

A Strong Regularity Result for Parabolic Equations

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Abstract: We consider a parabolic equation with a drift term $\Delta u + \mathbf{b}\nabla u - u_t = 0$. Under the condition $div \mathbf{b} = 0$, we prove that solutions possess dramatically better regularity than those provided by standard theory. For example, we prove continuity of solutions when not even boundedness is expected.

1. Introduction

We aim to study the parabolic equation

$$\Delta u(x, t) + \mathbf{b}(x, t)\nabla u(x, t) - u_t(x, t) = 0, \quad (x, t) \in \mathbf{R}^n \times (0, \infty), \quad (1.1)$$

where Δ is the standard Laplacian, \mathbf{b} is a vector valued function and $n \geq 2$. Standard existence and regularity theory for this kind of equations has existed for several decades. For instance when $|\mathbf{b}| \in L^p(\mathbf{R}^n)$, $p > n$, the fundamental solution of (1.1) has a local in time Gaussian lower and upper bound ([A]). Hence bounded solutions are Hölder continuous. In this paper we study the regularity problem of (1.1) for much more singular functions \mathbf{b} .

Several factors provide strong motivations for studying these kind of problems. The first is to investigate a possible gain of regularity in the presence of the singular drift term \mathbf{b} . This line of research has been followed in the papers [St, KS, O, CrZ, Se]. Under the condition $|\mathbf{b}| \in L^n(\mathbf{R}^n)$, Stampacchia [St] proved that bounded solutions of $\Delta u + \mathbf{b}\nabla u = 0$ are Hölder continuous. In the paper [CrZ], Cranston and Zhao proved that solutions to this equation are continuous when \mathbf{b} is in a suitable Kato class, i.e. $\lim_{r \rightarrow 0} \sup_x \int_{|x-y| \leq r} \frac{|\mathbf{b}(y)|}{|x-y|^{n-1}} dy = 0$. In the paper [KS] Kovalenko and Semenov proved the Hölder continuity of solutions when $|\mathbf{b}|^2$ is independent of time and is sufficiently small in the form sense. See the next paragraph for a statement of their condition. This result was recently generalized in [Se] to equations with leading term in divergence form. In [O], Osada proved, among other things, that the fundamental solution of (1.1)

has global Gaussian upper and lower bound when \mathbf{b} is the derivative of bounded functions (in distribution sense) and $div \mathbf{b} = 0$. More recently in the paper [LZ], Hölder continuity of solutions to (1.1) was established when $|\mathbf{b}|^2$ is form bounded and $div \mathbf{b} = 0$. See the next page for a description. We should mention that many authors have also studied the regularity property of the related heat equation

$$\Delta u + Vu - u_t = 0.$$

Here V is a singular potential. We refer the reader to the papers by Aizenman and Simon [AS], Simon [Si] and references therein. It is worth remarking that the current situation exhibits fundamentally new phenomena comparing with that case.

Another motivation comes from the study of nonlinear equations involving gradient structures. These include the Navier Stokes equations, which can be regarded as systems of parabolic equations with very singular first order terms. Our result provides a different proof of the well known fact that weak solutions to the two dimensional Navier-Stokes equations are smooth (Corollary 2). For the three dimensional Navier-Stokes equations, it is interesting to note that the singularity of the velocity field is covered by our theorem (see the discussion at the end of the introduction).

In this paper we actually go much beyond the above kinds of singularities. A special case of our result states that, under the assumption that $div \mathbf{b} = 0$, weak solutions of (1.1) are bounded as long as $\mathbf{b} \in L^p_{loc}(\mathbf{R}^n)$ with $p > n/2$, $n \geq 4$. For a time dependent vector field $\mathbf{b} \in L^2_{loc}(\mathbf{R}^n \times (0, \infty))$, it suffices to assume the general form bounded condition: for a fixed $m \in (1, 2]$, and $\phi \in C^\infty_0(\mathbf{R}^n \times (0, \infty))$,

$$\int \int_{\mathbf{R}^n} |\mathbf{b}(x, t)|^m \phi^2 dx dt \leq k \int \int_{\mathbf{R}^n} |\nabla_x \phi(x, t)|^2 dx dt,$$

where k is independent of ϕ . When $m = 2$ and k is sufficiently small, we are in a situation covered by the paper [KS] and [LZ]. The most interesting case is when m is close to 1. It is widely assumed that solutions of (1.1) can be regular only if the above inequality holds for $m \geq 2$. However Theorem 1.1 below proves that weak solutions to (1.1) are bounded as long as $m > 1$. In fact they are Lipschitz in the spatial direction. Hence \mathbf{b} can be *almost twice as singular* as allowed by standard theory, provided that $div \mathbf{b} = 0$.

The above class of the drift term \mathbf{b} includes and much exceeds the generalized Kato class that has been studied in several interesting papers [ChZ, CrZ, G]. These functions in general are not the derivative of bounded functions considered in [O] either (see Remark 1.1 below). Here is an example. Let $\mathbf{b} = \mathbf{b}(x_1, x_2, x_3)$ be a vector field in \mathbf{R}^3 . If \mathbf{b} has a local singularity of the form $\frac{c}{|x|^{1+\epsilon}}$ with a small $\epsilon > 0$, then \mathbf{b} is in this class. In contrast all previous results at best allow singularities in the form of $\frac{c}{|x|}$.

In addition to the unexpected regularity result, we also prove that the fundamental solution of (1.1) satisfies a global Gaussian like upper bound. An interesting feature of the bound is that it is not just a perturbation. This means that the global bound holds when \mathbf{b} satisfies (1.2) with no additional smallness condition. Using the approximation result in Sect. 2, the fundamental solution here means the minimum of pointwise limits of the fundamental solution of (1.1) with \mathbf{b} replaced by a sequence of smooth and divergence free vector fields. An interesting fact is that the bound is no longer Gaussian, reflecting the contribution given by the singularity of the drift term \mathbf{b} .

In the Kato class case, local in time Gaussian bounds for the heat kernel with singular drift terms were obtained in [Z], which was extended in [LS] recently.

In this paper we use the following definition of weak solutions.

