_i\}_{i=1}^k$ be a collection of designated depths. Bremer and Sage [5] defined a moduli space $\widetilde{\mathscr{M}}^*(\mathbf{S}, \mathbf{r})$ of meromorphic GL_n-connections with specified fixed combinatorics $(\mathbf{S}, \mathbf{r})$ corresponding to a set of irregular singularities. Furthermore, they have realized this moduli space as an explicit Poisson manifold. This type of construction is of particular interest in the study of *isomonodromic deformations* (see, e.g., [2, 5]). I have further generalized the construction $\widetilde{\mathscr{M}}^*(\mathbf{S}, \mathbf{r})$ for GSp_{2n}-connections, as described in Theorem 2 below (my **second main result**).

Theorem 2. *(L., in preparation) The moduli space* $\widetilde{\mathscr{M}}^*(\mathbf{S}, \mathbf{r})$ *is a Poisson reduction of its corresponding local pieces. The symplectic leaves of this Poisson manifold are the connected components of the framed extended orbits* $\widetilde{\mathscr{M}}(\mathbf{A})$.

5. Future directions

The seminal papers of Boalch, Bremer, and Sage established a foundation for a remarkably rich area of mathematics. Now further work needs to be done to expand the theory from that foundation. A few next-step projects are listed below.

(1) A first project involves the extension of Theorems 1 and 2 to flat G-bundles on $\mathbb{P}^1$ for arbitrary reductive groups G. I anticipate that much of my work with GSp_{2n}-connections should generalize directly.
(2) A second project involves the study of the isomonodromy equations for meromorphic G-connections. I anticipate that these equations can be explicitly computed as integrable systems in the Poisson moduli space of connections with fixed combinatorics, further extending the work of Bremer and Sage [5].
(3) As a third project, the geometries of extended orbits and moduli spaces merit further study. For example, the Deligne–Simpson Problem (see, e.g., [13]) is the determination of necessary and sufficient conditions for which the moduli space is nonempty. Another problem is the determination of necessary and sufficient conditions for which the moduli space is **rigid;** i.e., reduced to a singleton. Other geometric features (e.g., smoothness of framable extended orbits $\mathscr{M}(A)$) also merit more thorough exploration.
(4) Boalch [1] has recently realized certain moduli spaces as quiver varieties. A fourth project involves investigating whether this can be done for moduli spaces of G-connections with nonsplit S-formal types.

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