Statement of Research Interest

Prasad Senesi

Introduction

My current research concerns the representation theory of Lie algebras. Since their introduction over a century ago, Lie groups and Lie algebras have been exploited in and connected with a wide array of mathematical pursuits, including algebraic geometry, differential geometry, topology and knot theory, foundations of quantum mechanics, and quantum field theory. My primary focus is on finite–dimensional representations of infinite–dimensional Lie algebras, in particular of Kac–Moody algebras and some of their generalizations.

A representation of some algebraic or combinatorial structure A (such as a group, ring, algebra, quiver) is (in most cases) a vector space V and a homomorphism from A to $\mathrm{End}(V)$, where $\mathrm{End}(V)$ is endowed with the same algebraic structure as A. More conceptually, a representation is some vector space (e.g., a space of states, for many physicists) on which A is allowed to act in some way that is compatable with the internal structure of A. In one sense a representation of A provides a concrete realization of A, loosely analogous to the way that matrices are 'realizations' of linear transformations or embeddings are 'realizations' of manifolds. When one studies a representation of A, one is interested in the vector space on which A is acting and the structures which are preserved by this action, but also in the manifestation of A as a set of transformations.

I have been studying the representations of a particular class of Lie algebras called the twisted affine Kac-Moody algebras. Kac-Moody algebras were introduced independently by Victor Kac and Robert Moody in the 1960s. They are generalizations of the well-known finite-dimensional simple Lie algebras. One of the three flavors of Kac-Moody algebras, the (infinite-dimensional) affine Kac-Moody algebras have, in recent decades, found applications in many areas of mathematics and physics. Accordingly, their representations are of great interest as well.

Together with Vyjayanthi Chari and Ghislain Fourier, I classified the finite–dimensional irreducible representations of a twisted affine Kac–Moody algebra (more precisely, of its derived algebra modulo the center). I then provided a block decomposition of the category of all of its finite–dimensional representations. In this research statement I will outline these results and describe the questions they have led to and which I am now pursuing, as well as some research goals I have in the near future.

The Loop Algebras

All algebras and representations are vector spaces over the complex numbers.

The Lie algebras with which we are concerned have a concrete description. In the following, unless otherwise noted $\mathfrak{g}$ will denote a simple finite–dimensional Lie algebra. Let $L(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t^{\pm}]$, with bracket given by $[x \otimes f, y \otimes g] = [x, y] \otimes fg$, for $x, y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[t^{\pm}]$. We call $L(\mathfrak{g})$ the *untwisted loop algebra* of $\mathfrak{g}$.

Now let $\sigma \in \mathrm{Aut}(\mathfrak{g})$ be a diagram automorphism of $\mathfrak{g}$, $|\sigma| = m$, and ζ a primitive m^{th} root of unity. Under the action of σ, $\mathfrak{g}$ decomposes as a sum of σ–eigenspaces:

$$\mathfrak{g} = \bigoplus_{i=0}^{m-1} \mathfrak{g}_i, \qquad \mathfrak{g}_i = \left\{x \in \mathfrak{g} : \sigma(x) = \zeta^i x\right\},$$

and the set $\mathfrak{g}_0$ of fixed points is a simple Lie algebra. This action of σ extends now to an automorphism of the corresponding loop algebra $L(\mathfrak{g})$ by setting $\sigma(x \otimes t^k) = \zeta^k \sigma(x) \otimes t^k$. The set of fixed points $L(\mathfrak{g})^\sigma$ of the loop algebra with respect to this action is a subalgebra of $L(\mathfrak{g})$, which we call the *twisted loop algebra* of $\mathfrak{g}$.

The representation theory

Let $\mathfrak{g}$ be a simple finite–dimensional Lie algebra and $\mathcal{F}$ be the category of finite–dimensional representations of the loop algebra $L(\mathfrak{g})$. If $\mathfrak{a}$ is any Lie subalgebra of $\mathfrak{g}$, then $L(\mathfrak{a}) := \mathfrak{a} \otimes \mathbb{C}[t^{\pm}]$ is a Lie subalgebra of $L(\mathfrak{g})$. In particular, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ a triangular decomposition of $\mathfrak{g}$, then we have a corresponding decomposition $L(\mathfrak{n}^-) \oplus L(\mathfrak{h}) \oplus L(\mathfrak{n}^+)$ of $L(\mathfrak{g})$. Let P and Q denote a weight lattice and root lattice, respectively, of $\mathfrak{g}$.

