FINITE TIME BLOW UP FOR WAVE EQUATIONS WITH A POTENTIAL

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ABSTRACT. First we give a truly short proof of the major blow up result [Si] on higher dimensional semilinear wave equations. Using this new method, we also establish blow up phenomenon for wave equations with a potential. This complements the recent interesting existence result by [GHK], where the blow up problem was left open.

1. Introduction

We study the blow up of solutions to the following semilinear wave equation:

$$\begin{cases} \Delta u - Vu - u_{tt} + |u|^p = 0 \quad \text{in} \quad \mathbf{R}^n \times (0, \infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $\Delta = \Sigma_{i=1}^n \partial^2/\partial x_i^2$ is the Laplace operator and $V = V(x)$ is a potential. We consider dimensions $n \geq 3$ and exponents $p \in (1,\ p_c(n))$, where $p_c(n)$ is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

The number $p_c(n)$ is known as the critical exponent of the semilinear wave equation with $V = 0$ (see [St] e.g.). The study of this equation has an interesting and exciting history. We will only give a brief summary here and refer the reader to [St], [L], [DL] and a recent paper [JZ] for details. Let the initial values be compactly supported and nonnegative. John [J] proved that for $n = 3$ and $1 < p < p_c(3)$, nontrivial solutions must blow up in finite time. If $p > p_c(3)$, global solutions exist for small initial values. Glassey [G1-2] established the same result in the case $n = 2$. Shaffer [Sc] proved that the critical power $p = p_c(n)$ also belongs to the blow up case when $n = 2$, 3. In [GLS] the authors showed that when $p > p_c(n)$ and $n \geq 3$, (1.1) has global solutions for small initial values (see also [LS] and [T]). When $n \geq 4$ and $1 < p < p_c(n)$, the blow up result was proven by Sideris in [Si]. The proof is quite delicate, using sophisticated computation involving spherical harmonics. His proof was simplified in the papers [R] and [JZ], where spherical harmonics still play an important role. In this paper we discover a truly short proof of the blow up result using only a simple test function. More importantly the proof carries over to the case when the potential V is positive. It is a well known fact that the presence of potentials greatly increases the complexity of wave motion. In fact there is not much progress in either the existence or blow up problems in higher dimensional cases of (1.1). In the three dimensional case, it is known that there exist global small solutions when $p > p_c(3)$ and $V \in C_0^\infty(\mathbf{R}^3)$ is nonnegative, see [GHK]. In the same case, [ST] establishes, among other things, a blow up result for some $V \leq 0$. The current paper complements the result of [GHK] in dimension $n = 3$ and shows the blow up of solutions in all dimensions $n \geq 3$ when $1 < p < p_c(n)$ and V is a nonnegative

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potential satisfying the following condition: There exist two functions $\phi_0,\ \phi_1 \in C^2(\mathbf{R}^n)$ such that

$$\left\{\begin{array}{ll}\Delta\phi_0 - V\phi_0 = 0, & C_0^{-1} \le \phi_0(x) \le C_0,\\ \Delta\phi_1 - V\phi_1 = \phi_1, & 0 < \phi_1(x) \le C_1(1+|x|)^{-(n-1)/2}e^{|x|},\end{array}\right. \tag{1.2}$$

with positive constants C_0 and C_1. We show (Lemma 3.1) that this condition is satisfied by nonnegative potentials under very mild additional assumptions about regularity and behavior at infinity.

We consider compactly supported nonnegative data $(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$:

$$u_0(x) \ge 0, \quad u_1(x) \ge 0 \quad \text{a.e.}, \quad u_0(x) = u_1(x) = 0 \quad \text{for} \quad |x| > R. \tag{1.3}$$

Our main result is the following theorem.

Theorem 1.1. *Let (u_0, u_1) satisfy (1.3) and V satisfy (1.2). Suppose problem (1.1) has a solution $(u,\ u_t) \in C([0,T),\ H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n))$ such that*

$$supp(u, u_t) \subset \{(x,t) : |x| \le t + R\}.$$

If $1 < p < p_c(n)$, then $T < \infty$.

