0. Introduction.

This manuscript is a detailed treatment of the classification of Weyl modules for the twisted loop (sub)-algebra of $A_2^{(2)}$. Most of the computational difficulty of the classification for a more general twisted loop algebra is contained in this case. Organization and proofs have been adapted from the paper *Weyl Modules of Quantum Affine Algebras* by Chari and Pressley. The generalization of these results can be found at

http://www.math.ucr.edu/~prasad/twist(g).pdf

1. Preliminaries and Some Identities

Let $g$ be a finite-dimensional complex simple Lie algebra of rank $n$. Let $I = \{1, 2, \cdots, n\}$, let $A = (a_{ij})_{i,j \in I}$ be a Cartan matrix of $g$, $\{\alpha_i\}_{i \in I}$ a set of simple roots, $\{x_i^+, h_i\}_{i \in I}$ a set of Chevalley generators and

$$g = n^- \oplus h \oplus n^+$$

a corresponding triangular decomposition of $g$. Let $R$ be the set of roots of $g$, and $R^+$ be the corresponding set of positive roots.

Let $W \subset \text{Aut}(h^*)$ be the Weyl group of $g$; it is well known that $W$ is generated by simple reflections $s_i$ ($i \in I$).

The loop algebra of $g$ is the Lie algebra

$$L(g) = g \otimes \mathbb{C}[t, t^{-1}],$$

with commutator given by

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$$

for $x, y \in g$, $r, s \in \mathbb{Z}$.

**Definition 1.1.** A diagram automorphism of $g$ is a Lie algebra automorphism $\sigma : g \to g$ induced by a bijection $\sigma : I \to I$ satisfying

$$(1.1)\quad a_{ij} = a_{\sigma(i)\sigma(j)}$$

for all $i, j \in I$.

**Remark.** For a bijection $\sigma$ of $I$ satisfying $1.1$, the induced Lie algebra automorphism satisfies

$$\sigma : \begin{cases} x_i^+ &\mapsto x_{\sigma(i)}^+; \\ h_i &\mapsto h_{\sigma(i)}; \quad i \in I. \end{cases}$$

For existence and uniqueness of $\sigma : g \to g$, see [Wan]. It is easy to see that a bijection $\sigma$ satisfies $1.1$ if and only if the Dynkin diagram corresponding to $A = (a_{ij})$ remains invariant under the action of $\sigma$. A case by case examination of the complex
simple finite-dimensional Lie algebras reveals that a diagram automorphism is either identity or of order 2 or 3.

Let \( \sigma \) be a diagram automorphism of \( g \); let \( m \) be the order of \( \sigma \). Fix a primitive \( m^{th} \) root of unity \( \zeta \in \mathbb{C} \). For \( r \in \mathbb{Z}/m\mathbb{Z} \), denote by \( g_r \) the eigenspace
\[
g_r = \{ x \in g : \sigma(x) = \zeta^r x \}.
\]
Then
\[
g = \bigoplus_{r \in \mathbb{Z}/m\mathbb{Z}} g_r
\]
is a \( \mathbb{Z}/m\mathbb{Z} \)-gradation of \( g \). The fixed point algebra \( g_0 \) is a simple Lie algebra, and the nodes of its Dynkin diagram are naturally indexed by the set of orbits \( I_\sigma \) of \( I \) under the action of \( \sigma \). For \( 0 \neq r \in \mathbb{Z}/m\mathbb{Z} \), \( g_r \) is an irreducible representation of \( g_0 \).

The automorphism \( \sigma : g \to g \) may be extended to a Lie algebra automorphism \( \tilde{\sigma} \) of the Loop algebra \( L(g) \) by defining
\[
\tilde{\sigma}(x \otimes t^k) = \zeta^k \sigma(x) \otimes t^k,
\]
for \( x \in g, k \in \mathbb{Z} \).

The twisted loop algebra \( L^\sigma(g) \) is the subalgebra of fixed points of \( L(g) \) under \( \tilde{\sigma} \):
\[
L^\sigma(g) = \{ a \in L(g) : \tilde{\sigma}(a) = a \}.
\]
In particular, if \( \sigma \) is a diagram automorphism of order 2, then \( g = g_0 \oplus g_1 \), where
\[
g_0 = \{ x + \sigma(x) \mid x \in g \}, \quad g_1 = \{ x - \sigma(x) \mid x \in g \},
\]
and
\[
L^\sigma(g) = g_0 \otimes \mathbb{C} \left[ t^{\pm 2} \right] \bigoplus g_1 \otimes \mathbb{C} \left[ t^{\pm 2} \right].
\]

**Remark.** Seems I should mention something here about a triangular decomposition for \( g \), so that I can generally define \( U(<), U(0), \) etc...

For any Lie algebra \( a \), the universal enveloping algebra of \( a \) is denoted by \( U(a) \). Let \( h_0 \) be a Cartan subalgebra of \( g_0 \). We set
\[
U = U(L^\sigma(g)),
\]
\[
U(>) = U(L(n^+)) \cap U(L^\sigma(g)),
\]
\[
U(<) = U(L(n^-)) \cap U(L^\sigma(g)),
\]
\[
U(0) = U(L(h)) \cap U(L^\sigma(g)),
\]
\[
U_0 = U(g_0),
\]
\[
U_0(>) = U(>) \cap U_0,
\]
\[
U_0(<) = U(<) \cap U_0.
\]

By the Poincare-Birkhoff-Witt theorem, we have
\[
U_0 = U_0(<)U(h_0)U_0(>,
\]
\[
U = U(<)U(0)U(>).
\]
2. The Twisted Loop Algebra $L^\sigma(A_2)$

For the remainder let $g = A_2$ and $\sigma : g \to g$ be the order 2 diagra automorphism of $g$. Let $\{\alpha_1, \alpha_2\}$ be a simple system of roots for $g$, so that $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. Let $\{X_i^\pm, H_i\}_{i=1,2}$ be a corresponding Chevalley basis of $g$, and set $X_3^\pm = \{X_1^\pm, X_2^\pm\}$, so that $g_{\pm(\alpha_1+\alpha_2)} = C X_3^\pm$.

The following is a vector space basis for $L^\sigma(g)$:

\[ x_k = (-1)^k \sqrt{2} (X_1^+ + (-1)^k X_2^-) \otimes t^k \]
\[ y_k = (-1)^k \sqrt{2} (X_1^- + (-1)^k X_2^+) \otimes t^k \]
\[ h_0 = 2(H_1 + H_2) \otimes 1 \]
\[ h_{2k+1} = -X_2^+ \otimes t^{2k+1} \]
\[ \tilde{h}_k = \begin{cases} (-1)^k h_k, & k \neq 0, \\ 0 & k = 0. \end{cases} \]

For $k \in \mathbb{Z}$, let

In this case,

\[ g_0 = C (X_1^+ + X_2^-) \oplus C (H_1 + H_2) \oplus C (X_1^- + X_2^+) \],
and may be identified with the subalgebra $(y_0, h_0, x_0)$ of $L^\sigma(g)$, which is isomorphic to $sl_2$. $L^\sigma(g)$ is generated as a Lie algebra by the elements $x_0, y_0, \tilde{x}_1, y_1$.

**Lemma 2.1.** The assignment $T(x_{\alpha_i,m}^\pm) = x_{\alpha_i,m\pm1}^\pm$, for $i \in I, \ m \in \mathbb{Z}$, defines an algebra automorphism of $U$. \hfill \Box

We next introduce the following power series in an indeterminate $u$, which will be useful for stating some identities in $U$:

\[ \tilde{H}(u) = \sum_{m=-\infty}^{\infty} \tilde{h}_m u^m \]
\[ \tilde{P}^\pm(u) = \sum_{m=0}^{\infty} \tilde{P}_{\pm m} u^m = \exp \left( -\sum_{k=1}^{\infty} \frac{\tilde{h}_{\pm k}}{k} u^k \right) \]
\[ \tilde{H}_2(u) = \sum_{m=-\infty}^{\infty} \tilde{h}_{2m} u^m \]
\[ \tilde{P}_2^\pm(u) = \sum_{m=0}^{\infty} \tilde{P}_{2,\pm m} u^m = \exp \left( -\sum_{m=1}^{\infty} \frac{\tilde{h}_{\pm 2m}}{m} u^m \right) \]
\[ Y_0(u) = \sum_{m=0}^{\infty} y_m u^{m+1} \]
\[ Y(u) = \sum_{m=1}^{\infty} y_m u^m \]
\[ Y_{2,0}(u) = \sum_{m=0}^{\infty} y_{2m-1} u^{m+1} \]
\[ Y_2(u) = \sum_{m=1}^{\infty} y_{2m-1} u^m \]
\[ \tilde{Y}_0(u) = \sum_{m=-\infty}^{\infty} y_m u^{m+1} \]
\[ \tilde{Y}(u) = \sum_{m=-\infty}^{\infty} y_m u^m \]
\[ \tilde{Y}_{2,0}(u) = \sum_{m=-\infty}^{\infty} y_{2m-1} u^{m+1} \]
\[ \tilde{Y}_2(u) = \sum_{m=-\infty}^{\infty} \tilde{y}_{2m-1} u^m \]

**Remark.** It is easy to see that $\tilde{P}_0 = \tilde{P}_{2,0} = 1$.

The next lemma follows easily from the definition of the $\tilde{P}_i$.

**Lemma 2.2.** The subalgebra $U(0)$ of $U$ is generated by the elements $\tilde{P}_m$, for $m \in \mathbb{Z}$. \hfill \Box
Lemma 2.3.