There are several important features of the category $\mathcal{F}$, which we describe here:

(1) **Weyl modules**. With respect to the triangular decomposition of $L(\mathfrak{g})$ described above we may define a 'loop highest weight' representation for $L(\mathfrak{g})$. In [11], the authors defined the Weyl modules of $L(\mathfrak{g})$, which are universal loop–highest weight modules; i.e., any finite–dimensional loop–highest weight module is a homomorphic image of some Weyl module.

(2) **Irreducible $L(\mathfrak{g})$–modules**. The irreducible $L(\mathfrak{g})$–modules in $\mathcal{F}$ have been parametrized and described in [4] and [12]; they are in bijective correspondence with the Weyl modules. We briefly describe their parametrization here. Let $n = \mathrm{rank}(\mathfrak{g})$. The Weyl modules, and hence the irreducible modules, are parametrized by the set of normalized n–tuples of polynomials

$$\mathcal{P} := \{\boldsymbol{\pi} = (\pi_1(u), \ldots, \pi_n(u)) : \pi_i(0) = 1\}.$$

The loops of the Cartan subalgebra, $L(\mathfrak{h})$, act diagonally on these representations, and their eigenvalues on a highest–weight vector are described by these polynomials. We denote the Weyl module (respectively, the irreducible module) corresponding to an element $\boldsymbol{\pi} \in \mathcal{P}$ by $W(\boldsymbol{\pi})$ (respectively, $V(\boldsymbol{\pi})$).

(3) **Block decomposition of $\mathcal{F}$.** The category $\mathcal{F}$ is *not* semisimple; there exist indecomposable objects in $\mathcal{F}$ with nontrivial quotients - in particular, the Weyl modules. However, $\mathcal{F}$ does decompose into a direct sum of indecomposable subcategories, the *blocks* of $\mathcal{F}$. In [8], these blocks are parametrized by the *spectral characters* Ξ of $L(\mathfrak{g})$, where $\Xi = \{f : \mathbb{C}^\times \to P/Q : |\mathrm{Supp}(f)| < \infty\}$.

Now let $\mathcal{F}^\sigma$ be the category of finite–dimensional representations of $L(\mathfrak{g})^\sigma$. If $\mathfrak{a}$ is a σ–invariant Lie subalgebra of $\mathfrak{g}$, then the fixed–point set $L(\mathfrak{a})^\sigma$ is a Lie subalgebra of $L(\mathfrak{g})^\sigma$; in particular, with respect to the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ for which σ is a diagram automorphism, we have a decomposition $L(\mathfrak{g})^\sigma = L(\mathfrak{n}^-)^\sigma \oplus L(\mathfrak{h})^\sigma \oplus L(\mathfrak{n}^+)^\sigma$. Using this decomposition, the loop–highest weight representations can be defined in $\mathcal{F}^\sigma$ as well, and all of the properties listed above for the category $\mathcal{F}$ have recently been extended to the category $\mathcal{F}^\sigma$. In [6], we construct the twisted Weyl modules and prove their expected universal property, and then prove the following

Theorem. [6]

(a) Let $k = \mathrm{rank}(\mathfrak{g}_0)$, and $\mathcal{P}^\sigma = \{\boldsymbol{\pi}^\sigma = (\pi_1(u), \ldots, \pi_k(u)) : \pi_i(0) = 1\}$. The (isomorphism classes of) irreducible $L(\mathfrak{g})^\sigma$–modules are parametrized by $\mathcal{P}^\sigma$.

(b) Every irreducible $L(\mathfrak{g})^\sigma$–module can be obtained by restricting some irreducible $L(\mathfrak{g})$–module to $L(\mathfrak{g})^\sigma$.

The k–tuple of polynomials corresponding to an irreducible (or a Weyl) module as in (a) describes the eigenvalues of $L(\mathfrak{h})^\sigma$ on a highest–weight vector, just as in the untwisted case. In this paper we describe in detail the correspondence between untwisted and twisted irreducible modules stated in (b).