In particular the conclusion holds if V is locally Hölder continuous and $0 \le V(x) \le \frac{C}{1+|x|^{2+\delta}}$ for some $C,\ \delta > 0$ and all $x \in \mathbf{R}^n$.

When $V = 0$, we choose the functions

$$\left\{\begin{array}{l}\phi_0(x) = 1,\\ \phi_1(x) = \int_{S^{n-1}} e^{x\cdot\omega}d\omega, \quad \phi_1(x) \sim C_n|x|^{-(n-1)/2}e^{|x|} \quad \text{as} \quad |x| \to \infty.\end{array}\right.$$

Since condition (1.2) holds, we can apply Theorem 1.1 and deduce the well known results of John [J] and Sideris [Si].

The proof of Theorem 1.1 is given in Sections 2 and 3. To outline the method, we introduce

$$\begin{aligned} F_0(t) &= \int u(x,t)\phi_0(x)dx,\\ F_1(t) &= \int u(x,t)\psi_1(x,t)dx, \qquad \psi_1(x,t) = \phi_1(x)e^{-t}. \end{aligned} \tag{1.4}$$

The assumptions on u imply that $F_0(t)$ and $F_1(t)$ are well-defined C^2-functions for all t. By a standard procedure, we derive a nonlinear differential inequality for $F_0(t)$. We also derive a linear differential inequality for $F_1(t)$ and combine these to obtain a polynomial lower bound on $F_0(t)$ as $t \to \infty$. Theorem 1.1 is a consequence of the lower bound and the blow up result about nonlinear differential inequalities in Lemma 2.1.

In Section 3 we prove the existence of ϕ_0 and ϕ_1 in (1.2) when V is locally Hölder continuous and $0 \le V(x) \le \frac{C}{1+|x|^{2+\delta}}$ for some $C, \delta > 0$ and all $x \in \mathbf{R}^n$. This relies on a latest sharp estimate of heat kernels with a potential.

2. Proof of Theorem 1.1

We will use the following well known ODE result from p386[Si] e.g. to show that $F_0(t)$ in (1.4) blows up in finite time.

Lemma 2.1. *Let $p > 1$, $a \geq 1$, and $(p-1)a > q-2$. If $F \in C^2([0,T))$ satisfies*

$$(a) \qquad F(t) \geq K_0(t+R)^a,$$

$$(b) \qquad \frac{d^2F(t)}{dt^2} \geq K_1(t+R)^{-q}[F(t)]^p,$$

with some positive constants K_0, K_1, and R, then $T < \infty$.

To show that F_0 satisfies the above differential inequality for suitable a, q, we multiply equation (1.1) by ϕ_0 and integrate over $\mathbf{R}^n$. Condition (1.2) on ϕ_0 yields

$$\frac{d^2F_0(t)}{dt^2} = \int |u(x,t)|^p \phi_0(x)dx.$$

Note that for a fixed t, $u(\cdot,t) \in H_0^1(D_t)$ where D_t is the support of $u(\cdot,t)$. Hence the above equality is justified using integration by parts.

Estimating the right side by the Hölder inequality, we have

$$\int |u(x,t)|^p \phi_0(x)dx \geq \frac{\left|\int u(x,t)\phi_0(x)dx\right|^p}{\left(\int_{|x|\leq t+R} \phi_0(x)dx\right)^{p-1}}.$$

By Condition(1.2),

$$\int_{|x|\leq t+R} \phi_0(x)dx \leq C_0 \mathrm{vol}\{x : |x| < t+R\} = C_0 \mathrm{vol}(\mathbf{B}^n)(t+R)^n.$$

Thus, we obtain the differential inequality

$$\frac{d^2F_0(t)}{dt^2} \geq L_1(t+R)^{-n(p-1)}|F_0(t)|^p \tag{2.1}$$

with some $L_1 > 0$.