Definition 1.1. Let $D \subseteq \mathbf{R}^n$ be a domain and $T \in (0, \infty]$. A function u such that $u, |\nabla u| \in L^2_{loc}(D \times [0, T])$ is a weak solution to (1.1) if: for any $\phi \in C_0^\infty(D \times (-T, T))$, there holds

$$\int_0^T \int_D (u \partial_t \phi - \nabla u \nabla \phi) dx dt + \int_0^T \int_D \mathbf{b} \nabla u \phi dx dt = - \int_D u_0(x) \phi(x, 0) dx.$$

The assumption that ∇u is square integrable can be weakened. However we are not seeking full generality here.

Now we are ready to state the main theorem of the paper.

Theorem 1.1. Suppose \mathbf{b} satisfies:

- (i) $\mathbf{b} \in L^2_{loc}(\mathbf{R}^n \times [0, \infty))$ and $div \mathbf{b}(\cdot, t) = 0$;
- (ii) for a fixed $m \in (1, 2]$ and any $\phi \in C^\infty(\mathbf{R}^n \times (0, \infty))$ with compact support in the spatial direction,

$$\int \int_{\mathbf{R}^n} |\mathbf{b}|^m \phi^2 dx dt \leq k \int \int_{\mathbf{R}^n} |\nabla \phi|^2 dx dt, \quad (1.2)$$

where k is independent of ϕ . Then the following statements hold.

- (i) Weak solutions of (1.1) are locally bounded.
- (ii) Weak solutions of (1.1) are Hölder continuous when $b = b(x)$ and $m = 2$.
- (iii) Let G be the fundamental solution of (1.1). There exist positive constants c_1 and c_2 such that, for any $x, y \in \mathbf{R}^n$ and $t > s > 0$,

$$G(x, t; y, s) \leq \begin{cases} \frac{c_1(t-s)}{[(t-s)|B(x, \sqrt{t-s})|]^m / (2(m-1))} \left[\exp\left(-c_2 \frac{|x-y|^m}{(t-s)^{m-1}}\right) + \exp\left(-c_2 \frac{|x-y|^2}{t-s}\right) \right], & t-s \leq 1, \\ \frac{c_1}{|B(x, \sqrt{t-s})|} \left[\exp\left(-c_2 \frac{|x-y|^m}{(t-s)^{m-1}}\right) + \exp\left(-c_2 \frac{|x-y|^2}{t-s}\right) \right], & t-s \geq 1. \end{cases}$$

Remark 1.1. Note that the above upper bound reduces to the standard Gaussian upper bound when $m = 2$. This case was recently investigated in [LZ]. Part (ii), follows from [LZ], is here for completeness. If $\mathbf{b} \in L^p(\mathbf{R}^n)$ with $p > n/2$, then it is well known that (1.2) is satisfied. See [Si]. Hence we have

Corollary 1. Let u be a weak solution of the elliptic equation $\Delta u + \mathbf{b} \nabla u = 0$. Suppose $\mathbf{b} \in L^p_{loc}(\mathbf{R}^n)$, $p > n/2$, $n \geq 4$, and $div \mathbf{b} = 0$. Then u is a bounded function.

Note that without the assumption of $div \mathbf{b} = 0$, it is known ([St]) that u is Hölder continuous when $\mathbf{b} \in L^p_{loc}$ with $p = n$.

Remark 1.2. Due to its importance and potential applications, we single out part of the result of Theorem 1.1 in the three dimensional case as a corollary.

Corollary 2. Let $D \subseteq \mathbf{R}^n$, $n = 2, 3$. Assume $|\mathbf{b}| \in L^\infty([0, T], L^2(D))$ and $div \mathbf{b} = 0$. Suppose u is a weak solution of (1.1) in $D \times [0, T]$. Then u is locally bounded. In particular weak solutions to the two dimensional Navier-Stokes equation is smooth when $t > 0$.

Proof. It is enough to prove that the above condition on \mathbf{b} alone implies that condition (1.2) is satisfied for some $m > 1$. Here is a proof when $n = 3$. The case when $n = 2$ is dealt with similarly. Let us take $m = 4/3$ and $p = 2/m = 3/2$. Then, by Hölder's inequality,

$$\begin{aligned}
\int_0^T \int_D |\mathbf{b}|^{4/3} \phi^2 dx dt &\leq \int_0^T \left(\int_D |\mathbf{b}|^{mp} dx \right)^{1/p} \left(\int_D \phi^{2p/(p-1)} dx \right)^{(p-1)/p} dt \\
&= \int_0^T \left(\int_D |\mathbf{b}|^2 dx \right)^{2/3} \left(\int_D \phi^6 dx \right)^{1/3} dt \\
&\leq \sup_{t \in [0, T]} \left(\int_D |\mathbf{b}|^2(x, t) dx \right)^{2/3} \int_0^T \left(\int_D \phi^6 dx \right)^{1/3} dt \\
&\leq C \sup_{t \in [0, T]} \left(\int_D |\mathbf{b}|^2(x, t) dx \right)^{2/3} \int_0^T \int_D |\nabla \phi|^2 dx dt.
\end{aligned}$$

The last step is by Sobolev imbedding.

Now let $u = (u_1(x, t), u_2(x, t))$ be a weak solution to the 2 - d Navier-Stokes equation

$$\Delta u - u \nabla u - \nabla P - u_t = 0, \quad \operatorname{div} u = 0.$$

Then the curl of u , denoted by w , is a scalar satisfying

$$\Delta w + u \nabla w - w_t = 0.$$

By definition, $u \in L^\infty((0, \infty), L^2(\mathbf{R}^2))$. So Theorem 1.1 shows that w is bounded when $t > 0$. Hence u is smooth too when $t > 0$. \square

Discussion. Here we would like to speculate on some possible links between the regularity problem of the 3-d Navier-Stokes equation and Corollary 2. Let u be a Leray-Hopf solution to the 3-d Navier-Stokes equation

$$\Delta u - u \nabla u - \nabla P - u_t = 0, \quad \operatorname{div} u = 0, \quad |u(\cdot, 0)| \in L^2(\mathbf{R}^3).$$

Then it is well known that $\|u(\cdot, t)\|_{L^2(\mathbf{R}^n)}$ is non-increasing and hence uniformly bounded. Therefore assuming only the pressure term P is sufficiently regular locally, then Corollary 2 implies that u is bounded and hence smooth. It seems that all previous regularity results either make some global restrictions on P or on the initial value u_0 . We mention the recent interesting result of Seregin and Sverak [SS]. The authors proved that u is smooth provided that $u_0 \in W^{1,2}$ and P is bounded from below. See also [BG]. It would be interesting to see how far the method in this paper may go for the system case.