To provide a block decomposition of the category $\mathcal{F}^\sigma$, I began with the decomposition given in [8] and introduced an equivalence relation $\sim_\sigma$ on the set of spectral characters Ξ defined there. Following the strategy used in [8], to each such 'twisted spectral character' χ I associated a subcategory $\mathcal{F}^\sigma_\chi$ of $\mathcal{F}^\sigma$ and proved the following

Theorem. [20] The category $\mathcal{F}^\sigma$ decomposes as a sum

$$\mathcal{F}^{\supset} = \bigoplus_{\chi \in \Xi/\sim_\sigma} \mathcal{F}^\sigma_\chi,$$

and each $\mathcal{F}^\sigma_\chi$ is a block of $\mathcal{F}^\sigma$.

σ–equivariance

Alternatively, we can recast all of the following results in terms of functions on $\mathbb{C}^\times$. Let $\mathfrak{h}$ be a cartan subalgebra for which σ is a diagram automorphism, Q a root lattice and $\{\omega_i\}$ the fundamental weights of $\mathfrak{h}^*$. The diagram automorphism σ provides us with an automorphism also of $\mathbb{C}^\times$ ($a \mapsto \zeta a$) and of the fundamental weight lattice $P = \bigoplus \mathbb{Z}\omega_i$ ($\omega_i \mapsto \omega_{\sigma(i)}$). The loops $L(\mathfrak{g})$ can be interpreted as the algebra of regular functions $\{f : \mathbb{C}^\times \to \mathfrak{g}\}$, and the twisted loops $L(\mathfrak{g})^\sigma$ as the (sub)algebra of σ–equivariant regular functions $\{f : \mathbb{C}^\times \to \mathfrak{g} : \sigma^{-1} f \sigma = f\}$. In addition, we have the following correspondences:

$$\begin{aligned}
\mathcal{P} &\longleftrightarrow \{f : \mathbb{C}^\times \to P : |\mathrm{Supp}(f)| < \infty\} \\
\mathcal{P}^\sigma &\longleftrightarrow \{f : \mathbb{C}^\times \to P : |\mathrm{Supp}(f)| < \infty,\ \sigma^{-1} f \sigma = f\} \\
\text{Blocks of } \mathcal{F} &\longleftrightarrow \{f : \mathbb{C}^\times \to P/Q : |\mathrm{Supp}(f)| < \infty\} \\
\text{Blocks of } \mathcal{F}^\sigma &\longleftrightarrow \{f : \mathbb{C}^\times \to P/Q : |\mathrm{Supp}(f)| < \infty,\ \sigma^{-1} f \sigma = f\}
\end{aligned}$$

All together, we have the following table

	$\mathfrak{g}$	$L(\mathfrak{g})$	$L^\sigma(\mathfrak{g})$
Lie algebra	$\{\cdot \to \mathfrak{g}\}$	$\{\mathbb{C}^\times \to \mathfrak{g}\}$	$\{\mathbb{C}^\times \to_\sigma \mathfrak{g}\}$
simple objects	$\{\cdot \to P^+\}$	$\{\mathbb{C}^\times \to P^+\}$	$\{\mathbb{C}^\times \to_\sigma P^+\}$
blocks	$\{\cdot \to P^+\}$	$\{\mathbb{C}^\times \to P/Q\}$	$\{\mathbb{C}^\times \to_\sigma P/Q\}$

with some comments:

1. By 'simple objects' in the first column, we mean isomorphism classes of simple objects. By 'blocks', we mean blocks of the corresponding category of finite–dimensional representations.
2. By the set $\{A \to B\}$ we mean some collection of maps from A to B; e.g., the entry $\{\mathbb{C}^\times \to \mathfrak{g}\}$ in the first row is the set of all *regular* maps, and the entry $\{\mathbb{C}^\times \to P^+\}$ in the second row is the set of all *finitely supported* maps.
3. The maps $\{A \to_\sigma B\}$ occurring in the fourth column are those maps from the second column which are σ–equivariant, as described above.
4. The domain ($\cdot$) occurring in the second column is just the singleton set.
5. Of course, these sets of maps have additional structure; in the first row, they have the structure of Lie algebras, and in the second and third rows, of additive monoids.
6. The distinction between the third and fourth columns is artificial and unnecessary. If we allow the trivial diagram automorphism $\sigma = \mathrm{Id}$ then we can elemininate the third column all together.