(i) Let $\varsigma, \tau : L^\beta(A_2) \to L^\beta(A_2)$ be defined as follows:

\[
\begin{align*}
\varsigma(x_k) &= x_{-k} & \tau(x_k) &= x_k \\
\varsigma(y_k) &= y_{-k} & \tau(y_k) &= y_k \\
\varsigma(x_{2k+1}) &= x_{-2k-1} & \tau(x_{2k+1}) &= x_{2k+1} \\
\varsigma(y_{2k+1}) &= y_{-2k-1} & \tau(y_{2k+1}) &= y_{2k+1} \\
\varsigma(h_k) &= h_{-k} & \tau(h_k) &= -h_k.
\end{align*}
\]

Then $\varsigma$ is an automorphism and $\tau$ an antiautomorphism of $L^\beta(A_2)$.

(ii) Let the elements $p_i \in U(0)$ be defined as follows:

\[ p_0 = 1, \quad p_n = \begin{cases} 
\frac{1}{n} \sum_{i=1}^{n} (-1)^i h_i p_{n-i}, & n > 0, \\
\varsigma(p_{-n}), & n < 0.
\end{cases} \]

Denote the extension of $\tau$ to $U(L^\beta(A_2))$ by $\tau$ as well. Then $\tau(p_m) = p_m$.

Proof. We will only prove (ii). Since $\tau$ is an order 2 automorphism, it is sufficient to prove that $\tau(P_m) = P_m$. It is well known that, if $\{r_i\}$ is a sequence of commuting elements from a ring $R$ with identity and $\{a_i\} \in R$ are defined recursively by

\[ a_0 = 1, \quad a_n = \sum_{i=1}^{n} r_i a_{n-i}, \]

then the $a_i$ are also given by the generating function with indeterminate $u$

\[ \exp \left( \sum_{k=1}^{\infty} \frac{r_k}{k} u_k \right). \]

Applying $\tau$ to the generating function for $P_n$, $n > 0$, yields

\[
\sum_{m=0}^{\infty} \tau(P_m) u^m = \tau \left( \sum_{m=0}^{\infty} P_m u^m \right)
= \tau \left( \exp \left( - \sum_{m=1}^{\infty} \frac{h_m}{m} u^m \right) \right)
= \exp \left( \sum_{m=1}^{\infty} \tau \left( (-1)^m h_m \right) u^m \right).
\]

But the coefficients $\tau(P_m)$ of this generating function are also given by the recursive relation

\[ \tau(P_0) = 1, \quad \tau(P_n) = \sum_{i=1}^{n} (-1)^i h_i \tau(P_{n-i}), \]

and this is the same relation used in (2.1) above to define the $p_n$ for $n \geq 0$. To prove the result for $n < 0$, first observe that applying the automorphism $\varsigma$ to the
power series (2.2) above yields

\begin{equation}
\exp \left( \sum_{m=1}^{\infty} \frac{(-1)^m h_m u^m}{m} \right).
\end{equation}

Since we have the identity

\[ \sum_{m=0}^{\infty} p_m u^m = \exp \left( \sum_{m=1}^{\infty} \frac{(-1)^m h_m u^m}{m} \right), \]

this shows us that the elements \( p_{-n} = \zeta(p_n), \ n > 0, \) are also given by the generating function (2.3), and hence can be defined recursively via

\[ p_0 = 1, \quad p_{-n} = \sum_{i=1}^{n} (-1)^i h_i p_{i-n}. \]

And now a proof analogous to that above, used to show \( \tau(P_n) = p_n \) for \( n > 0, \) can be used to show that \( \tau(P_{-n}) = p_{-n}. \) \( \square \)

**Definition 2.1.** For \( k \in \mathbb{Z}^+ \),

(i) let \( D^-_k : \mathbb{Z}^+ \to U(>) \) be defined by

\[ D^-_0(0) = 1, \quad D^-_n(0) = 0 \text{ if } n > 0, \quad D^-_n(r) = 0 \text{ if } r < 0, \]

and

\[ \left( \frac{n + r}{2} \right) D^-_n(r) = \sum_{0 \leq 4i \leq n} y_{1+2i} D^-_{n-4i}(r-2) - \sum_{0 \leq 2i \leq n-1} (i+1) y_{1+i} D^-_{n-(2i+1)}(r-1). \]

(ii) let \( D^+_k : \mathbb{Z}^+ \to U(<) \) be defined by

\[ D^+_0(0) = 1, \quad D^+_n(0) = 0 \text{ if } n > 0, \quad D^+_n(r) = 0 \text{ if } r < 0, \]

and

\[ r D^+_n(r) = \sum_{i=0}^{n} D^+_{n-i}(r-1) x_i - 2 \sum_{0 \leq 2i \leq n-1} D^+_{n-(2i+1)}(r-2) x_{2i+1}. \]

(iii) Let \( D_k : \mathbb{Z}_{\geq 0} \to UU(>) \) be defined by

\[ D_k = \sum_{i=0}^{k} D^+_{k-i} p_i. \]

**Lemma 2.4.** \( D^-_{2k+1}(1) = -y_{k+1} \) for \( k \geq 0. \)

**Remark.** *Should I include a proof of this?*

For any power series \( f \) in \( u \) with coefficients in an algebra \( A, \) let \( f_m \) be the coefficient of \( u^m \) \( (m \in \mathbb{Z}) \) and let \( f' \) denote the derivative of \( f \) with respect to \( u. \)

For \( x \in U, \ r \in \mathbb{Z}^+, \) set

\[ x^{(r)} = \frac{x^r}{r!}. \]

For an algebra \( A, \) let \( A_+ \) denote the augmentation ideal.

**Lemma 2.5.**
(i) Let $0 \leq s \leq 2r$. Then
\[ x_0^{(s)} y_1^{(r)} = \left( \sum_{\rho=0}^{\lfloor s/2 \rfloor} \tau \left( D_{s-2\rho}^{-2r-s} \right) \mathcal{P}_\rho \right) \mod \mathbb{U}\mathbb{U}(>)_+ . \]

(ii) Let $0 \leq s \leq r$. Then
\[ x_1^{(s)} y_1^{(r)} = (-1)^s \left( (Y_2(u))^{(r-s)} \mathcal{P}_2(u) \right) \tau \mod \mathbb{U}\mathbb{U}(>)_+ . \]

**Proof.** It is a result of Fisher-Vasta that
\[ (2.5) \]
\[ x_0^{(s)} y_1^{(r)} = \sum_{i=0}^{\text{min}(2r,s)} \sum_{\rho=0}^{\text{min}(i-s, 2r-i)} \mathcal{D}_\rho (s-i-\rho) D_{i-\rho}^{-2r} (2r-i-\rho). \]

To prove the lemma we must first apply to (2.4) the antiautomorphism $\tau$ to re-order the left-hand side:
\[ (2.5) \]
\[ x_0^{(s)} y_1^{(r)} = \sum_{i=0}^{\text{min}(2r,s)} \sum_{\rho=0}^{\text{min}(s-i, 2r-i)} \tau \left( D_{s-\rho}^{-2r} (2r-i-\rho) \right) \mathcal{D}_\rho (s-i-\rho) \tau \mod \mathbb{U}\mathbb{U}(>)_+ . \]

We are interested only in those terms from (2.5) which are not in $\mathbb{U}\mathbb{U}(>)_+$. To isolate these terms, note that $\tau(\mathbb{D}_k(n)) = \tau \left( \sum_{i=0}^{k} D_{k-i}^{-1} (n) p_i \right) = \sum_{i=0}^{k} \mathcal{P}_i \tau \left( D_{k-i}^{-1} (n) \right)$ and that $D_m^+(n)$ (and hence $\tau(D_m^+(n))$) is contained in $\mathbb{U}(>)_+$ for $m > 0$.

**Remark.** Should I include a proof of this?

Therefore,
\[ \tau(\mathbb{D}_\rho(s-i-\rho)) = \mathcal{P}_\rho \tau \left( D_0^+(s-i-\rho) \right) \mod \mathbb{U}\mathbb{U}(>)_+ . \]

And it is also easy to verify that $D_0^+(n) \in \mathbb{U}\mathbb{U}(>)_+$ for $n > 0$. So the only element $D_m^+(n)$ which is not contained in $\mathbb{U}\mathbb{U}(>)_+$ is $D_0^+(0) = 1$, and therefore the only nonzero terms in the sum (2.5) which are not contained in $\mathbb{U}\mathbb{U}(>)_+$ are those for which $i + \rho = s$. So (2.5) becomes
\[ x_0^{(s)} y_1^{(r)} = \sum_{i=0}^{\text{min}(2r,s)} \sum_{\rho=0}^{\text{min}(s-i, 2r-i)} \tau \left( D_{s-2\rho}^{-2r} (2r-s) \right) \mathcal{P}_\rho \mod \mathbb{U}\mathbb{U}(>)_+ , \]

where $M = \text{min}(i, s-i, 2r-i)$. For $0 \leq s \leq 2r$, this becomes
\[ x_0^{(s)} y_1^{(r)} = \left( \sum_{\rho=0}^{\lfloor s/2 \rfloor} \tau \left( D_{s-2\rho}^{-2r} (2r-s) \right) \mathcal{P}_\rho \right) \mod \mathbb{U}\mathbb{U}(>)_+ . \]

For the proof of (ii), let $L(sl_2) = \langle X_m, Y_n, H_l \rangle$, where $\{X, Y, H\}$ is a set of Chevalley generators for $sl_2$. The proof of the lemma relies on the isomorphism
\[ \phi : \langle X_m, Y_n, H_l \rangle \cong \langle X_{2m+1}, Y_{2n+1}, H_2 \rangle \]
where $\phi$ is given by
and a result from [CP]:

\[(2.6) \quad X_0^{(s)} Y_1^{(r)} = (-1)^s \left( \left( \sum_{m=1}^{\infty} Y_m u^m \right)^{(r-s)} \left( \sum_{n=0}^{\infty} \Lambda_n u^n \right) \right)_r \mod \mathbf{U}(\mathbf{U}(\mathbf{s})),\]

where \( \mathbf{U} = \mathbf{U} L(\mathbf{L}(\mathbf{s})), \mathbf{U}(\mathbf{s}) = \langle X_m, Y_n, H_l \rangle \) denotes the \( r \)-th coefficient of a power series, and \( \Lambda_n \in \mathbf{U}(\mathbf{L}(\mathbf{s})) \) are defined via

\[(2.7) \quad \sum_{n=0}^{\infty} \Lambda_n u^n = \exp \left( - \sum_{k=1}^{\infty} \frac{H_k}{k} u^k \right).\]

**Remark.** Is there a better notation here for \( \langle X_m, Y_n, H_l \rangle \)? For \( \mathbf{U} = \mathbf{U} L(\mathbf{L}(\mathbf{s})) \)? For \( P_n \in \mathbf{U}(\mathbf{L}(\mathbf{s})) \)? There is some possibility for confusion because all of these are the same symbols as those I’ve used for \( \mathbf{L}_\sigma(\mathbf{A}_2) \).

