STABILITY OF THE CHENG-YAU GRADIENT ESTIMATE

QI S. ZHANG

Abstract. We prove that the Cheng-Yau gradient estimate on positive harmonic functions on manifolds with non-negative Ricci curvature is globally stable under certain perturbation of the metric. In some cases, one only needs the condition $Ricci(x) \ge -\frac{\epsilon}{1+d(x)^{2+\delta}}$ with $\delta > 0$ and $\epsilon (> 0)$ being sufficiently small. Whether such stability holds is a question that has been circulating for some time.

1. Introduction

One of the most useful inequalities in geometric analysis is the Cheng-Yau estimates on the gradient of positive harmonic functions.

Theorem *(Cheng-Yau). Let $\mathbf{M}$ be a complete manifold with dimension $n \ge 2$, $Ricci(\mathbf{M}) \ge -k$, $k \ge 0$. Suppose u is any positive harmonic function in a geodesic ball $B(x_0, r) \subset \mathbf{M}$. There holds*

$$\sup_{B(x_0,r/2)} \frac{|\nabla u|}{u} \le \frac{c_n}{r} + c_n\sqrt{k}, \tag{1.1}$$

where c_n depends only on the dimension n.

When the manifold $\mathbf{M}$ has nonnegative Ricci curvature, i.e. $k = 0$, then the Cheng-Yau estimate becomes

$$\sup_{B(x_0,r/2)} \frac{|\nabla u|}{u} \le \frac{c_n}{r}, \tag{1.2}$$

which is sharp as indicated in the Euclidean case. However even if $\mathbf{M}$ contains a small compact region where the Ricci curvature is not nonnegative, estimate (1.1) becomes very much different from (1.2) when r is large, due to the presence of the $\sqrt{k}$ term. Whether estimate (1.2) is stable under perturbation has been an open question for some time, in light of the known stability results on weaker properties of harmonic functions, such as the Harnack inequality.

The goal of the paper is to confirm that (1.2) is stable when the nonpositive part of the Ricci curvature is sufficiently small in an integral sense.

Let us mention that some smallness for the nonpositive part of the Ricci curvature is *necessary* for gradient estimate (1.2) to hold. For instance if the non-positive part of the Ricci curvature is so large that $\mathbf{M}$ admits a bounded nonconstant harmonic function, then clearly (1.2) can not hold.

Throughout the paper Δ is the Laplace-Beltrami operator, $d(x, y)$ is the distance between x and y; and $d(x)$ is the distance between x and a fixed reference point. $|B(x, r)|$ denotes the volume of the geodesic ball of radius r centered at x.

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Let us layout the basic assumptions to be used in the paper. As to be explained later, these assumptions are stable under certain perturbation of the metric.

Assumption (A). M is a complete noncompact Riemannian manifold satisfying the volume doubling property.

$$|B(x, 2r)| \le 2^\nu |B(x, r)|$$

for all $x \in \mathbf{M}$, $r > 0$ and some $\nu > 0$. $n = dim(\mathbf{M}) \ge 2$.

Assumption (B). The heat kernel G of the Laplace-Beltrami operator satisfies a Gaussian upper bound:

$$G(x, t; y, 0) \le \frac{B_1}{|B(x, \sqrt{t})|} e^{-b_1 d^2(x,y)/t},$$

for some $b_1, B_1 > 0$, and all $x, y \in \mathbf{M}$ and $t > 0$.

Several conditions are well known to be equivalent to assumptions (A) and (B). For instance, it was proved in [5] that assumptions (A) and (B) together are equivalent to the following relative Faber-Krahn inequality:

(FK). For all $x \in M$, $r > 0$, and every non-empty subset $\Omega \subset B(x, r)$,

$$\lambda_1(\Omega) \ge \frac{c}{r^2} \left(\frac{|B(x, r)|}{|\Omega|} \right)^{2/\nu}.$$

Here $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Ω and $c > 0$.