Multiloop algebras

The twisted loop algebras $L(\mathfrak{g})^\sigma$ described above are only a special case of the following:

Definition 1. Let $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_k)$ be a collection of commuting finite–order Lie algebra automorphisms of $\mathfrak{g}$, $|\sigma_i| = m_i$, and ζ_i a primitive m_i^{th} root of unity. We then have $\mathfrak{g} = \bigoplus_{\mathbf{s} \in \mathbb{Z}^k} \mathfrak{g}_{\mathbf{s}}$, where, for $\mathbf{s} = (s_1, \dots, s_k) \in \mathbf{Z}^k$, $\mathfrak{g}_k = \{x \in \mathfrak{g} : \sigma_i(x) = \zeta_i^{s_i} x\}$. Then we define

$$L(\mathfrak{g}, \boldsymbol{\sigma}) = \bigoplus_{\mathbf{s} \in \mathbb{Z}^k} \mathfrak{g}_{\mathbf{s}} \otimes t^{\mathbf{s}},$$

where $t^{\mathbf{s}} = t_1^{s_1} \cdots t_k^{s_k} \in \mathbb{C}\left[t_1^{\pm}, \ldots, t_k^{\pm}\right]$.

$L(\mathfrak{g}, \boldsymbol{\sigma})$ is called a *multiloop algebra* (of $\mathfrak{g}$). The representation theory of these multiloop algebras is currently under development, and so far nothing is known about the structure of their categories.

These multiloop algebras play a fundamental role in the description of the extended affine Lie algebras (the EALAs), which are generalizations of both the multiloop algebras and of the affine Kac-Moody algebras. The classification of these Lie algebras is the subject of a number of recent papers (for example, [1], [2], [3], [18], [19]), although - like the multiloop algebras - not so much is known of the (finite–dimensional) representation theory.

My immediate research goals are as follows:

(1) Extend the finite–dimensional 'loop–highest weight' theory developed for the untwisted and (in the case of a diagram automorphism) twisted loop algebras to the multiloop algebras; find the multiloop analogs of the Weyl modules for these algebras.

(2) Describe the block decomposition of the category of finite–dimensional representations of a multiloop algebra. Like the categories $\mathcal{F}$ and $\mathcal{F}^{\sigma}$, this category should be non–semisimple, and there should be some natural extension of the spectral characters used to describe the blocks in the 'single loop' cases.

In both of these pursuits the language of equivariant functions will be used to describe the algebras, representations, and blocks. At present it is not clear in what generality these goals can be accomplished. Finding a suitable triangular decomposition of $L(\mathfrak{g}, \boldsymbol{\sigma})$ with respect to which a representation should be 'loop–highest weight' is one obstacle. Given a commuting collection of automorphisms $\{\sigma_i\} \subseteq \mathrm{Aut}(\mathfrak{g})$, if we are so fortunate as to have a decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ which is stable under the action of all $\sigma_i \in \boldsymbol{\sigma}$, this decomposition of $\mathfrak{g}$ extends to one of $L(\mathfrak{g}, \boldsymbol{\sigma})$ (as is the case when $\boldsymbol{\sigma}$ consists of a single diagram automorphism of $\mathfrak{g}$).

Future directions in research

Existence of $\mathbb{Z}$–forms

A $\mathbb{Z}$–form or $\mathbb{Z}$–basis for an algebra R allows one to work with R over characteristic p. A similar construction can sometimes be made for representations. $\mathbb{Z}$–forms have been shown to exist for the universal enveloping algebras of twisted affine algebras (see [15], [17]), but not all representations can be equipped with a $\mathbb{Z}$–form. A necessary condition is known for the Weyl modules of the untwisted loop algebras, but beyond this little is known concerning the existence or presentation of $\mathbb{Z}$–forms for twisted loop algebra modules.

Weyl modules of quantized enveloping algebras

Although Weyl modules have been presented here and are studied elsewhere (see [16]) as representations of classical (i.e. affine Kac–Moody) Lie algebras, they originated as $q \to 1$ specializations of certain representations of the quantized universal enveloping algebra $U_q(\hat{\mathfrak{g}})$ of an affine Lie algebra.