Applying \( \phi \) to (2.7) gives us

\[ \sum_{n=0}^{\infty} \phi(\Lambda_n) u^n = \exp \left( - \sum_{k=1}^{\infty} \frac{\tilde{h}_k}{k} u^k \right) = \sum_{m=0}^{\infty} P_{2,m} u^m; \]

so that \( \phi(\Lambda_n) = P_{2,n} \). Then applying \( \phi \) to (2.6) gives us

\[ \pi_1^{(s)} \gamma_1^{(r)} = (-1)^s \left( \left( \sum_{m=1}^{\infty} \phi_2 m u^m \right)^{(r-s)} \left( \sum_{n=0}^{\infty} P_{2,n} u^n \right) \right)_r \mod \mathbf{U}(\mathbf{U}(\mathbf{s})), \]

which is part (ii) of the lemma. \( \square \)

We conclude this section with some elementary properties of integrable \( L^\mathcal{L}(\mathbf{A}_2) \)-modules.

A representation \( V \) of \( \mathbf{U} \) is called integrable if the generators \( x_0, y_0, \pi_1, y_{-1} \), act locally nilpotently on \( V \) and

\[ V = \bigoplus_{\lambda \in (\mathcal{C} \mathbf{h}_0)^*} V_\lambda, \]

where

\[ V_\lambda = \{ v \in V : h_0 v = \lambda(h_0) v \}. \]

**Remark.** Should I replace \( (\mathcal{C} \mathbf{h}_0)^* \) here with just \( \mathbb{C} \)? Should I explain how, more generally, \( (\mathcal{C} \mathbf{h}_0)^* \) is replaced by a C.S.A. of the fixed point algebra \( \mathbf{g}_0 \)?

Set

\[ V_\lambda^+ = \{ v \in V_\lambda : \mathbf{U}(\langle \rangle)_+ v = 0 \ \forall \ m, n \in \mathbb{Z} \}. \]

If \( V \) is integrable, then \( V_\lambda^+ \neq 0 \) (resp. \( V_\lambda^+ \neq 0 \)) only if \( \lambda(h_0) \in 2\mathbb{Z} \) (resp. \( \lambda(h_0) \in 2\mathbb{Z}_{\geq 0} \)). To see this, note that for a fixed \( n \in \mathbb{Z} \) and a nonzero \( w \in V_\lambda \),

\[ \left( \phi_{2n+1}, \gamma_{-2n-1}, \frac{h_0}{2} \right) \cong \mathbf{s} \mathbf{l}_2. \]
and the submodule \( \langle \pi_{2n+1}, \pi_{-2n-1}, \frac{h_2}{2} \rangle \cdot w \) is isomorphic to a nonzero finite-dimensional representation \( V(\lambda) \) of \( sl_2 \). Then standard results from \( sl_2 \)-representation theory tell us that \( \frac{\lambda(h_0)}{2} = \lambda \left( \frac{h_2}{2} \right) \in \mathbb{Z} \), and if \( w \in V_\lambda^+ \), then \( \langle \pi_{2n+1}, \pi_{-2n-1}, \frac{h_2}{2} \rangle \cdot w \) is a

 nonzero finite-dimensional highest weight \( sl_2 \)-module, in which case \( \frac{\lambda(h_0)}{2} \in \mathbb{Z} \) (see, for example, [Hum]).

Further, if \( v \in V_\lambda^+ \), then

\[ V_\lambda \neq 0 \implies V_w \lambda \neq 0 \ \forall \ w \in W_0, \]

where \( W_0 \) is the Weyl group of the fixed point simple Lie algebra \( g_0 \cong sl_2 \).

If \( \lambda \in P_\ast \), let \( V^{fin}(\lambda) \) be the finite-dimensional irreducible \( U^{fin}_0 \)-module with highest weight \( \lambda \). If \( V \) is an integrable \( U \)-module and \( 0 \neq v \in V_\lambda^+ \), then \( U^{fin}_0 \cdot v \) is a \( U^{fin}_0 \)-submodule of \( V \) isomorphic to \( V^{fin}(\lambda) \).

**Lemma 2.6.** Let \( r \geq \frac{\lambda(h_0)}{2} + 1 \) and \( 0 \neq v \in V_\lambda^+ \). Then

(i) \[ \left( \sum_{k=1}^{r} y_k P_{r-k} \right) \cdot v = 0; \]

(ii) \[ \left( \sum_{k=1}^{r} \pi_{2k-1} P_{r-k} \right) \cdot v = 0. \]

**Proof.** Taking \( r = \frac{\lambda(h_0)}{2} + 1 \) and \( s = 2r - 1 \) in lemma 2.5 (i), we have \( \left\lfloor \frac{s}{2} \right\rfloor = r - 1 \) and

\[ 0 = \pi_0^{(\frac{\lambda(h_0)}{2})} \pi_1^{(\frac{\lambda(h_0)}{2} + 1)} \cdot v \]

\[ = \left( \sum_{\rho=0}^{r-1} x (D_{s_2}(1)) P_{\rho} \right) \cdot v \]

\[ = \left( \sum_{\rho=0}^{r-1} (-y_{s_2+1}) P_{\rho} \right) \cdot v \]

\[ = - (y_r + y_{r-1} P_1 + y_{r-2} P_2 + \ldots + y_1 P_{r-1}) \cdot v, \]

which proves (i) of the lemma.

Using lemma 2.5 (ii) with \( r = \frac{\lambda(h_0)}{2} + 1 \), \( s = r - 1 \) gives us

\[ 0 = \pi_1^{(\frac{\lambda(h_0)}{2})} \pi_1^{(\frac{\lambda(h_0)}{2} + 1)} \cdot v \]

\[ = (-1)^s (Y_2(u) P_2(u)) \pi_1^{\lambda(h_0)+1} \cdot v \]

\[ = (-1)^s \left( \pi_1^{\lambda(h_0)+1} + \pi_1^{\lambda(h_0)-1} P_2 + \pi_1^{\lambda(h_0)-2} P_3 + \pi_1^{\lambda(h_0)-3} P_4 + \ldots + \pi_1^{\lambda(h_0)-r} P_{r} \right) \cdot v, \]

which proves (ii) of the lemma. \( \square \)

**Proposition 2.1.** Let \( V \) be an integrable \( U \)-module. Let \( \lambda(h_0) \in \mathbb{Z}_{\geq 0} \) and \( 0 \neq v \in V_\lambda^+ \). Then:

(i) \( \pi_m \cdot v = 0, \ m > \frac{\lambda(h_0)}{2}, \)
Proof. Taking \( r \geq \frac{\lambda(h_0)}{2} + 1 \) and \( s = 2r \) in lemma 2.5 (i) gives us
\[
0 = \left( \sum_{\rho=0}^{r} \tau (D_{2r-2\rho}(0)) P_{\rho} \right) .v = P_r .v,
\]
since \( D_n(0) = 0 \) for \( n > 0 \) and \( D_0(0) = 1 \). This proves part (i) of the proposition.

Lemma 2.6 (i) gives us, for \( r = \frac{\lambda(h_0)}{2} + 1 \),
\[
\left( \sum_{m=0}^{\lambda(h_0)} y_{m+1} P_{\lambda(h_0)/2-m} \right) .v = 0.
\]
Let \( s \in \mathbb{Z} \). Acting on the left of this expression by \( x_{-s-1} \) and noting that \([x_{-s-1}, y_{m+1}] = 2\bar{h}_{m-s} \) and \( U(0)V_{\lambda}^+ \subseteq V_{\lambda}^+ \), we get
\[
\left( \sum_{m=0}^{\lambda(h_0)} \bar{h}_{m-s} P_{\lambda(h_0)/2-m} \right) .v = 0,
\]
Replacing \( s \) with \( \frac{\lambda(h_0)}{2} - s \) and using part (i) of the proposition gives us
\[
0 = \left( \sum_{m=0}^{\lambda(h_0)} \bar{h}_{m-s} P_{\lambda(h_0)/2-m} \right) .v = \left( \sum_{m=0}^{\lambda(h_0)} \bar{h}_{m-s} P_{\lambda(h_0)/2-m} \right) .v.
\]
for arbitrary \( s \in \mathbb{Z} \), which is part (ii).