To show that F_0 admits the lower bound in Lemma 2.1 (a), we relate d^2F_0/dt^2 to F_1 using again equation (1.1) and the Hölder inequality:

$$\frac{d^2F_0(t)}{dt^2} = \int |u(x,t)|^p\phi_0(x)dx \geq \frac{\left|\int u(x,t)\psi_1(x,t)dx\right|^p}{\left(\int_{|x|\leq t+R}[\phi_0(x)]^{-1/(p-1)}[\psi_1(x,t)]^{p/(p-1)}dx\right)^{p-1}}.$$

By (1.2), the above becomes

$$\frac{d^2F_0(t)}{dt^2} \geq \frac{C_0\,|F_1(t)|^p}{\left(\int_{|x|\leq t+R}[\psi_1(x,t)]^{p/(p-1)}dx\right)^{p-1}}. \tag{2.1$'$}$$

The following lemmas estimate the numerator and denominator, respectively, and provide a lower bound on d^2F_0/dt^2.

Lemma 2.2. *Let V satisfy (1.2) and (u_0,u_1) satisfy (1.3). Assume that u meets the conditions of Theorem 1.1. Then for all $t \geq 0$,*

$$F_1(t) \geq \frac{1}{2}(1-e^{-2t})\int [u_0(x)+u_1(x)]\phi_1(x)dx + e^{-2t}\int u_0(x)\phi_1(x)dx \geq c > 0.$$

Lemma 2.3. *Let $p > 1$. Assume that ϕ_0 and ϕ_1 satisfy condition (1.2) Then for all $t \geq 0$*

$$\int_{|x|\leq t+R} [\psi_1(x,t)]^{p/(p-1)} dx \leq C(t+R)^{n-1-(n-1)p'/2},$$

where $p' = p/(p-1)$.

Taking the two lemmas for granted, we combine them with (2.1′) to obtain

$$\frac{d^2F_0(t)}{dt^2} \geq L_2(t+R)^{n-1-(n-1)p/2}, \quad t \geq 0,$$

where $L_2 > 0$. Integrating twice, we have the final estimate

$$F_0(t) \geq L_0(t+R)^{n+1-(n-1)p/2} + \frac{dF_0(0)}{dt} t + F_0(0)$$

with some $L_0 > 0$. When $1 < p < p_c(n)$, it is easy to check that $n+1-(n-1)p/2 > 1$. Hence the following estimate is valid when t is large:

$$F_0(t) \geq L_0(t+R)^{n+1-(n-1)p/2}. \tag{2.2}$$

Estimates (2.1), (2.2), and Lemma 2.1 with parameters

$$a \equiv n+1-(n-1)p/2 \quad \text{and} \quad q \equiv n(p-1)$$

imply Theorem 1.1 for all exponents p such that

$$(p-1)(n+1-(n-1)p/2) > n(p-1) - 2 \quad \text{and} \quad p > 1.$$

It is easy to see that the solution set is $p \in (1,\, p_c(n))$. The proof of Theorem 1.1 is complete, assuming Lemma 2.2, 2.3 and the validity of (1.2). □

Proof of Lemma 2.2. We multiply equation (1.1) by a test function $\psi \in C^2(\mathbf{R}^{n+1})$ and integrate over $\mathbf{R}^n \times [0,\, t]$.

$$\begin{aligned} &\int_0^t \int u(\Delta\psi - V\psi - \psi_{ss}) dxds + \int_0^t \int |u|^p \psi \, dxds \\ &\quad = \int (u_s\psi - u\psi_s) dx|_{s=t} - \int (u_s\psi - u\psi_s) dx|_{s=0}. \end{aligned} \tag{2.3}$$

We will apply this identity to $\psi = \psi_1$. Notice that for a fixed t, $u(\cdot, t) \in H_0^1(D_t)$ where D_t is the support of $u(\cdot, t)$. Hence all terms involving lateral boundary vanish during integration by parts. Notice also that

$$(\psi_1)_t = -\psi_1, \quad \Delta\psi_1 - V\psi_1 - (\psi_1)_{tt} = 0,$$

and

$$\begin{aligned} \int (u_s\psi_1 - u(\psi_1)_s) dx|_{s=t} &= \int (u_t\psi_1 + u(\psi_1)_t) dx - 2\int u(\psi_1)_t dx \\ &= \frac{d}{dt}\int u\psi_1 dx + 2\int u\psi_1 dx. \end{aligned}$$