The rest of the paper is organized as follows. In Sect. 2 we show some approximation results of solutions of (1.1) under some singular drift term. Theorems 1.1 will be proven in Sect. 3.

2. Preliminaries

Since the drift term \mathbf{b} in (1.1) can be much more singular than those allowed by the standard theory, the existence and uniqueness of weak solutions of (1.1) can not be taken for granted. In order to proceed first we need to prove some approximation results. The next proposition shows that Eq. (1.1) possesses weak solutions even when \mathbf{b} satisfies the assumption of Theorem 1.1.

Proposition 2.1. *Let \mathbf{b} be given as in Theorem 1.1 and \mathbf{b}_k be a sequence of smooth divergence free vector fields. Suppose $\mathbf{b}_k \rightarrow \mathbf{b}$ in $L^2(D \times [0, T])$ norm and let u_k be the unique solution to*

$$\begin{cases} \Delta u_k + \mathbf{b}_k \nabla u_k - \partial_t u_k = 0, & \text{in } D \times [0, T] \\ u_k(x, t) = 0, & (x, t) \in \partial D \times [0, T], \\ u_k(x, 0) = u_0(x), & u_0 \in L^2(\mathbf{R}^n). \end{cases} \quad (2.1)$$

Then there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, which converges weakly in $L^2(D \times [0, T])$ to a solution of (1.1).

Proof. Since $\operatorname{div} \mathbf{b}_k = 0$, multiplying Eq. (2.1) by u_k and integrating, one easily obtains

$$\int_0^T \int_D |\nabla u_k|^2 dx dt + \int_D u_k^2(x, T) dx = \int_D u_0^2(x) dx.$$

Hence there exists a function u such that $u, |\nabla u| \in L^2(D \times [0, T])$ and a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that

$$\begin{aligned} u_k &\rightarrow u, & \text{weakly in } L^2(D \times [0, T]); \\ \nabla u_k &\rightarrow \nabla u, & \text{weakly in } L^2(D \times [0, T]). \end{aligned}$$

We will prove that u is a solution to (1.1).

Clearly u_k satisfies, for any $\phi \in C_0^\infty(D \times [0, T])$,

$$\begin{aligned} &\int_0^T \int_D (u_k \partial_t \phi - \nabla u_k \nabla \phi) dx dt + \int_0^T \int_D \mathbf{b}_k \nabla u_k \phi dx dt \\ &= - \int_D u_0(x) \phi(x, 0) dx. \end{aligned} \quad (2.2)$$

By the weak convergence of u_k and ∇u_k , we have

$$\int_0^T \int_D (u_k \partial_t \phi - \nabla u_k \nabla \phi) dx dt \rightarrow \int_0^T \int_D (u \partial_t \phi - \nabla u \nabla \phi) dx dt, \quad k \rightarrow \infty. \quad (2.3)$$

Next, notice that

$$\begin{aligned} &\int_0^T \int_D \mathbf{b}_k \nabla u_k \phi dx dt - \int_0^T \int_D \mathbf{b} \nabla u \phi dx dt \\ &= \int_0^T \int_D (\mathbf{b}_k - \mathbf{b}) \nabla u_k \phi dx dt + \int_0^T \int_D \mathbf{b} (\nabla u_k - \nabla u) \phi dx dt. \end{aligned}$$

By the strong convergence of \mathbf{b}_k and the weak convergence of ∇u_k , we see that

$$\int_0^T \int_D u_k \mathbf{b}_k \nabla \phi dx dt - \int_0^T \int_D \mathbf{b} \nabla u \phi dx dt \rightarrow 0, \quad k \rightarrow \infty. \quad (2.4)$$

By (2.2) and (2.4) we obtain

$$\int_0^T \int_D (u \partial_t \phi - \nabla \phi) dx dt + \int_0^T \int_D \mathbf{b} \nabla u \phi dx dt = - \int_D u_0(x) \phi(x, 0) dx,$$

i.e. u is a solution to (1.1). \square

Proposition 2.2. *Suppose $\mathbf{b} \in C^\infty(\mathbf{R}^n \times [0, \infty)) \cap L^\infty$ and $\operatorname{div} \mathbf{b} = 0$. Let G be the fundamental solution of (1.1). Then, for any $x \in \mathbf{R}^n$ and $t > s > 0$,*

$$\int_{\mathbf{R}^n} G(x, t; y, s) dy = 1, \quad \int_{\mathbf{R}^n} G(x, t; y, s) dx = 1.$$

Proof. Since \mathbf{b} is smooth and bounded, G is smooth and has local Gaussian upper bound. Hence we have

$$\frac{d}{ds} \int_{\mathbf{R}^n} G(x, t; y, s) dy = \int_{\mathbf{R}^n} [-\Delta_y G(x, t; y, s) + \mathbf{b}(y, s) \nabla_y G(x, t; y, s)] dy = 0.$$

The other equality is proved similarly. \square

Proposition 2.3. *Let $Q \equiv D \times [0, T]$ with $D \subseteq \mathbf{R}^n$ being a smooth domain and $T > 0$. Suppose that $\mathbf{b} \in C^\infty(Q) \cap L^\infty(Q)$ and $f \in L^1(Q)$. Suppose u is a weak solution to*

$$\begin{cases} \Delta u + \mathbf{b} \nabla u - u_t = f, & \text{in } Q \\ u(x, t) = 0, & (x, t) \in \partial D \times [0, T] \\ u(x, 0) = 0. \end{cases} \quad (2.5)$$

Here the boundary condition is in the sense that $u \in L^2([0, T], W_0^{1,2}(D))$. Then

$$u(x, t) = - \int_0^t \int_D G(x, t; y, s) f(y, s) dy ds.$$

Here G is the Green's function of (1.1) with initial Dirichlet boundary condition in Q .