For the remainder of this proof, let \( P(u) = P^+(u) \) and \( \tilde{P}(u) = P^{-}(u^{-1}) \), so that
\[
\tilde{P}(u) = \exp \left( -\sum_{k=1}^{\infty} \frac{\bar{h}_k}{k} u^{-k} \right).
\]
Then
\[
\frac{\tilde{P}'(u)}{\tilde{P}(u)} = -\sum_{k=1}^{\infty} \bar{h}_k u^k;
\]
\[
\frac{\tilde{P}'(u)}{\tilde{P}(u)} = \sum_{k=1}^{\infty} \bar{h}_k u^{-k}.
\]
and so

\[
\left( \frac{\lambda(h_0)}{2} - u \frac{\mathcal{P}'(u)}{\mathcal{P}(u)} + u \frac{\tilde{\mathcal{P}}'(u)}{\mathcal{P}(u)} \right) \cdot v = \left( \sum_{k=-\infty}^{\infty} \tilde{h}_k u^k \right) \cdot v = \tilde{H}(u) \cdot v
\]

for \( v \in V_h^+ \); i.e., as operators on \( V_h^+ \), we have

\[
\left( \frac{\lambda(h_0)}{2} - u \frac{\mathcal{P}'(u)}{\mathcal{P}(u)} + u \frac{\tilde{\mathcal{P}}'(u)}{\mathcal{P}(u)} \right) = \tilde{H}(u).
\]

Multiplying this relation by \( \mathcal{P}(u) \), and noting that \( \mathcal{P}(u) \) is invertible, gives us

\[
\mathcal{P}(u) \left( \frac{\lambda(h_0)}{2} - u \frac{\mathcal{P}'(u)}{\mathcal{P}(u)} + u \frac{\tilde{\mathcal{P}}'(u)}{\mathcal{P}(u)} \right) = \tilde{H}(u) \mathcal{P}(u) = 0,
\]

the last equality by part (ii) of the proposition. We then have

\[
\left( \frac{\mathcal{P}(u)}{\mathcal{P}(u)} \right)' = \left( \frac{\lambda(h_0)}{2} - u \frac{\mathcal{P}'(u)}{\mathcal{P}(u)} + u \frac{\tilde{\mathcal{P}}'(u)}{\mathcal{P}(u)} \right),
\]

and so

\[
\frac{\mathcal{P}(u)}{\mathcal{P}(u)} = \lambda(h_0) \frac{2}{\mathcal{P}(u)},
\]

where \( A \) is some operator on \( V_h^+ \) independent of \( u \). Equating coefficients of \( u^{\lambda(h_0)/2} \) shows that \( A = \mathcal{P}^{\lambda(h_0)} \) and then the equation (of operators on \( V_h^+ \))

\[
\mathcal{P}^{\lambda(h_0)} \tilde{\mathcal{P}}(u) = u^{-\lambda(h_0)/2} \mathcal{P}(u),
\]

which proves parts (iii) and (iv).

Lemma 2.5 (ii) with \( r > \lambda(h_0)/2 \), \( s = r \) gives us part (v). The proof of (vi), (vii), and (viii) is identical to that of (ii), (iii), and (iv), with \( \mathcal{P}^\pm, \tilde{H} \) replaced by \( \mathcal{P}_2^\pm \) and \( \tilde{H}_2 \), respectively.

The remaining lemmas prepare us for the final proposition of the section. For the remainder of this section, let \( V \) be an integrable representation of \( U \) and fix a vector \( v \in V_h^+ \).

**Lemma 2.7.** Let \( m, n, N \in \mathbb{Z} \). Then
\[(y_m, h_n) = \begin{cases} 
2y_m, & n = 0 \\
y_{m+n}, & n \neq 0, n \text{ even}; \\
-3y_{m+n}, & n \text{ odd}; 
\end{cases}\]

(ii)

\[[\mathcal{g}_{2N+1}, U(<)] = 0\]

Proof. Follows directly from the definitions of \(y_i, \mathcal{g}_m,\) and \(h_n\). \(\square\)
Lemma 2.8. Let $N \in \mathbb{Z}$. Then

(i) 
$$\gamma_N U(0).v \subseteq sp \{ \gamma_k U(0).v \}_{0 \leq k < \frac{\lambda(h_0)}{2}}$$

(ii) 
$$\gamma_{2N+1} U(0).v \subseteq sp \{ \gamma_{2k+1} U(0).v \}_{0 \leq k < \frac{\lambda(h_0)}{2}}.$$

Proof. Let $N \geq 0$. If $N \leq \frac{\lambda(h_0)}{2}$, then (i) is immediate. If $N > \frac{\lambda(h_0)}{2}$, then since $U(0).v \in V_\lambda^+$, a straightforward proof by induction using lemma 2.6 (i) proves the result.

Now, if $N < 0$, another induction will prove the result. Let $w = \mathcal{P}_{-\frac{\lambda(h_0)}{2}}.v \in V_\lambda^+$. We will prove the result first for $N = -1$. Lemma 2.6 (i) again gives us 

$$\left( \sum_{k=1}^{r} y_k \mathcal{P}_{r-k} \right) .w = 0,$$

and acting on the left of this relation by $h_{-2}$ yields 

$$0 = \left( \sum_{k=1}^{r} (y_k h_{-2} + [h_{-2}, y_k]) \mathcal{P}_{r-k} \right) .w = \left( \sum_{k=1}^{r} (y_k h_{-2} - y_{k-2}) \mathcal{P}_{r-k} \right) .w = \left( \sum_{k=1}^{r} y_k \mathcal{P}_{r-k} \right) h_{-2}.w + \left( \sum_{k=1}^{r} -y_{k-2} \mathcal{P}_{r-k} \right) .w = \left( \sum_{k=1}^{r} -y_{k-2} \mathcal{P}_{r-k} \right) .w,$$

since $h_{-2}w$ is still in $V_\lambda^+$. Taking $r = \frac{\lambda(h_0)}{2} + 1$, this gives us 

$$- \left( \sum_{k=1}^{r-1} y_{k-1} \mathcal{P}_{r-k-1} \right) .w = y_{-1} \mathcal{P}_{\frac{\lambda(h_0)}{2}}.w = y_{-1} \mathcal{P}_{\frac{\lambda(h_0)}{2}} \mathcal{P}_{-\frac{\lambda(h_0)}{2}}.v = y_{-1} \mathcal{P}_{\frac{\lambda(h_0)}{2}} \frac{\lambda(h_0)}{2}.v = y_{-1} \mathcal{P}_{0}.v = y_{-1}.v,$$

where we have used proposition 2.1 (iii) for the third equality. So we have 

$$y_{-1}.v = - \left( \sum_{k=1}^{r-1} y_{k-1} \mathcal{P}_{r-k-1} \right) .w \in sp \{ \gamma_k U(0).v \}_{0 \leq k < \lambda(h_0)+1},$$

as desired. For the inductive step, let $N < -1$. We begin again with $w = \mathcal{P}_{-\frac{\lambda(h_0)}{2}}.v \in V_\lambda^+$ and the relation 

$$\left( \sum_{k=1}^{r} y_k \mathcal{P}_{r-k} \right) .w = 0,$$
this time acting on the left by $h_{N-1}$. Let $[h_{N-1}, y_k] = c_N y_{N+k-1}$, where $c_N = -1, -2$ or 3, according to lemma 2.7 (i). This gives us

$$0 = \left( \sum_{k=1}^{r} (y_k h_{N-1} + [h_{N-1}, y_k]) P_{r-k} \right) \cdot w$$

$$= \left( \sum_{k=1}^{r} (y_k h_{N-1} + c_N y_{N+k-1}) P_{r-k} \right) \cdot w$$

$$= c_N \left( \sum_{k=1}^{r} y_{N+k-1} P_{r-k} \right) \cdot w.$$ 

Again taking $r = \lambda(h_0)/2 + 1$, we have

$$- \left( \sum_{k=1}^{r-1} y_{N+k} P_{r-k-1} \right) \cdot w = y_N P_{\lambda(h_0)/2} \cdot w$$

$$= y_N P_{\lambda(h_0)/2} \left( P_{\lambda(h_0)/2} \cdot w \right)$$

$$= y_N P_{\lambda(h_0)/2} \cdot \lambda(h_0) \cdot w$$

$$= y_N P_{\lambda(h_0)/2} \cdot v$$

$$= y_N \cdot v,$$

and then by the induction hypothesis we have

$$y_N \cdot v \in \text{sp} \{ y_k U(0) \cdot v \}_{0 \leq k < \lambda(h_0) + 1}.$$

Let $S_\lambda$ be the subspace of $V$ spanned by the elements

$$\overline{y}_{2m_1+1} \overline{y}_{2m_2+1} \cdots \overline{y}_{2m_r+1} y_1 y_2 \cdots y_n, U(0) \cdot v$$

for $0 < 2m_1 + 1 \leq \lambda(h_0) + 1, 0 \leq n_j \leq \lambda(h_0) + 1; r, s \geq 0$.

**Lemma 2.9.** For all $N \in \mathbb{Z}$, $\overline{y}_{2N+1} S_\lambda \subseteq S_\lambda$.

**Proof.** This follows from lemmas 2.7 (ii) and 2.8 (ii). Let $w \in U(0) \cdot v, 0 \leq 2m_i \leq \lambda(h_0)/2, 0 \leq n_j \leq \lambda(h_0) + 1$. Then

$$\overline{y}_{2N+1} \left( \overline{y}_{2m_1+1} \cdots \overline{y}_{2m_r+1} y_1 \cdots y_n \right) \cdot w$$

$$= \overline{y}_{2m_1+1} \cdots \overline{y}_{2m_r+1} y_1 \cdots y_n \left( \overline{y}_{2N+1} \cdot w \right)$$

$$\in \text{sp} \{ \overline{y}_{2m_1+1} \cdots \overline{y}_{2m_r+1} y_1 \cdots y_n, U(0) \cdot w \}_{0 \leq 2m \leq \lambda(h_0)/2}$$

$$= \text{sp} \{ \overline{y}_{2m+1} \overline{y}_{2m+1} \cdots \overline{y}_{2m+1} y_1 \cdots y_n, U(0) \cdot w \}_{0 \leq 2m \leq \lambda(h_0)/2}$$

$$\subseteq S_\lambda.$$

**Lemma 2.10.** For all $N \in \mathbb{Z}, 0 \leq n_1 \leq \lambda(h_0) + 1, s \geq 0$,

$$y_N (y_{n_1} \cdots y_n, U(0) \cdot v) \subseteq S_\lambda.$$

**Proof.** By induction on $s$. Let $N \in \mathbb{Z}, 0 \leq n_1 \leq \lambda(h_0) + 1$. Then
The first expression is contained in $S_\lambda$ by lemma 2.8 (i), while the second expression is contained in $S_\lambda$ by lemmas 2.8 (ii) and 2.9.