Hence, (2.3) becomes

$$\frac{dF_1(t)}{dt} + 2F_1(t) = \int [u_0(x) + u_1(x)]\phi_1(x) dx + \int_0^t \int |u(x,s)|^p \psi_1(x,s) dxds.$$

Since $\psi_1 > 0$, we conclude that

$$\frac{dF_1(t)}{dt} + 2F_1(t) \geq \int [u_0(x) + u_1(x)]\phi_1(x)dx.$$

We multiply by e^{2t} and integrate on $[0,\, t]$. Then

$$e^{2t}F_1(t) - F_1(0) \geq \frac{1}{2}(e^{2t} - 1)\int [u_0(x) + u_1(x)]\phi_1(x)dx.$$

Dividing through by e^{2t}, we obtain the lower bound in Lemma 2.2.

Proof of Lemma 2.3. Let $I(t)$ be the integral in Lemma 2.3. Condition (1.2) shows that

$$I(t) \leq \text{area}(\mathbf{S}^{n-1})C_1^{p/(p-1)}e^{-p't}\int_0^{t+R}(1+r)^{-(n-1)p'/2}e^{p'r}r^{n-1}dr,$$

where $p' = p/(p-1)$. Since $r < r+1$, it is sufficient to show that

$$I(t) \leq Ce^{-p't}\int_0^{t+R}(1+r)^{n-1-(n-1)p'/2}e^{p'r}dr \leq C(t+R)^{n-1-(n-1)p'/2}.$$

This estimate is evident after splitting the last integral into two parts:

$$\begin{aligned}\int_0^{(t+R)/2}(1+r)^{n-1-(n-1)p'/2}e^{rp'}dr &\leq (1+t+R)^{q_1}\int_0^{(t+R)/2}e^{p'r}dr \\ &\leq \frac{e^{p'R/2}}{p'}(1+t+R)^{q_1}e^{p't/2},\end{aligned}$$

where $q_1 = \max(0,\, n-1-(n-1)p'/2)$, and

$$\begin{aligned}\int_{(t+R)/2}^{t+R}(1+r)^{n-1-(n-1)p'/2}e^{p'r}dr &\leq 2^{-q_2}(1+t+R)^{n-1-(n-1)p'/2}\int_{(t+R)/2}^{t+R}e^{p'r}dr \\ &\leq \frac{2^{-q_2}e^{p'R}}{p'}(1+t+R)^{n-1-(n-1)p'/2}e^{p't},\end{aligned}$$

where $q_2 = \min(0,\, n-1-(n-1)p'/2)$. This proves Lemma 2.3. □

To complete the proof of Theorem 1.1, it remains to prove the next Lemma 3.1. In the special case $V = 0$, the next section is redundant.

3. Existence of the two functions in (1.2)

In this section we prove

Lemma 3.1. *Suppose V is locally Hölder continuous and $0 \leq V(x) \leq \frac{C}{1+|x|^{2+\delta}}$ for some $C, \delta > 0$ and all $x \in \mathbf{R}^n$. Then exist two functions ϕ_0 and ϕ_1 satisfying (1.2), i.e.,*

$$\begin{cases} \Delta\phi_0 - V\phi_0 = 0, & C_0^{-1} \leq \phi(x) \leq C_0, \\ \Delta\phi_1 - V\phi_1 = \phi_1, & 0 < \phi_1(x) \leq C_1(1+|x|)^{-(n-1)/2}e^{|x|}. \end{cases}$$

Proof. Let H_0 and H be the fundamental solutions of

$$\Delta u - u - u_t = 0, \qquad \Delta u - u - Vu - u_t = 0$$

in $\mathbf{R}^n \times (0,\infty)$, respectively. Then $H_0 = e^{-t}G_0$ and $H = e^{-t}G$, where G_0 and G are the fundamental solution of

$$\Delta u - u_t = 0, \qquad \Delta u - Vu - u_t = 0.$$

By Theorem 1.1 (a) and Remark 1.1 in [Z1], there exists a positive constant c such that

$$cG_0(x,t;y,0) \le G(x,t;y,0) \le G_0(x,t;y,0) = \frac{c_n}{t^{n/2}} e^{-\frac{|x-y|^2}{4t}}$$

for all $x, y \in \mathbf{R}^n$ and $t > 0$. We should mention that the global lower bound is nontrivial since one needs to keep the exact coefficient $1/4$ in each exponential term.