Proof. This result is trivial when f is bounded and smooth. When f is just L^1 , it is known too. Here we present a proof for completeness.

Let ψ be a smooth function in Q . Since \mathbf{b} is bounded and smooth, standard theory shows that the following backward problem has a unique smooth solution:

$$\begin{cases} \Delta \eta - \mathbf{b} \nabla \eta + \eta_t = \psi, & \text{in } Q \\ \eta(x, t) = 0, & (x, t) \in \partial D \times [0, T] \\ \eta(x, T) = 0. \end{cases} \quad (2.6)$$

Moreover

$$\eta(y, s) = - \int_s^T \int_D G(x, t; y, s) \psi(x, t) dx dt. \quad (2.7)$$

Regularity

Since u is a weak solution to (2.5), we have, by definition

$$\int_Q [-\nabla u \nabla \eta + \mathbf{b} \nabla u \eta + u \eta_t] dx dt = \int_Q f \eta dx dt.$$

Using integration by parts we have

$$\int_Q u [\Delta \eta - \mathbf{b} \nabla \eta + \eta_t] dx dt = \int_Q f \eta dx dt.$$

By this, (2.6) and (2.7), we deduce

$$\int_Q u \psi dx dt = - \int_0^T \int_D f(y, s) \int_s^T \int_D G(x, t; y, s) \psi(x, t) dx dt.$$

That is

$$\int_Q u \psi dx dt = - \int_0^T \int_D \int_0^t \int_D G(x, t; y, s) f(y, s) dy ds \psi(x, t) dx dt.$$

The proposition follows since ψ is arbitrary. \square

Proposition 2.4. *Suppose u is a weak solution of Eq. (1.1) in the cube $Q = D \times [0, T]$, where \mathbf{b} satisfies the condition in Theorem 1.1. Here D is a domain in \mathbf{R}^n . Then u is the L^1_{loc} limit of functions $\{u_k\}$. Here $\{u_k\}$ is a weak solution of (1.1) in which \mathbf{b} is replaced by smooth, divergence free \mathbf{b}_k such that $\mathbf{b}_k \rightarrow \mathbf{b}$ strongly in $L^2(Q)$, $k \rightarrow \infty$.*

Proof. First we select a sequence of smooth, bounded, divergence free \mathbf{b}_k such that $\mathbf{b}_k \rightarrow \mathbf{b}$ strongly in $L^2(Q)$, $k \rightarrow \infty$. Let $D' \subset D$ be a smooth sub-domain of D . Then the following problem has a weak solution u_k :

$$\begin{cases} \Delta u_k + \mathbf{b}_k \nabla u_k - (u_k)_t = 0, & \text{in } Q' = D' \times (0, T) \\ u_k(x, t) = u(x, t), & (x, t) \in \partial D' \times [0, T] \\ u_k(x, 0) = u(x, 0). \end{cases} \quad (2.8)$$

Clearly $u_k - u$ is a weak solution to the following:

$$\begin{cases} \Delta(u_k - u) + \mathbf{b}_k \nabla(u_k - u) - (u_k - u)_t = (\mathbf{b} - \mathbf{b}_k) \nabla u, & \text{in } Q' = D' \times (0, T) \\ (u_k - u)(x, t) = 0, & (x, t) \in \partial D' \times [0, T] \\ (u_k - u)(x, 0) = 0. \end{cases} \quad (2.9)$$

Here the boundary condition is in the sense that $u_k - u \in L^2([0, T], W_0^{1,2}(D'))$.

By our assumptions on \mathbf{b} , \mathbf{b}_k and ∇u , we know that

$$(\mathbf{b} - \mathbf{b}_k) \nabla u \in L^1(Q').$$

Since \mathbf{b}_k is bounded and smooth, Proposition 2.3 shows that

$$(u_k - u)(x, t) = - \int_0^t \int_{D'} G'_k(x, t; y, s) (\mathbf{b} - \mathbf{b}_k) \nabla u(y, s) dy ds.$$

Here G'_k is the Green's function of $\Delta u + \mathbf{b}_k \nabla u - u_t = 0$ in Q' with Dirichlet initial boundary value condition. By Proposition 2.2, or the local version of it, we have

$$\int_{D'} G'_k(x, t; y, s) dx \leq 1.$$

Hence

$$\int_{D'} |u_k - u|(x, t) dx \leq \int_0^t \int_{D'} |\mathbf{b} - \mathbf{b}_k| |\nabla u(y, s)| dy ds.$$

Hence

$$\int_{D'} |u_k - u|(x, t) dx \leq \|\mathbf{b} - \mathbf{b}_k\|_{L^2(Q')} \|\nabla u\|_{L^2(Q')} \rightarrow 0.$$

This proves the proposition. \square

3. Proof of Theorem 1.1

Using the approximation result of Sect. 2, we may and do assume that the vector field \mathbf{b} is bounded and smooth.

The beginning of the proof generally follows the classical strategy of using test functions to establish $L^2 - L^\infty$ bounds and weighted estimates for solutions of (1.1). However it is well known that this method does not provide a sharp global upper bound in the presence of lower order terms and the vector field \mathbf{b} can not be as singular as we are assuming. For instance there is usually an extraneous e^{ct} term when t is large. Nevertheless, by using the special structure of the drift term and exploiting a special role of the divergence of \mathbf{b} , we show that this classical method can be refined to derive sharp global bounds. In order to overcome the singularity of the drift term \mathbf{b} , we need to construct a refined test function. This is the key step in proving the bounds.

We divide the proof into five steps. For the sake of clarity we draw a flow chart for the proof:

- Step 1: Energy estimates using refined test function
- \Rightarrow Step 2: L^∞ bound for weak solutions ((i) of Theorem 1.1)
- \Rightarrow Step 3: Weighted estimates
- \Rightarrow Step 4: Gaussian like upper bound ((iii) of Theorem 1.1);
- Step 5: Proof of (ii)

Step 1. Caccioppoli inequality (energy estimates).

Let u be a solution of (1.1) in the parabolic cube $Q_{\sigma r} = B(x, \sigma r) \times [t - (\sigma r)^2, t]$. Here $x \in \mathbf{R}^n$, $\sigma > 1$, $r > 0$ and $t > 0$.