Now let $0 \leq n_1, \ldots, n_s, n_{s+1} \leq \lambda(h_0) + 1$; $N \in \mathbb{Z}$. Then

$$y_n y_{n+1} y_n \cdots y_{n_1} U(0).v = \begin{cases} y_{n_1} y_N U(0).v & \text{if } N \equiv n_1 \pmod{2}, \\ (y_{n_1} y_N + \overline{y}_{N+n_1}) U(0).v & \text{if } N \not\equiv n_1 \pmod{2}. \end{cases}$$

In the first case, the expression is contained in $S_\lambda$ by the induction hypothesis. In the second case, we have

$$\begin{align*}
(y_{n_1} y_N + \overline{y}_{N+n_1}) y_n \cdots y_{n_1} U(0).v &= y_{n_1} y_N y_n \cdots y_{n_1} U(0).v \\
&\quad + \overline{y}_{N+n_1} y_n \cdots y_{n_1} U(0).v.
\end{align*}$$

The first summand on the right is contained in $S_\lambda$ again by the induction hypothesis, while the second is contained in $S_\lambda$ by lemma 2.9.

**Lemma 2.11.** $U(<)U(0).v \subseteq S_\lambda$.

**Proof.** This follows from lemmas 2.9 and 2.10. □

**Proposition 2.2.** Let $V$ be an integrable $U$-module, let $\lambda \in \mathcal{P}_+$ and let $0 \neq v_+ \in V^{\lambda+}$ be such that $V = U.v_+$.

(i) If $V_\mu \neq 0$, then $\mu = \lambda - \eta$ for some $\eta \in Q_+$ such that $V^{fin}(\lambda)_{\lambda-\eta} \neq 0$.

(ii) $V$ is spanned by the elements

$$\overline{y}_{2m_1+1} \overline{y}_{2m_2+1} \cdots \overline{y}_{2m_r+1} y_n y_{n_2} \cdots y_{n_1} U(0).v_+,$$

for $0 < 2m_i + 1 \leq \frac{\lambda(h_0)}{2} + 1$, $0 \leq n_j \leq \lambda(h_0) + 1$; $r, s \geq 0$.

**Proof.** Since $U(>)_+v_+ = 0$, it follows that $V = U(<)U(0).v_+$ and hence $V_\mu \neq 0 \implies \mu = \lambda - \eta$,

for some $\eta \in Q^+$. Choose $\sigma \in W$ such that $\sigma(\lambda - \eta) \in \mathcal{P}_+$. Since $V$ is integrable, $V_{\sigma(\mu)} \neq 0$, hence $\sigma(\lambda - \eta) = \lambda - \eta'$ for some $\eta' \in Q_+$. This implies that $V^{fin}(\lambda)_{\sigma(\lambda-\eta)} \neq 0$, and hence that $V^{fin}(\sigma(\lambda-\eta)) \neq 0$.

Since $V = U(<)U(0).v_+$, (ii) follows immediately from lemma 2.11. □

3. **Maximal Integrable and Maximal Finite-Dimensional Modules**

In this section we define, for every $\lambda \in 2\mathcal{P}_+$ an integrable $U$-module $W(\lambda)$. Further, for any polynomial $\pi(u)$ in an indeterminate $u$ with constant term $1$ and degree $\lambda(h_0)$, we define a finite-dimensional quotient $U$-module $W(\pi)$ of $W(\lambda)$.

For $\lambda \in 2\mathcal{P}_+$, let $I_\lambda$ be the left ideal in the subalgebra $U(<)U(0)$ of $U$ generated by the following elements:
Remark. In the original paper, instead of \( U(<)U(0) \) you have \( U(<)U(0)U(h) \). I'm not sure why the \( U(h) \) is included here, since it's contained in \( U(0) \).

Let \( \tilde{I}_\lambda \) be the left ideal in \( U \) generated by \( I_\lambda \) and \( x_m, \bar{x}_{2n+1} \) for all \( m, n \in \mathbb{Z} \). Set

\[
W(\lambda) = U/\tilde{I}_\lambda.
\]

Clearly, \( W(\lambda) \) is a left \( U \)-module through left multiplication. Let \( w_\lambda \) be the image of 1 in \( W(\lambda) \). Then, \( \mathbf{U}(>)_+ w_\lambda = 0, \; W(\lambda) = U.w_\lambda \).

Since \( \tilde{I}_\lambda U(0) \subset \tilde{I}_\lambda \), we can do regard \( W(\lambda) \) as a right \( U(0) \)-module as well. For \( \eta \in Q^+ \), we set

\[
W(\lambda)[\eta] = W(\lambda)\lambda^{-\eta}.
\]

Clearly \( W(\lambda)[\eta] \) is a right \( U(0) \)-module for all \( \eta \in Q^+ \) and we have

\[
W(\lambda) = \bigoplus_{\eta \in Q^+} W(\lambda)[\eta]
\]
as right \( U(0) \)-modules.

Let \( I_\lambda(0) = I_\lambda \cap U(0) \). It is easy to see that

\[
U(0)/I_\lambda(0) \cong \mathbb{C}[P_m, P^{-1}_{\lambda(h_0)} : 1 \leq m \leq \lambda(h_0)].
\]

In particular,

\[
W(\lambda)[0] \cong \mathbb{C}[P_m, P^{-1}_{\lambda(h_0)} : 1 \leq m \leq \lambda(h_0)]
\]
as right \( U(0) \)-modules. It follows immediately from proposition (????) that, for all \( \eta \in Q^+ \), \( W(\lambda)[\eta] \) is a finitely-generated \( U(0) \)-module.

Next, let \( \pi(u) \) be a polynomial in an indeterminate \( u \) with constant term 1, and define an element \( \lambda_\pi \in (h_0)^* \) by setting \( \lambda_\pi(\eta_H) = \deg \pi \). Set

\[
\pi^+(u) = \pi(u), \quad \pi^-(u) = \frac{u^{\deg \pi(u-1)} - u^{-\deg \pi(u-1)}}{u^{\deg \pi(u-1)}|_{u=0}}.
\]

Let \( I_\pi(0) \) be the maximal ideal in \( U(0) \) generated by

\[
(P^\pm(u) - \pi^\pm(u))_s \quad (s \geq 0),
\]
and let \( C_\pi = U(0)/I_\pi(0) \) be the one-dimensional \( U(0) \)-module.

Set

\[
W(\pi) = W(\lambda_\pi) \otimes_{U(0)} C_\pi.
\]
Then, \( W(\pi) \) is a left \( U \)-module with \( x \in U \) acting as \( x \otimes 1 \). Let \( w_\pi \) be the image of 1 in \( W(\pi) \). Note that \( P^\pm(u).w_\pi = \pi^\pm(u)w_\pi \). The assignment \( \lambda_\pi \mapsto w_\pi \) extends to a surjective \( U \)-module homomorphism \( W(\lambda_\pi) \rightarrow W(\pi) \).

**Theorem 1.**
(i) Let \( \lambda \in P_1 \). Then, \( W(\lambda) \) is an integrable \( U \)-module.
(ii) Let \( \pi \) be a polynomial with constant term one. Then, \( W(\pi) \) is a finite-dimensional \( U \)-module.

**Proof.** We must show that the elements \( x_0, y_0, \bar{\pi}, 1, \bar{y}_1 \) all act locally nilpotently on \( W(\lambda) \).

**Lemma 3.1.** Let \( A \) be an associative algebra, \( a, b \in A \), \( (ada)(b) = [a, b], N \geq 0 \).

Then

\[
a^N b = \sum_{j=0}^{N} \binom{N}{j} (ada)^j(b)a^{N-j}.
\]

**Proof.** Proof by induction. The statement is immediate for \( N = 0 \).

\[
a^{N+1}b = a a^N b
\]

\[
= a \left( \sum_{j=0}^{N} \binom{N}{j} (ada)^j(b)a^{N-j} \right)
\]

\[
= \sum_{j=0}^{N} \binom{N}{j} ((ada)^j(b)a + (ada)^{j+1}(b))a^{N-j}
\]

\[
= \sum_{j=0}^{N} \binom{N}{j} (ada)^j(b)a^{N-j+1} + \sum_{j=1}^{N+1} \binom{N}{j-1} (ada)^j(b)a^{N-j+1}
\]

\[
= \sum_{j=0}^{N+1} \binom{N+1}{j} (ada)^j(b)a^{N+1-j}.
\]

\( \square \)

**Lemma 3.2.** Let \( m \in \mathbb{Z}; k > 0 \). For any \( a \in L^\sigma(A_2), n > 0 \), set \( a^{-n} = 0 \). Then

\[
y_{k}^{m} y_{m} \in U y_{k}^{m-1} \quad y_{k}^{m} \bar{y}_{2m+1} \in U y_{k/2}^{m},
\]

\[
\bar{y}_{k}^{m} y_{m} \in U \bar{y}_{k}^{m-1} \quad \bar{y}_{k}^{m} \bar{y}_{2m+1} \in U \bar{y}_{k/2}^{m},
\]

\[
x_{k}^{m} x_{m} \in U x_{k}^{m-3} \quad x_{k}^{m} \bar{x}_{2m+1} \in U x_{k}^{m-4},
\]

\[
\bar{x}_{k}^{m} x_{m} \in U \bar{x}_{k}^{m-1} \quad \bar{x}_{k}^{m} \bar{x}_{2m+1} \in U \bar{x}_{k}^{m-1}.
\]

**Lemma 3.3.** \( U(0).w_\lambda \subseteq W(\lambda)_+ \).

**Proof.** It is easy to verify that, for \( m \in \mathbb{Z}, x_m U(0) \subseteq U(0)x_m \) and \( \bar{x}_{2m+1} U(0) \subseteq U(0) \bar{x}_{2m+1} \). Therefore \( x_m U(0).w_\lambda = \bar{x}_{2m+1} U(0).w_\lambda = 0 \). And since \( [h_0, U(0)] = 0 \) and \( w_\lambda \in W(\lambda)_+ \), it follows that \( U(0).w_\lambda \subseteq W(\lambda)_+ \).