Hence we have the following global bounds:

$$cH_0(x,t;y,0) \le H(x,t;y,0) \le H_0(x,t;y,0). \tag{3.1}$$

Let μ_0 be a positive solution of $\Delta\mu_0 - \mu_0 = 0$ such that $\mu_0(x) \sim e^{|x|}/(1+|x|)^{(n-1)/2}$. The existence of such μ_0 is well known and explained in the introduction. Consider the function

$$u(x,t) \equiv \int_{\mathbf{R}^n} H(x,t;y,0)\mu_0(y)dy. \tag{3.2}$$

Since for fixed (x,t), $H(x,t;y,0)$ decays super exponentially near infinity, the above integral is well defined. Moreover, u is a solution to

$$\Delta u - u - Vu - u_t = 0. \tag{3.3}$$

By (3.1) and (3.2) we have

$$c\int_{\mathbf{R}^n} H_0(x,t;y,0)\mu_0(y)dy \le u(x,t) \le \int_{\mathbf{R}^n} H_0(x,t;y,0)\mu_0(y)dy.$$

Since $\Delta\mu_0 - \mu_0 = 0$, it is clear from differentiation that

$$\mu_0(x) = \int_{\mathbf{R}^n} H_0(x,t;y,0)\mu_0(y)dy,$$

even though the righthand side apparently depends on time. Indeed,

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbf{R}^n} H_0(x,t;y,0)\mu_0(y)dy &= -\int_{\mathbf{R}^n} (\Delta_y - 1)H_0(x,t;y,0)\mu_0(y)dy \\ &= -\int_{\mathbf{R}^n} H_0(x,t;y,0)(\Delta_y - 1)\mu_0(y)dy = 0.\end{aligned}$$

Here we observe that integration by parts is legitimate since, for fixed $t > 0$ and x, $H_0(x,t;y,0)$ has super exponential decay near infinity while μ_0 only grows exponentially.

Hence

$$c\mu_0(x) \le u(x,t) \le \mu_0(x). \tag{3.4}$$

Differentiating (3.3) with respect to t, we obtain

$$\begin{cases}\Delta u_t - u_t - Vu_t - (u_t)_t = 0, & (x,t) \in \mathbf{R}^n \times (0,\infty), \\ u_t|_{t=0} = \Delta\mu_0 - \mu_0 - V\mu_0 \le 0.\end{cases}$$

Here we remark that under our assumption that V is locally Hölder continuous, it is not clear whether u_{tt} exists. However, we can work on the finite difference of u_t and use a standard approximation argument to achieve the same result.

By the maximum principle, we know that $u_t(x,t) \le 0$ everywhere. This and (3.4) show that $u(x,t)$ converges to a function $\phi_1 = \phi_1(x)$ as $t \to \infty$. Moreover,

$$c\mu_0(x) \le \phi_1(x) \le \mu_0(x). \tag{3.5}$$

We are going to show that

$$\Delta\phi_1 - \phi_1 - V\phi_1 = 0. \tag{3.6}$$

To this end, let us consider the function $w = w(x,t) = \int_t^{t+1} u(x,s)ds$. Direct computation shows that

$$\Delta w(x,t) - w(x,t) - V(x)w(x,t) = u(x,t+1) - u(x,t).$$

It is also clear that $w(x,t) \to \phi_1(x)$ when $t \to \infty$. Let $\eta = \eta(x)$ be any function in $C_0^\infty(\mathbf{R}^n)$. Then we obtain

$$\int_{\mathbf{R}^n} [w(x,t)\Delta\eta(x) - w(x,t)\eta(x) - V(x)w(x,t)\eta(x)]dx = \int_{\mathbf{R}^n} [u(x,t+1) - u(x,t)]\eta(x)dx.$$