By direct computation, for any rational number $p \geq 1$, which can be written as the quotient of two integers with the denominator being odd, one has

$$\Delta u^p + \mathbf{b} \nabla u^p - \partial_t u^p = p(p-1) |\nabla u|^2 u^{p-2}. \quad (3.1)$$

Here the condition on p is to ensure that u^p makes sense when u changes sign. One can also just work on positive solutions now and prove the boundedness of all solutions later. See Step 6 at the end of the section.

Regularity

Choose $\psi = \phi(y)\eta(s)$ to be a refined cut-off function satisfying

$$\text{supp } \phi \subset B(x, \sigma r); \quad \phi(y) = 1, \quad y \in B(x, r); \quad \frac{|\nabla\phi|}{\phi^\delta} \leq \frac{C}{((\sigma - 1)r)}, \quad 0 \leq \phi \leq 1;$$

here $\delta \in (0, 1)$. By scaling it is easy to show that such a function exists,

$$\begin{aligned} \text{supp } \eta &\subset (t - (\sigma r)^2, t); \quad \eta(s) = 1, \\ s &\in [t - r^2, t]; \quad |\eta'| \leq 2/((\sigma - 1)r)^2; \quad 0 \leq \eta \leq 1. \end{aligned}$$

Denoting $w = u^p$ and using $w\psi^2$ as a test function on (3.1), one obtains

$$\int_{Q_{\sigma r}} (\Delta w + \mathbf{b}\nabla w - \partial_s w)w\psi^2 dy ds = p(p-1) \int_{Q_{\sigma r}} |\nabla u|^2 w^2 u^{-2} \geq 0.$$

Using integration by parts, one deduces

$$\int_{Q_{\sigma r}} \nabla(w\psi^2)\nabla w dy ds \leq \int_{Q_{\sigma r}} \mathbf{b}\nabla w(w\psi^2) dy ds - \int_{Q_{\sigma r}} (\partial_s w)w\psi^2 dy ds. \quad (3.2)$$

By direct calculation,

$$\begin{aligned} \int_{Q_{\sigma r}} \nabla(w\psi^2)\nabla w dy ds &= \int_{Q_{\sigma r}} \nabla[(w\psi)\psi]\nabla w dy ds \\ &= \int_{Q_{\sigma r}} [\nabla(w\psi)(\nabla(w\psi) - (\nabla\psi)w) + w\psi\nabla\psi\nabla w] dy ds \\ &= \int_{Q_{\sigma r}} [|\nabla(w\psi)|^2 - |\nabla\psi|^2 w^2] dy ds. \end{aligned}$$

Substituting this to (3.2), we obtain

$$\begin{aligned} \int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dy ds &\leq \int_{Q_{\sigma r}} \mathbf{b}\nabla w(w\psi^2) dy ds - \int_{Q_{\sigma r}} (\partial_s w)w\psi^2 dy ds \\ &\quad + \int_{Q_{\sigma r}} |\nabla\psi|^2 w^2 dy ds. \end{aligned} \quad (3.3)$$

Next notice that

$$\begin{aligned} \int_{Q_{\sigma r}} (\partial_s w)w\psi^2 dy ds &= \frac{1}{2} \int_{Q_{\sigma r}} (\partial_s w^2)\psi^2 dy ds \\ &= - \int_{Q_{\sigma r}} w^2 \phi^2 \eta \partial_s \eta dy ds + \frac{1}{2} \int_{B(x, \sigma r)} w^2(y, t) \phi^2(y) dy. \end{aligned}$$

Combining this with (3.3), we see that

$$\begin{aligned} \int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dy ds &+ \frac{1}{2} \int_{B(x, \sigma r)} w^2(y, t) \phi^2(y) dy \\ &\leq \int_{Q_{\sigma r}} (|\nabla\psi|^2 + \eta \partial_s \eta) w^2 dy ds + \int_{Q_{\sigma r}} \mathbf{b}(\nabla w)(w\psi^2) dy ds \equiv T_1 + T_2. \end{aligned} \quad (3.4)$$

The first term on the right-hand side of (3.4) is already in good shape. So let us estimate the second term as follows:

$$\begin{aligned}
T_2 &= \int_{Q_{\sigma r}} \mathbf{b}(\nabla w)(w\psi^2) dy ds \\
&= \frac{1}{2} \int_{Q_{\sigma r}} \mathbf{b}\psi^2 \nabla w^2 dy ds = -\frac{1}{2} \int_{Q_{\sigma r}} \operatorname{div}(\mathbf{b}\psi^2) w^2 dy ds \\
&= -\frac{1}{2} \int_{Q_{\sigma r}} \operatorname{div} \mathbf{b} (\psi w)^2 dy ds - \frac{1}{2} \int_{Q_{\sigma r}} \mathbf{b} \nabla (\psi^2) w^2 dy ds \\
&= -\frac{1}{2} \int_{Q_{\sigma r}} \operatorname{div} \mathbf{b} (\psi w)^2 dy ds - \int_{Q_{\sigma r}} \mathbf{b}(\nabla \psi) \psi w^2 dy ds \\
&= - \int_{Q_{\sigma r}} \mathbf{b}(\nabla \psi) \psi w^2 dy ds.
\end{aligned}$$

Here we just used the assumption that $\operatorname{div} \mathbf{b} = 0$.

The next paragraph contains the key argument of the paper.