\( \square \)

**Lemma 3.4.** \( y_0^{\lambda(h_0)+1} U(0).w_\lambda = \bar{y}_1^{\lambda(h_0)+1} U(0).w_\lambda = 0 \).

**Proof.** Let \( v \in U(0).w_\lambda \). By the above lemma, \( x_0.v = 0 \) and \( h_0.v = \lambda(h_0)v; \lambda(h_0) \in \mathbb{Z} \). Standard \( sl_2 \) results then tell us that \( y_0^{\lambda(h_0)+1}.v = 0 \). Similarly, since \( \bar{x}_{-1}.v = 0 \), \( h_{-1}^{\frac{1}{2}}v = \lambda(h_{-1}^{\frac{1}{2}})v \) and \( \lambda(\frac{h_{-1}}{2}) \in \mathbb{Z} \), it follows that \( \bar{y}_1^{\lambda(h_0)+1}.v = 0 \).

\( \square \)
Let 

\[ S_1 = \text{sp} \left\{ y_{m_k} \ldots y_{m_1} U(0).w_\lambda \right\} \]

for \( k \geq 1, m_i \in \mathbb{Z} \). We claim that all four generators \( x_0, y_0, \pi_{-1}, \bar{y}_1 \) act locally nilpotently on \( S_1 \). Since \([\bar{y}_1, U(\langle \cdot \rangle)] = 0\), \( \bar{y}_1^{\lambda(h_0)}+1 S_1 = 0 \) by the above lemma. Proving the claim for the other three generators requires induction on \( k \).

We begin by considering the subspace \( y_m U(0).w_\lambda \) for some \( m \in \mathbb{Z} \). By lemma (??),

\[ y_0^{\lambda(h_0)+2} y_m \in U y_0^{\lambda(h_0)+1}, \]

and so

\[ y_0^{\lambda(h_0)+2} y_m U(0).w_\lambda \subseteq U y_0^{\lambda(h_0)+1} U(0).w_\lambda = 0. \]

By the same lemma we also have

\[ x_0^4 y_m U(0).w_\lambda = x_{-1}^2 y_m U(0).w_\lambda = 0. \]

Now let \( k > 1; m_1, \ldots, m_k \in \mathbb{Z} \). By the induction hypothesis there exists \( N \in \mathbb{Z}^+ \) such that

\[ y_0^N y_{m_{k-1}} \ldots y_{m_1} U(0).w_\lambda = 0. \]

Then, again using lemma (??),

\[ y_0^{N+1} y_{m_k} y_{m_{k-1}} \ldots y_{m_1} U(0).w_\lambda \subseteq U y_0^N y_{m_{k-1}} \ldots y_{m_1} U(0).w_\lambda \]

\[ = 0. \]

Therefore \( y_0 \) acts locally nilpotently on \( S_1 \). The inductive proof that this is so for \( x_0 \) and \( \pi_{-1} \) as well is similar.

Now let

\[ S_2 = \text{sp} \left\{ \bar{y}_{2n_1+1} \ldots \bar{y}_{2n_k+1} S_1 \right\} \]

for \( k > 0, n_i \in \mathbb{Z} \). We will now show that \( x_0, y_0, \pi_{-1}, \bar{y}_1 \) act locally nilpotently on \( S_2 \). This will conclude the proof that \( W(\lambda) \) is integrable, since

\[ W(\lambda) = U(\langle \cdot \rangle) U(0).w_\lambda \subseteq S_2. \]

Again, since \([\bar{y}_1, U(0)] = 0\), we have

\[ \bar{y}_1^{\lambda(h_0)} \bar{y}_{2n_1+1} \ldots \bar{y}_{2n_k+1} S_1 = \bar{y}_{2n_1+1} \ldots \bar{y}_{2n_k+1} \bar{y}_1^{\lambda(h_0)+1} S_1 = 0, \]

so \( \bar{y}_1^{\lambda(h_0)+1} S_2 = 0 \). The inductive proof that \( x_0, y_0, \pi_{-1} \) also act locally nilpotently on \( S_2 \) is similar to that used above for \( S_1 \).

For the proof of (ii), first note that, by lemma 2.2 (ii), \( W(\pi) \) is spanned by

\[ \bar{y}_{2m_1+1} \bar{y}_{2m_2+1} \ldots \bar{y}_{2m_r+1} y_{n_1} y_{n_2} \ldots y_{n_s} \otimes 1, \]

for \( 0 < 2m_i + 1 \leq \frac{\lambda(h_0)}{2} + 1, 0 \leq n_j \leq \lambda(h_0) + 1; r, s \geq 0 \). But since \( W(\pi) \) is a quotient of \( W(\Lambda_\infty) \), hence integrable, there is an upper bound on the sum \( r + s \). Therefore the above spanning set is finite, and so \( \text{dim}(W(\pi)) < \infty \).

The modules \( W(\lambda) \) and \( W(\pi) \) have certain universal properties.

**Proposition 3.1.**

(i) Let \( V \) be any integrable \( U \)-module generated by a non-zero element \( v \in V_\lambda^+ \).

Then, \( V' \) is a quotient of \( W(\lambda) \).
(ii) Let $V$ be a quotient $U$-module of $W(\lambda)$, and assume that $\dim V_\lambda = 1$. Then, $V$ is a quotient of $W(\pi)$ for some choice of $\pi$.

(iii) Let $V$ be finite-dimensional $U$-module generated by a vector $v \in V_\lambda^+$ and such that $\dim V_\lambda = 1$. Then, $V$ is a quotient of $W(\pi)$ for some $\pi$.

Proof. Let $V = U.v$ be integrable, $v \in V_\lambda^+$. We must show $\tilde{I}_\lambda v = 0$. Since $\langle y_0, h_0, x_0 \rangle.v$ and $\langle y_1, h_0, x_0 \rangle . v$ are both finite-dimensional highest weight representations of $sl_2$ with highest weight $\lambda$, it follows that $y_0^{\lambda(h_0)} + k.v = \frac{y_0^{\lambda(h_0)} + k}{y_0^{\lambda(h_0)}}$ for $k > 0$. By proposition 2.1, we also have $P_m.v = \pi_m v$, for some scalars $\pi_m \in \mathbb{C}$. By proposition 2.1, it follows that $\pi_m = 0$ for $|m| > \frac{\lambda(h_0)}{2}$. By proposition 2.1(vii) shows that $P(\pm u).v = \pi(\pm u)v$.

This shows that $V$ is a quotient of $W(\lambda)$ quotient.

To prove (ii), let $v \neq 0$ be the image of $w_\lambda$ in $V$. Notice that $\dim V_\lambda = 1$ implies that $P_m.v = \pi_m v$, for some scalars $\pi_m \in \mathbb{C}$. By proposition 2.1, it follows that $\pi_m = 0$ for $|m| > \frac{\lambda(h_0)}{2}$.

Set

$$\pi(u) = \sum_{k=0}^{\lambda(h_0)} \pi_k u^k.$$ 

Then $\pi(u)$ is a polynomial with constant term 1 and Proposition 2.1(vii) shows that

$$P(\pm u).v = \pi(\pm u)v.$$ 

This shows that $V$ is a quotient of $W(\pi)$, where $\pi(u)$ is the polynomial defined above.

Finally, (iii) follows immediately from (i) and (ii).

4. A tensor product theorem for $W(\pi)$

The main result of this section is the following theorem.

**Theorem 2.** Let $\pi(\pm u)$ and $\tilde{\pi}(\pm u)$ be coprime polynomials with constant term 1; i.e., if

$$\pi(u) = \prod_{i=0}^{M} (u - a_i)^{r_i}, \quad \tilde{\pi}(u) = \prod_{i=0}^{N} (u - b_i)^{s_i},$$ 

then $|a_i| \neq |b_j|$ for any $i, j$. Then

$$W(\pi \tilde{\pi}) \cong W(\pi) \otimes W(\tilde{\pi})$$ 

as $U$-modules.
**Definition 4.1.** Let $\pi(u)$ be a polynomial with constant term 1, and define $\pi_2(u) = \pi(u)\pi(-u)$. Define $M(\pi)$ to be the left $U$-module obtained by taking the quotient of $U$ by the left ideal generated by the following:

\[
h_0 - 2\deg \pi, \quad x_k, \quad \pi_{2k+1} \quad (k \in \mathbb{Z}), \quad \left(\mathcal{P}^\pm(u) - \pi^\pm(u)\right)_s \quad (s \geq 0), \quad \left(\tilde{\mathcal{Y}}(u)\pi(u)\right)_s, \quad \left(\tilde{\mathcal{Y}}_2(u)\pi_2(u)\right)_s \quad (s \in \mathbb{Z})
\]

Let $m_\pi$ be the image of 1 in $M(\pi)$. It is clear that, for all $s \in \mathbb{Z}$,

\[
\left(\pi(u)\tilde{H}(u)\right)_s m_\pi = \left(\pi_2(u)\tilde{H}_2(u)\right)_s m_\pi = 0.
\]

Let

\[
\pi_2(u) = \sum_{k=0}^{\lambda(h_0)} \pi_{2k}u^{2k}.
\]

Then the coefficients of $\pi_2(u)$ can be expressed using the coefficients of $\pi(u)$ via

\[
\pi_{2,2k} = \sum_{i+j=2k} (-1)^i \pi_i \pi_j.
\]

It is also clear that

\[
\left(\tilde{\mathcal{Y}}(u)\pi(u)\right)_s = \sum_{k=0}^{\deg \pi} y_{s-k}\pi_k, \\
\left(\tilde{\mathcal{Y}}_2(u)\pi_2(u)\right)_s = \begin{cases} 0, & \text{seven} \\
\sum_{k=0}^{\lambda(h_0)} y_{s-2k}\pi_{2,2k}, & \text{sodd}
\end{cases}
\]

Set $\lambda_\pi(h_0) = 2\deg \pi$.