Provided that Weyl modules can be defined for the multiloop algebras, perhaps one can construct a quantized universal enveloping algebra of a multiloop algebra and define a quantized multiloop Weyl module whose $q \to 1$ specialization behaves as we would expect.

Equivariance with respect to other algebras

The shift in point of view described above from polynomials to functions on $\mathbb{C}^\times$ as descriptions of the loop algebras and their irreducible modules suggests the possibility of replacing the torus - or its coordinate ring of Laurent polynomials with the coordinate ring of some other variety. The moral of the story is now that the main object of study is the Lie algebra of equivariant maps defined on some geometric object. Do we always have finite–dimensional representations of such algebras? If so, is there some analog of the evaluation representation which allows us to realize these modules as $\mathfrak{g}$–modules? Is the semisimplicity of the corresponding category always lost, as in the loop cases?

Finite–dimensional representation theory of extended affine Lie algebras

The structure of these generalizations of affine Kac-Moody lie algebras is still being investigated, but a good understanding of the representation theory of the multiloop algebras is certainly a first step towards that of the EALAs.

References

[1] B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. 126 #603 (1997).

[2] B. Allison, S. Berman, J. Faulkner, A. Pianzola, *Realization of graded-simple algebras as loop algebras*, Forum Math. 20 No. 3 (2008), 395-432.

[3] B. Allison, S. Berman, J. Faulkner, A. Pianzola, *Multiloop realization of extended affine Lie algebras and Lie tori*, math.RA/0709.0975.

[4] V. Chari, *Integrable representations of affine Lie algebras*, Invent. Math 85 (1986), no. 2, 317 - 335.

[5] V. Chari, *On the fermionic formula and the Kirillov-Reshetikhin conjecture*, Internat. Math. Res. Notices 2001, no. 12, 629–654

[6] V. Chari, G. Fourier and P. Senesi, *Weyl Modules for the twisted loop algebras*, J. Algebra 319 (2008), no. 12, 5016–5038.

[7] V. Chari and S. Loktev, *Weyl, Demazure and fusion modules for the current algebra of* $\mathfrak{sl}_{r+1}$, Adv. Math. 207 (2006), no. 2, 928–960.

[8] V. Chari and A. Moura, *Spectral characters of finite–dimensional representations of affine algebras*,

[9] V. Chari and A. Pressley, *Integrable Representations of Twisted Affine Lie Algebras*, J. Algebra 113 No. 2 (1988), 438–464.

[10] V. Chari and A. Pressley, *Twisted quantum affine algebras*, Comm. in Math. Phys. 196, No. 2 (1998), 461–476.

[11] V. Chari and A. Pressley, *Weyl modules for classical and quantum affine algebras*, Represent. Theory 5 (2001), 191–223

[12] V. Chari and A. Pressley, *New unitary representations of loop groups*, Math. Ann. 275 (1986), 87–104.

[13] P. Etingof and A. Moura, *Elliptic Central Characters and Blocks of Finite Dimensional Representations of Quantum Affine Algebras*, Represent. Theo. 7 (2003), 346–373.

[14] B. Feigin, A. N. Kirillov and S. Loktev, *Combinatorics and geometry of higher level Weyl modules*, math.QA/0503315.

[15] T.Fisher-Vasta, *Presentations of Z-Forms for the Universal Enveloping Algebras of Affine Lie Algebras* University of California Riverside Ph.D. Dissertation (1999).

[16] G. Fourier and P. Littelmann, *Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions*, Adv. Math. 211 no. (2007), 566-593.

[17] D. Mitzman, *Integral Bases for Affine Lie Algebras and Their Universal Enveloping Algebras*, Contemporary Mathematics 40 (1983).

[18] E. Neher, *Lie tori*, C.R. Math. Acad. Sci. Soc. R. Can. 26 (2004), 8489.

[19] E. Neher, *Extended affine Lie algebras*, C.R. Math. Acad. Sci. Soc. R. Can 26 (2004), 9096.

[20] P. Senesi, *A block decomposition of finite-dimensional representations of twisted loop algebras*, math.RT/0807.4116.