Notice that for $\delta \in (0, 1)$, $a \in (0, 2)$ and $m \in (1, 2]$,

$$\begin{aligned}
T_2 &\leq \left| \int_{Q_{\sigma r}} \mathbf{b}(\nabla \psi) \psi w^2 dy ds \right| \\
&= \left| \int_{Q_{\sigma r}} \mathbf{b} \psi^{1+\delta} |w|^{2-a} \frac{\nabla \psi}{\psi^\delta} |w|^a dy ds \right| \\
&\leq \left[\int_{Q_{\sigma r}} |\mathbf{b}|^m \psi^{(1+\delta)m} |w|^{(2-a)m} dy ds \right]^{1/m} \\
&\quad \times \left[\int_{Q_{\sigma r}} \left(\frac{|\nabla \psi|}{\psi^\delta} \right)^{m/(m-1)} |w|^{am/(m-1)} dy ds \right]^{(m-1)/m}.
\end{aligned}$$

Take a, δ so that

$$(2-a)m = 2, \quad (1+\delta)m = 2.$$

Then

$$am/(m-1) = a \frac{2}{2-a} \Big/ \left(\frac{2}{2-a} - 1 \right) = 2, \quad \delta = (2/m) - 1 < 1.$$

These and the assumption on the cut-off function ψ show that

$$T_2 \leq \left[\int_{Q_{\sigma r}} |\mathbf{b}|^m (\psi w)^2 dy ds \right]^{1/m} \left[\int_{Q_{\sigma r}} \frac{c}{[(\sigma-1)r]^{m/(m-1)}} w^2 dy ds \right]^{(m-1)/m}.$$

This implies for any $\epsilon > 0$,

$$T_2 \leq \epsilon^m \int_{Q_{\sigma r}} |\mathbf{b}|^m (\psi w)^2 dy ds + C \frac{\epsilon^{-m/(m-1)}}{[(\sigma-1)r]^{m/(m-1)}} \int_{Q_{\sigma r}} w^2 dy ds. \quad (3.5)$$

By our assumptions on \mathbf{b} ,

$$\int_{Q_{\sigma r}} |\mathbf{b}|^m (\psi w)^2 dy ds \leq k \int_{Q_{\sigma r}} |\nabla(\psi w)|^2 dy ds.$$

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Substituting the above to (3.5), we can find $k_1 < 1/2$ and $k_2 > 0$ such that

$$\begin{aligned} |T_2| &= \left| \int_{Q_{\sigma r}} \mathbf{b}(\nabla w)(w\psi^2)dyds \right| \\ &\leq k_1 \int_{Q_{\sigma r}} |\nabla(\psi w)|^2 dyds + k_2 \frac{1}{((\sigma-1)r)^{m/(m-1)}} \int_{Q_{\sigma r}} w^2 dyds. \end{aligned} \quad (3.6)$$

Combining (3.4) with (3.6), we reach

$$\begin{aligned} &\int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dyds + \int_{B(x, \sigma r)} w^2(y, t)\phi^2(y)dy \\ &\leq \frac{C}{((\sigma-1)r)^{m/(m-1)}} \int_{Q_{\sigma r}} w^2 dyds, \quad r \leq 1, \end{aligned} \quad (3.7)$$

$$\int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dyds + \int_{B(x, \sigma r)} w^2(y, t)\phi^2(y)dy \leq \frac{C}{((\sigma-1)r)^2} \int_{Q_{\sigma r}} w^2 dyds, \quad r \geq 1. \quad (3.7')$$

Step 2. $L^2 - L^\infty$ bounds. It is known that (3.7) implies the following $L^2 - L^\infty$ estimate via Moser's iteration.

$$\sup_{Q_r} u^2 \leq C \frac{1}{|Q_r|^{m/(2(m-1))}} \int_{Q_{2r}} u^2 dyds, \quad r \leq 1. \quad (3.8)$$

Here $m > 1$. Also, (3.7') shows

$$\sup_{Q_r} u^2 \leq C \frac{1}{|Q_r|} \int_{Q_{2r}} u^2 dyds, \quad r \geq 1. \quad (3.8')$$

Indeed, by Hölder's inequality,

$$\int_{\mathbf{R}^n} (\phi w)^{2(1+(2/n))} \leq \left(\int_{\mathbf{R}^n} (\phi w)^{2n/(n-2)} \right)^{(n-2)/n} \left(\int_{\mathbf{R}^n} (\phi w)^2 \right)^{2/n}.$$

Using the Sobolev inequality, one obtains

$$\int_{\mathbf{R}^n} (\phi w)^{2(1+(2/n))} \leq C \left(\int (\phi w)^2 \right)^{2/n} \left(\int_{\mathbf{R}^n} |\nabla(\phi w)|^2 \right).$$

The last inequality, together with (3.7) implies, for some $C_1 > 0$,

$$\int_{Q_{\sigma' r}} u^{2p\theta} \leq C \left(C_1 (r\tau)^{-m/(m-1)} \int_{Q_{\sigma r}} u^{2p} \right)^\theta,$$

where $\theta = 1 + (2/n)$.

When the dimension n is odd or $u \geq 0$, we set $\tau_i = 2^{-i-1}$, $\sigma_0 = 1$, $\sigma_i = \sigma_{i-1} - \tau_i = 1 - \sum_1^i \tau_j$, $p = \theta^i$. The above then yields, for some $C_2 > 0$,

$$\int_{Q_{\sigma_{i+1}r}(x,t)} u^{2\theta^{i+1}} \leq C \left(C_2^{i+1} r^{-m/(m-1)} \int_{Q_{\sigma_i r}(x,t)} u^{2\theta^i} \right)^\theta.$$

After iterations the above implies

$$\left(\int_{Q_{\sigma_{i+1}r}(x,t)} u^{2\theta^{i+1}} \right)^{\theta^{-i-1}} \leq C^{\Sigma\theta^{-j-1}} C_2^{-\Sigma(j+1)\theta^{-j-1}} (r^{-m/(m-1)})^{\Sigma\theta^{-j}} \int_{Q_r} u^2.$$

Letting $i \rightarrow \infty$ and observing that $\Sigma_{j=0}^{\infty} \theta^{-j} = (n+2)/2$, we obtain

$$\sup_{Q_{r/2}} u^2 \leq \frac{C}{r^{m(n+2)/(2(m-1))}} \int_{Q_r} u^2.$$

This proves (3.8) either for odd n or for all n and nonnegative u . Similarly one proves (3.8').

In case n is even and u changes sign, we just regard u as a solution of Eq. (1.1) in $\mathbf{R}^{n+1} \times (0, T)$. Then the L^∞ bound of u follows from the above.

Step 3. Weighted estimate. Let G be the heat kernel of (1.1). For a fixed $\lambda \in \mathbf{R}$ and a fixed bounded function ψ such that $|\nabla\psi| \leq 1$, we write

$$f_s(y) = e^{\lambda\psi(y)} \int G(y, s; z, 0) e^{-\lambda\psi(z)} f(z) dz.$$

Here and later the integral takes place in \mathbf{R}^n if no integral region is specified.