**Lemma 4.1.** Let $\alpha_0$ be twice the fundamental weight corresponding to $h_0$, i.e., $\alpha_0(h_0) = 2$. We have

\[
M(\pi) = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} M(\pi)_{\lambda_r - r\alpha_0},
\]

and $\dim M(\pi)_{\lambda_r - r\alpha_0} < \infty$.

**Proof.** The sets

\[
L_\pi = \{y_r : 0 \leq r < \lambda_\pi\left(\frac{h_0}{2}\right)\} \cup \{y_{2r+1} : 0 \leq r < \lambda_\pi\left(\frac{h_0}{2}\right)\}, \\
L^\pi = \{\left(\tilde{\mathcal{Y}}(u)\pi(u)\right)_s : s \in \mathbb{Z}\} \cup \{\left(\tilde{\mathcal{Y}}_2(u)\pi_2(u)\right)_s : s \in \mathbb{Z}\}
\]

contain all basis elements $y_m, y_{2n+1}$ ($m, n \in \mathbb{Z}$) of $L^\pi(A_2)(\mathfrak{n}^-)$, and so by the PBW basis theorem we can write

\[
\mathcal{U}(\mathfrak{g}) = \mathcal{U}_\pi \mathcal{U}^\pi,
\]
where $U_{\pi}$ (resp. $U^\pi$) consists of ordered monomials from $L_{\pi}$ (resp. $L^\pi$). The relations
\[
\left(\tilde{Y}(u)_{\pi}(u)\right)_s.m_{\pi} = \left(\tilde{Y}_2(u)_{\pi_2}(u)\right)_s.m_{\pi} = 0
\]
for all $s \in \mathbb{Z}$ imply that $(U^\pi)_+.m_{\pi} = 0$. A further use of the PBW theorem now shows that

\[
M(\pi) \cong U_{\pi}
\]
as vector spaces.

**Remark.** Requires proof!!

Moreover, this isomorphism takes $M(\pi)_{\lambda_{\pi'} - \eta}$ to
\[
U_{\pi}(\eta) = \{x \in U_{\pi} : [h_0, x] = \eta(h_0)x\}.
\]
Since this space is clearly finite-dimensional, the second statement of the lemma follows.

**Lemma 4.2.** The $U$-module $W(\pi)$ is a quotient of $M(\pi)$, and any finite-dimensional quotient of $M(\pi)$ is a quotient of $W(\pi)$.

**Proof.** Let $V$ be a finite-dimensional quotient of $M(\pi)$, let $v \in V$ be the image of $m_{\pi}$, and let $\lambda = \lambda_{\pi}$. Then, $\dim V = \dim M(\pi)_{\lambda} = 1$, so by proposition 3.1(ii), $V$ is a quotient of some $W(\tilde{\pi})$ with $\lambda = \lambda_{\tilde{\pi}}$. Since $\dim W(\tilde{\pi})_{\lambda} = 1$, $v$ is a scalar multiple of the image of $w_{\pi'}$. But then by comparing the action of $P^{\pm}(u)$ on $w_{\pi'}$ and on $m_{\pi}$, we see that $\pi = \tilde{\pi}$.

**Remark.** This requires some proof as well.

To show that $W(\pi)$ is a quotient of $M(\pi)$, it is clear from the definitions of these modules that we only need to show that
\[
\left(\tilde{Y}(u)_{\pi}(u)\right)_s.w_{\pi} = \left(\tilde{Y}_2(u)_{\pi_2}(u)\right)_s.w_{\pi} = 0
\]
for all $s \in \mathbb{Z}$.

Since $W(\pi)$ is a $\lambda_{\pi}$-highest weight module, generated by $w_{\pi} \in (W(\pi))_{\lambda_{\pi}}$, lemma 2.6 gives us
\[
(4.1) \quad \left(\sum_{k=0}^{r-1} y_{r-k-\pi_k}\right).w_{\pi} = 0,
\]
\[
(4.2) \quad \left(\sum_{k=0}^{r-1} y_{2r-2k-1-\pi_2,2k}\right).w_{\pi} = 0,
\]
where $\lambda = \lambda_{\pi}$ and $r = \frac{\lambda(h_0)}{2} + 1$. Let $s \in \mathbb{Z}$. Acting by $h_{s-r}$ by left multiplication on (4.1) gives us
\[
\left(\tilde{Y}(u)_{\pi}(u)\right)_s.m_{\pi} = \left(\sum_{k=0}^{\lambda(h_0)} y_{s-k-\pi_k}\right).w_{\pi} = 0,
\]
and similarly acting by $b_{2s-2r}$ on (4.2) gives us

$$
\left(\tilde{Y}_2(u)\pi_2(u)\right)_{2s-1}.m_\pi = \left(\sum_{k=0}^{\frac{m}{2}} \tilde{g}_{2s-2k-1}\pi_{2,2k}\right).w_{\pi} = 0.
$$

This completes the proof of the lemma, since all even coefficients of $\tilde{Y}_2(u)\pi_2(u)$ are zero. □

Denote by $\Delta : U \to U \otimes U$ the comultiplication of $U$ defined by extending to an algebra homomorphism the assignment $x \mapsto x \otimes 1 + 1 \otimes x$, for all $x \in L^*(A_2)$. The following lemma is proved in ???.

**Lemma 4.3.**

$$\Delta(P^\pm) = P^\pm \otimes P^\pm,$$

where

$$P^\pm \otimes P^\pm = \sum_{k,m \geq 0} (P_{\pm k} \otimes P_{\pm m})u^{k+m}.$$

□

Theorem 2 is now clearly a consequence of the following proposition.

**Proposition 4.1.** Assume that $\pi(\pm u)$ and $\tilde{\pi}(\pm u)$ are coprime polynomials with constant term 1. Then:

(i) $M(\pi \tilde{\pi}) \cong M(\pi) \otimes M(\tilde{\pi})$;

(ii) every finite-dimensional quotient $U$-module of $M(\pi) \otimes M(\tilde{\pi})$ is a quotient of $W(\pi) \otimes W(\tilde{\pi})$.

**Proof.** Set $\lambda = \lambda_\pi + \lambda_{\tilde{\pi}}$. It is clear from the proof of Lemma 4.1 that, for all $\eta \in Q_+$, we have

$$M(\pi \tilde{\pi})_{\lambda-\eta} \cong (M(\pi) \otimes M(\tilde{\pi}))_{\lambda-\eta}$$

as (finite-dimensional) vector spaces. To prove (i), it therefore suffices to prove that there exists a surjective homomorphism of $U$-modules $M(\pi \tilde{\pi}) \to M(\pi) \otimes M(\tilde{\pi})$. It is easy to see, using Lemma 4.3, that the element $m_\pi \otimes m_{\tilde{\pi}}$ satisfies the defining relations of $M(\tilde{\pi})$, so there exists a $U$-module map $M(\pi \tilde{\pi}) \to M(\pi) \otimes M(\tilde{\pi})$ that sends $m_\pi \otimes m_{\tilde{\pi}}$. Thus, to prove (i), we must show that, if $\pi_i$ and $\tilde{\pi}_j$ have no roots in common, the element $m_\pi \otimes m_{\tilde{\pi}}$ generates $M(\pi) \otimes M(\tilde{\pi})$ as a $U$-module.

Set

$$N = U.(m_\pi \otimes m_{\tilde{\pi}}).$$

Assume that, for all $0 \leq r, \tilde{r} < s$, we have

$$M(\pi)_{\lambda_\pi - r\alpha_0} \otimes M(\tilde{\pi})_{\lambda_{\tilde{\pi}} - \tilde{r}\alpha_0} \subset N.$$

We shall prove that

$$(y_k.M(\pi)_{\lambda_\pi - r\alpha_0}) \otimes M(\tilde{\pi})_{\lambda_{\tilde{\pi}} - \tilde{r}\alpha_0} \subset N;$$

$$M(\pi)_{\lambda_\pi - r\alpha_0} \otimes (y_k.M(\tilde{\pi})_{\lambda_{\tilde{\pi}} - \tilde{r}\alpha_0}) \subset N,$$

for all $k \in \mathbb{Z}$. This will prove that

$$M(\pi)_{\lambda_\pi - r\alpha_0} \otimes M(\tilde{\pi})_{\lambda_{\tilde{\pi}} - \tilde{r}\alpha_0} \subset N,$$
when \( r \leq s, \tilde{r} \leq s \), and hence, by induction on \( s \), that \( N = M(\pi) \otimes M(\tilde{\pi}) \).