Letting $t \to \infty$, we have

$$\int_{\mathbf{R}^n} [\phi_1(x)\Delta\eta(x) - \phi_1(x)\eta(x) - V(x)\phi_1(x)\eta(x)]dx = 0.$$

Since η is arbitrary and ϕ_1 is locally bounded, we know that ϕ_1 is a classical solution to (3.6), which also satisfies (3.5). This proves the existence of ϕ_1 in (1.2). The existence of ϕ_0 under our assumption is well known (see [Z2] Theorem B e.g.). In fact it can be proven by exactly the same method except that we drop the term $-u$ everywhere. □

Remark 3.1. The decay condition for V is Lemma 3.1 can be generalized. In [Z1], a necessary and sufficient condition for the validity of the sharp comparison result right before (3.1) was found for all nonnegative V. This class of V resembles the Kato class in mathematical physics. It overlaps with $L^{n/2}(\mathbf{R}^n)$. But they are not the same.

References

[DL] Deng, Keng; Levine, Howard A. *The role of critical exponents in blow-up theorems: the sequel.* J. Math. Anal. Appl. 243 (2000), no. 1, 85–126.

[GHK] Georgiev, Vladimir; Heiming, Charlotte; Kubo, Hideo, *Supercritical semilinear wave equation with non-negative potential.* Comm. Partial Differential Equations 26 (2001), no. 11-12, 2267–2303.

[Gl1] Glassey, Robert T. *Finite-time blow-up for solutions of nonlinear wave equations.* Math. Z. 177 (1981), no. 3, 323–340.

[Gl2] Glassey, Robert T. *Existence in the large for $\Box u = F(u)$ in two space dimensions,* Math. Z. 178 (1981) 233–261.

[GLS] V. Georgiev, H. Lindblad, C.D. Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math. 119 (6) (1997) 1291–1319.

[J] John, Fritz, *Blow-up of solutions of nonlinear wave equations in three space dimensions.* Manuscripta Math. 28 (1979), no. 1-3, 235–268.

[JZ] Jiao, Hengli; Zhou, Zhengfang, *An elementary proof of the blow-up for semilinear wave equation in high space dimensions.* J. Differential Equations 189 (2003), no. 2, 355–365.

[L] Levine, Howard A. *The role of critical exponents in blowup theorems.* SIAM Rev. 32 (1990), no. 2, 262–288.

[LS] H. Lindblad, C. Sogge, *Long-time existence for small amplitude semilinear wave equations,* Amer. J. Math. 118 (5) (1996) 1047–1135.

[R] Rammaha, M. A. *Finite-time blow-up for nonlinear wave equations in high dimensions.* Comm. Partial Differential Equations 12 (1987), no. 6, 677–700.

[Sc] J. Schaeffer, The equation $\Box u = |u|^p$ for the critical value of p, Proc. Roy. Soc. Edinburgh 101A (1985) 31–44.

[Si] Sideris, Thomas C. *Nonexistence of global solutions to semilinear wave equations in high dimensions.* J. Differential Equations 52 (1984), no. 3, 378–406.

[St] Strauss, Walter A. *Nonlinear wave equations.* CBMS Regional Conference Series in Mathematics, 73. AMS, Providence, RI, 1989.

[ST] Strauss, Walter A.; Tsutaya, Kimitoshi, *Existence and blow up of small amplitude nonlinear waves with a negative potential.* Discrete Contin. Dynam. Systems 3 (1997), no. 2, 175–188.

[T] Tataru, Daniel, *Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation.* Trans. Amer. Math. Soc. 353 (2001), no. 2, 795–807

[Z1] Zhang, Qi S. *A sharp comparison result concerning Schrdinger heat kernels.* Bull. London Math. Soc. 35 (2003), no. 4, 461–472.

[Z2] Zhang, Qi S. *An optimal parabolic estimate and its applications in prescribing scalar curvature on some open manifolds with* Ricci ≥ 0. Math. Ann. 316 (2000), no. 4, 703–731.

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