Direct computation shows that

$$\begin{aligned} \partial_s \|f_s\|_2^2 &= 2 \int (\partial_s f_s(y)) f_s(y) dy \\ &= 2 \int e^{\lambda\psi(y)} f_s(y) \int \partial_s G(y, s; z, 0) e^{-\lambda\psi(z)} f(z) dz dy \\ &= 2 \int e^{\lambda\psi(y)} f_s(y) \Delta_y \left(\int G(y, s; z, 0) e^{-\lambda\psi(z)} f(z) dz \right) dy \\ &\quad + 2 \int e^{\lambda\psi(y)} f_s(y) \mathbf{b}(y) \nabla_y \left(\int G(y, s; z, 0) e^{-\lambda\psi(z)} f(z) dz \right) dy \\ &\equiv J_1 + J_2. \end{aligned} \tag{3.9}$$

Following standard computation, we see that

$$J_1 \leq -2 \int |\nabla f_s(y)|^2 dy + 2c\lambda^2 \int f_s(y)^2 dy. \tag{3.10}$$

Next we estimate J_2 . For simplicity we write

$$u(y, s) = e^{-\lambda\psi(y)} f_s(y),$$

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which is a solution to $\Delta u + b\nabla u - \partial_t u = 0$ in $\mathbf{R}^n \times (0, \infty)$. Then

$$\begin{aligned}
J_2 &= 2 \int e^{\lambda\psi(y)} f_s(y) \mathbf{b}(y) \nabla_y u(y, s) dy \\
&= -2 \int \operatorname{div}(e^{\lambda\psi(y)} f_s(y) \mathbf{b}(y)) u(y, s) dy \\
&= -2 \int \nabla(e^{\lambda\psi(y)} f_s(y)) \mathbf{b}(y) u(y, s) dy - 2 \int e^{\lambda\psi(y)} f_s(y) u(y, s) \operatorname{div} \mathbf{b}(y) dy \\
&= -2\lambda \int e^{\lambda\psi(y)} f_s(y) u(y, s) \nabla\psi(y) \mathbf{b}(y) dy \\
&\quad - 2 \int e^{\lambda\psi(y)} u(y, s) \nabla f_s(y) \mathbf{b}(y) dy - 2 \int f_s(y)^2 \operatorname{div} \mathbf{b}(y) dy \\
&= -2\lambda \int f_s(y)^2 \nabla\psi(y) \mathbf{b}(y) dy - 2 \int f_s(y) \nabla f_s(y) \mathbf{b}(y) dy - 2 \int f_s(y)^2 \operatorname{div} \mathbf{b}(y) dy \\
&= -2\lambda \int f_s(y)^2 \nabla\psi(y) \mathbf{b}(y) dy - \int f_s(y)^2 \operatorname{div} \mathbf{b}(y) dy.
\end{aligned}$$

In the last step we have used integration by parts.

Hence

$$J_2 = -2\lambda \int f_s(y)^2 \nabla\psi(y) \mathbf{b}(y) dy. \quad (3.11)$$

Using an argument similar to that in the middle of Step 2, we see that

$$\begin{aligned}
J_2 &\leq 2\lambda \int_{Q_{\sigma r}} |\mathbf{b} f_s^{2-a} \nabla\psi f_s^a| dy \\
&\leq \left[\int |\mathbf{b}|^m f_s^{(2-a)m} dy \right]^{1/m} 2\lambda \left[\int (|\nabla\psi|)^{m/(m-1)} f_s^{am/(m-1)} dy \right]^{(m-1)/m}.
\end{aligned}$$

Here, as before $a \in (0, 2)$ and $(2-a)m = 2$, $am/(m-1) = 2$.

It follows that, for any $\epsilon > 0$,

$$J_2 \leq \epsilon \int |\mathbf{b}|^m f_s(y)^2 dy + c\epsilon^{-1/(m-1)} \lambda^{m/(m-1)} \int f_s(y)^2 dy.$$

Combining this with the estimate for J_1 , we have

$$\partial_s \|f_s\|_2^2 \leq -2 \int |\nabla f_s(y)|^2 dy + c_1 (\lambda^{m/(m-1)} + \lambda^2) \int f_s(y)^2 dy + \epsilon \int |\mathbf{b}|^m f_s(y)^2 dy.$$

Here c_1 may depend on ϵ .

Writing

$$F(s) \equiv \|f_s\|_2^2, \quad H(s) \equiv -2 \int |\nabla f_s(y)|^2 dy + \epsilon \int |\mathbf{b}|^m f_s(y)^2 dy,$$

the above differential inequality can be written as

$$\partial_s F(s) \leq c_1 (\lambda^{m/(m-1)} + \lambda^2) F(s) + H(s).$$

Hence

$$F(s) \leq e^{cz(\lambda)s} F(0) + e^{Cz(\lambda)s} \int_0^s e^{-z(\lambda)\tau} H(\tau) d\tau,$$

where

$$z(\lambda) \equiv \lambda^{m/(m-1)} + \lambda^2.$$

That is

$$\begin{aligned} F(s) &\leq e^{cz(\lambda)s} F(0) + e^{Cz(\lambda)s} \left[-2 \int_0^s \int |\nabla(f_\tau(y) e^{-z(\lambda)\tau/2})|^2 dy d\tau \right. \\ &\quad \left. + \epsilon \int_0^s \int |\mathbf{b}|^m (f_s(y) e^{-z(\lambda)\tau/2})^2 dy d\tau \right]. \end{aligned}$$

Taking ϵ sufficiently small and using the condition on \mathbf{b} we conclude that

$$\|f_s\|_2^2 \leq e^{cz(\lambda)s} \|f\|_2^2 = e^{c(\lambda^{m/(m-1)} + \lambda^2)s} \|f\|_2^2. \quad (3.12)$$