Note that, since \( \pi(\pm u) \) and \( \tilde{\pi}(\pm u) \) are coprime, \( \pi_2(u) \) and \( \tilde{\pi}_2(u) \) are coprime. Therefore we can choose polynomials \( R(u), \tilde{R}(u) \) such that

\[
R(u)\pi_2(u) + \tilde{R}(u)\tilde{\pi}_2(u) = 1.
\]

Also note that, for all \( s \in \mathbb{Z} \),

\[
(R(u)\pi_2(u)Y(u))_s.m_\pi = 0,
\]

\[
(\tilde{R}(u)\tilde{\pi}_2(u)Y(u))_s.m_\bar{\pi} = 0;
\]

the coefficients of \( R(u)\pi_2(u)Y(u) = R(u)\pi(-u)\pi(u)Y(u) \) being \( R(u)\pi(-u) \)–linear combinations of the coefficients of \( \pi(u)Y(u) \), and similarly for \( \tilde{R}(u)\tilde{\pi}_2(u)Y(u) \).

**Lemma 4.4.** Let \( P(u) \in \mathbb{C}\mathbb{[u^{\pm 2}]} \), \( s, j \in \mathbb{Z} \). Then

\[
\left[ (P(u)\tilde{Y}(u))_s, Y_j \right] = \begin{cases} 4(-1)^{s+j} (P(u)\tilde{Y}_2(u))_{s+j}, & s \not\equiv j \mod 2; \\ 0, & s \equiv j \mod 2. \end{cases}
\]

**Lemma 4.5.** Let \( r \in \mathbb{Z}, s \in 2\mathbb{Z} + 1 \). Then

\[
\left( \pi_2(u)\tilde{Y}(u) \right)_s.M(\pi) = 0,
\]

\[
\left( \pi_2(u)\tilde{Y}_2(u) \right)_s.M(\pi) = 0.
\]

**Proof.** For the first equality, we will prove by induction on \( r \) that

\[
\left( \pi_2(u)\tilde{Y}(u) \right)_s.M(\pi)_{\lambda_n - r\alpha} = 0.
\]

Since \( \left( \pi_2(u)\tilde{Y}(u) \right) = \left( \pi(-u)\pi(u)\tilde{Y}(u) \right) \), the statement is immediately true for \( r = 0 \) by the definition of \( M(\pi) \). For the inductive step, let \( k_1, \ldots, k_n \in \mathbb{Z} \). Then

\[
\left( \pi_2(u)\tilde{Y}(u) \right)_s.y_{k_n}y_{k_{n-1}} \cdots y_{k_1}.m_\pi = y_{k_n}\left( \pi_2(u)\tilde{Y}(u) \right)_s.y_{k_{n-1}} \cdots y_{k_1}.m_\pi + \left( \pi_2(u)\tilde{Y}(u) \right)_s.y_{k_{n-1}} \cdots y_{k_1}.m_\pi.
\]

The first summand is zero by the induction hypothesis. If \( s \equiv k_n \mod 2 \), the second summand is zero as well. If \( s \not\equiv k_n \mod 2 \), the second summand is

\[
4(-1)^{s+1} \left( \pi_2(u)\tilde{Y}_2(u) \right)_{s+k_n}y_{k_{n-1}} \cdots y_{k_1}.m_\pi
\]

\[
= 4(-1)^{s+1}y_{k_{n-1}} \cdots y_{k_1}\left( \pi_2(u)\tilde{Y}_2(u) \right)_{s+k_n}.m_\pi
\]

\[
= 0,
\]

again by the definition of \( M(\pi) \).

The second equality is immediate since \([y_{\alpha}, U(<)] = 0\). \( \square \)
Lemma 4.6. Let \( r \in \mathbb{Z}, s \in 2\mathbb{Z} + 1, w \in M(\pi), \tilde{w} \in M(\hat{\pi}) \). Then

\[
(4.5) \quad \left( \bar{R}(u)\tilde{\pi}_2(u)\bar{Y}(u) \right)_r \cdot w \otimes \tilde{w} = (y_r \cdot w) \otimes \tilde{w},
\]

\[
(4.6) \quad \left( \bar{R}(u)\tilde{\pi}_2(u)\bar{Y}_2(u) \right)_s \cdot w \otimes \tilde{w} = (\bar{y}_s \cdot w) \otimes \tilde{w},
\]

\[
(4.7) \quad \left( R(u)\pi_2(u)\bar{Y}(u) \right)_r \cdot w \otimes \tilde{w} = w \otimes (y_r \cdot \tilde{w}),
\]

\[
(4.8) \quad \left( R(u)\pi_2(u)\bar{Y}_2(u) \right)_s \cdot w \otimes \tilde{w} = w \otimes (\bar{y}_s \cdot \tilde{w}).
\]

Proof. We will prove only (4.5).

\[
\left( \bar{R}(u)\tilde{\pi}_2(u)\bar{Y}(u) \right)_r \cdot (w \otimes \tilde{w}) = \left( \bar{R}(u)\tilde{\pi}_2(u)\bar{Y}(u) \right)_r \cdot w \otimes \tilde{w} + w \otimes \left( \bar{R}(u)\tilde{\pi}_2(u)\bar{Y}(u) \right)_r \cdot \tilde{w} = w \otimes \left( 1 - R(u)v_2(w) \bar{Y}(u) \right)_r \cdot \tilde{w} = w \otimes \bar{Y}(u)_r \cdot \tilde{w} = w \otimes y_r \cdot \tilde{w}.
\]

Taking \( w \in M(\pi)_{\lambda_{z-\eta}} \) and \( \tilde{w} \in M(\hat{\pi})_{\lambda_{z-\eta}} \), so that \( w \otimes \tilde{w} \in N \), it follows that \( w \otimes y_r \cdot \tilde{w} \in N \) for all \( m \in \mathbb{Z} \).

Suppose that \( V \) is a finite-dimensional quotient of \( M(\pi) \otimes M(\hat{\pi}) \) with kernel \( K \). We shall prove that, for all \( r > \lambda(\frac{h_0}{2}), \tilde{r} > \lambda(\frac{h_0}{2}) \),

\[
(4.9) \quad \langle y^{(2r)}_0 \rangle . m_\pi \otimes M(\hat{\pi}), \quad \langle \bar{y}^{(r)}_1 \rangle . m_\pi \otimes M(\hat{\pi}) \subset K,
\]

\[
(4.10) \quad M(\pi) \otimes \langle y^{(2r)}_0 \rangle . m_\hat{\pi}, \quad M(\pi) \otimes \langle \bar{y}^{(r)}_1 \rangle . m_\hat{\pi} \subset K,
\]

where \( \langle y \rangle \) is the left ideal in \( U \) generated by \( y \in U \). Since the sum of these subspaces is the kernel of the quotient map

\[ M(\pi) \otimes M(\hat{\pi}) \rightarrow W(\pi) \otimes W(\hat{\pi}), \]

**Remark.** This needs proof!

it follows that \( V \) is a quotient of \( W(\pi) \otimes W(\hat{\pi}) \), which proves part (ii).

To prove the containments (4.3), (4.4), it suffices to prove that

\[ (x^{-}_{i,m})^{r+1} . m_\pi \otimes m_{\hat{\pi}} \in K, \]

where \( r_i = \deg \pi_i = \lambda_\pi(h_i) \). Since \( V \) is finite-dimensional, the element \( (x^{-}_{i,m})^{r} . m_\pi \otimes m_{\hat{\pi}} \in K \) for some \( r \geq 0 \). Let \( r_0 \) be the smallest value of \( r \) with this property. Since

\[ x^{-}_{i,m}(x^{-}_{i,m})^{r} . m_\pi \otimes m_{\hat{\pi}} = (r_i - r_0 + 1)(x^{-}_{i,m})^{r_0+1} . m_\pi \otimes m_{\hat{\pi}}, \]

it follows by the minimality of \( r_0 \) that \( r_i + 1 = r_0 \).

Equation (3.3) is proved similarly, and we are done. \( \square \)

Note that, since \( \dim W(\pi)_{\lambda_\pi} = 1 \), it follows that \( W(\pi) \) has a unique irreducible quotient \( V(\pi) \). Write \( \pi \) as a product

\[ \pi = \pi^{(1)} \cdots \pi^{(k)}, \]

where \( i_i \leq 0 \) for all \( i \). Let \( k \) be the smallest index such that \( i_k = 0 \). Then, setting

\[ m_\pi = \pi^{(1)} \cdots \pi^{(k)}, \quad M(\pi) \Rightarrow m_\pi \Rightarrow M(\hat{\pi}), \]

we have a quotient map

\[ V(\pi) \rightarrow V(\hat{\pi}), \]

where \( V(\pi) \) is the quotient of \( V(\pi) \) by the kernel of the map

\[ V(\pi) \rightarrow \bigwedge^r V(\pi), \]

for \( R \in M(\pi) \). Since \( \dim V(\hat{\pi})_{\lambda_{\pi \hat{\pi}}} = 1 \), it follows that \( V(\hat{\pi}) \) has a unique irreducible quotient \( V(\hat{\pi}) \). Write \( \hat{\pi} \) as a product
where $\pi^{(j)}$ is such that
\[ \pi^{(j)}_{\theta_j} = (1 - a_j u)^{m_j}, \]
for some $m_j > 0$ and $a_j \in \mathbb{C}^\times$ with $a_j \neq a_k$ if $j \neq k$. The following result was proved in \[?\].

**Proposition 4.2.** With the above notation,
\[ V(\pi) \cong V(\pi^{(1)}) \otimes V(\pi^{(2)}) \otimes \cdots \otimes V(\pi^{(k)}) \]
as $L(g)$-modules. Further,
\[ V(\pi^{(j)}) \cong V^{fin}(\lambda_{\pi_j}) \]
as $g$-modules. \[\square\]