Step 4. Gaussian-like upper bound. For simplicity we only prove the bound for $G(x, t; y, 0)$. We just prove the inequality $t \leq 1$. When $t \geq 1$, the situation is simpler and the proof is omitted. Now consider the function

$$u(y, s) = e^{-\lambda\psi(y)} f_s(y)$$

which is a solution to $\Delta u + \mathbf{b}\nabla u - \partial_t u = 0$ in $\mathbf{R}^n \times (0, \infty)$. Here ψ is a function such that $|\nabla\psi| \leq 1$ and whose precise values are to be chosen later. Applying (3.8) with $Q_{\sqrt{t}/2}(x, t) = B(x, \sqrt{t}/2) \times (3t/4, t)$, we obtain

$$u(x, t)^2 \leq C \frac{1}{|Q_{\sqrt{t}/2}(x, t)|^{m/(2(m-1))}} \int_{3t/4}^t \int_{B(x, \sqrt{t}/2)} u^2.$$

From (3.12), it follows that

$$\begin{aligned} e^{2\lambda\psi(x)} u(x, t)^2 &\leq C e^{2\lambda\psi(x)} \frac{1}{|Q_{\sqrt{t}/2}(x, t)|^{m/(2(m-1))}} \int_{3t/4}^t \int_{B(x, \sqrt{t}/2)} u^2 \\ &= C \frac{1}{|Q_{\sqrt{t}/2}(x, t)|^{m/(2(m-1))}} \int_{3t/4}^t \int_{B(x, \sqrt{t}/2)} e^{2\lambda[\psi(x) - \psi(z)]} f_s^2 \\ &\leq C e^{2\lambda\sqrt{t}} \frac{t}{[t|B(x, \sqrt{t})|]^{m/(2(m-1))}} e^{c(\lambda^{m/(m-1)} + \lambda^2)t} \|f\|_2^2. \end{aligned}$$

Taking the supremum over all $f \in L^2(B(y, \sqrt{t}))$ with $\|f\|_2 = 1$, we find that

$$\begin{aligned} e^{2\lambda[\psi(x) - \psi(y)]} \int_{B(y, \sqrt{t}/2)} G(x, t; z, 0)^2 dz \\ \leq C e^{4\lambda\sqrt{t} + c(\lambda^{m/(m-1)} + \lambda^2)t} \frac{t}{[t|B(x, \sqrt{t})|]^{m/(2(m-1))}}. \end{aligned} \quad (3.13)$$

Note that the second entries of the heat kernel G satisfies the equation

$$\Delta u - \nabla(\mathbf{b}u) + \partial_s u = 0.$$

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Hence it satisfies

$$\Delta u - \mathbf{b}\nabla u + \partial_s u = 0.$$

Therefore we can use (3.8) backward on the second entries of the heat kernel to conclude, from (3.13), that

$$\begin{aligned} G(x, t; y, 0)^2 &\leq C \left[\frac{1}{|Q_{\sqrt{t}/2}(y, t)|^{m/(2(m-1))}} \right] \int_0^{t/4} \int_{B(y, \sqrt{t}/2)} G(x, t; z, s)^2 dz ds \\ &\leq C \frac{t^2}{[t|B(x, \sqrt{t})|]^{m/(m-1)}} e^{4\lambda\sqrt{t} + c(\lambda^{m/(m-1)} + \lambda^2)t - 2\lambda[\psi(x) - \psi(y)]}. \end{aligned}$$

This shows, since $\lambda\sqrt{t} \leq c_1 + c_2\lambda^2 t$,

$$G(x, t; y, 0)^2 \leq C \frac{t^2}{[t|B(x, \sqrt{t})|]^{m/(m-1)}} e^{c(\lambda^{m/(m-1)} + \lambda^2)t - 2\lambda[\psi(x) - \psi(y)]}.$$

Now we select ψ so that $\psi(x) - \psi(y) = |x - y|$. Then it follows

$$G(x, t; y, 0)^2 \leq C \frac{t^2}{[t|B(x, \sqrt{t})|]^{m/(m-1)}} e^{c(\lambda^{m/(m-1)} + \lambda^2)t - 2\lambda|x - y|} \equiv C(t)e^{Q(\lambda)}. \quad (3.14)$$

Here for simplicity, we write

$$Q(\lambda) \equiv c(\lambda^{m/(m-1)} + \lambda^2)t - 2\lambda|x - y|.$$

Now we choose λ to be a positive number satisfying

$$\lambda^{1/(m-1)} + \lambda = a|x - y|/t, \quad (3.15)$$

where $a > 0$ will be chosen in a moment. Then

$$Q(\lambda) = c\lambda(\lambda^{1/(m-1)} + \lambda)t - 2\lambda|x - y| = (ca - 2)\lambda|x - y|.$$

Taking $a = 1/c$, we see that

$$Q(\lambda) = -\lambda|x - y|. \quad (3.16)$$

Next we consider two separate cases.

Case 1. $|x - y|/t \geq 1$. Then from (3.15), there exists $c_0 > 0$ such that $\lambda \geq c_0$. Hence $\lambda \leq c_1\lambda^{1/(m-1)}$ because $m \leq 2$. By (3.15), $\lambda \geq c_2(|x - y|/t)^{m-1}$. This shows, via (3.16),

$$Q(\lambda) \leq -c_3 \frac{|x - y|^m}{t^{m-1}}. \quad (3.17)$$

Case 2. When $|x - y|/t \leq 1$. In this case (3.15) implies that $\lambda \leq c_0$ and hence $\lambda^{1/(m-1)} \leq c_1\lambda$. Therefore, by (3.15), $\lambda \geq c_2|x - y|/t$. Hence

$$Q(\lambda) \leq -c_3|x - y|^2/t. \quad (3.18)$$

Substituting (3.17) and (3.18) to (3.14), we obtain

$$G(x, t; y, 0) \leq \frac{c_1 t}{[t|B(x, \sqrt{t})|]^{m/(2(m-1))}} \left[\exp\left(-c_2 \frac{|x - y|^m}{t^{m-1}}\right) + \exp\left(-c_2 \frac{|x - y|^2}{t}\right) \right].$$

This proves the upper bound for G .

Step 5. Proof of (ii). Since the proof is identical to that in [LZ], we omit the details.

Final Remark. Using a Nash type estimate, it is easy to prove that $G(x, t; y, s) \leq c/(t - s)^{n/2}$. It would be interesting to combine this bound with the bound in (iii) to get a sharper bound. The same could be said about the lower bound.

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