It is also known that $(A) + (B) \Leftrightarrow (A) + (B')$, where (B') is:

Assumption (B'). The following Sobolev inequality holds for all $\phi \in C_0^\infty(B(x, r))$, $x \in \mathbf{M}$, $r > 0$, and a fixed $\alpha > 2$.

$$\Big(\int \phi^{2\alpha/(\alpha-2)} dy \Big)^{(\alpha-2)/\alpha} \le S_0 |B(x, r)|^{-2/\alpha} \int \big(r^2 |\nabla \phi|^2 + \phi^2 \big) dy.$$

See [11]. Moreover (A)+(B) is also equivalent with (A) + a Poicaré inequality (See [11] and [5]). Further (A)+(B) are equivalent to (A)+ a mean value inequality (see [8].) There is an extensive literature on manifolds satisfying various global conditions including the ones mentioned above. We refer the interested reader to [6], [12] and the reference therein.

There exist many manifolds satisfying assumptions (A) and (B), among them manifolds which are quasi isometric to manifolds with nonnegative Ricci curvature; connected sums of two copies of $\mathbb{R}^n$. See for example [2]

Next we introduce the conditions on the non-positive part of the Ricci curvature that will imply the global Cheng-Yau estimate (1.2). The conditions, in general integral form first, will be elucidated in the Corollary below by simple conditions. Essentially, the non-positive part of the Ricci curvature is required to be small and decay sufficiently fast near infinity.

Let $\lambda = \lambda(x)$ be the lowest eigenvalue of $Ric(x)$, $x \in \mathbf{M}$. We will use the notation

$$V(x) = \lambda^-(x) = (|\lambda(x)| - \lambda(x))/2. \tag{1.3}$$

Assumption (C). $V \in L^\infty(\mathbf{M})$ and there exists $\epsilon_0 > 0$ and $K > 0$ such that

(1.4)

$$\begin{cases} (1). N(V) \equiv \sup_{x \in \mathbf{M}} \int_0^\infty \int_{\mathbf{M}} \frac{e^{-d(x,y)^2/t}}{|B(x,\sqrt{t})|} V(y) dy dt < \epsilon_0, \\ (2). \text{ for all } \phi \in C_0^\infty(B(x, r)), \quad \int_{\mathbf{M}} V(y) \phi^2(y) dy < \frac{1}{11n} \int_{\mathbf{M}} |\nabla \phi(x)|^2 dx + \frac{K}{r^2} \int_{\mathbf{M}} \phi^2(y) dy, \end{cases}$$

Here is the main result of the paper.

Theorem 1.1. *Suppose M satisfies assumptions (A), (B). There exists $\epsilon_0 > 0$ depending only on the parameters in (A) and (B) so that if assumption (C) for Ricci curvature holds then the following statement holds.*

Let u be a positive harmonic function in the ball $B(x,r)$, then

$$\sup_{y\in B(x,r/2)} \frac{|\nabla u(y)|}{u(y)} \le \frac{C_0}{r}.$$

Here C_0 depends only on the parameters in the assumptions : ν, b_1, B_1, ϵ_0, K and n.

Remark 1.1. The condition on V in the theorem in many cases simply means that

$$Ric(x) \ge -\frac{\epsilon}{1+d(x)^{2+\delta}}$$

for some sufficiently small $\epsilon > 0$ and $\delta > 0$. This is indicated in Corollary 1.1 below. In general, item one in condition (1.4) is a Kato type condition and item 2 takes the form of Hardy's inequality i.e. for $f \in C_0^\infty(\mathbb{R}^n)$, $n \ge 3$,

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{1}{|x|^2} f(x)^2 dx \le \int_{\mathbb{R}^n} |\nabla f|^2(x) dx.$$

For versions of Hardy's inequality in manifolds, see the paper [1]. In many situations the first item implies the second one, as in the case of the corollary.

Corollary 1. *Suppose M satisfies the Sobolev inequality (B') with $\alpha = n$ and that $|B(x,r)|$ is comparable with r^n, $n > 2$, i.e. there exists $a > 0$ such that $a^{-1}r^n \le |B(x,r)| \le ar^n$ for all $x \in \mathbf{M}$ and $r > 0$. Then the gradient bound (1.2) holds if*

$$Ric(x) \ge -\frac{\epsilon}{1+d(x)^{2+\delta}}$$

for a sufficiently small $\epsilon > 0$ and $\delta > 0$. Here $\epsilon = \epsilon(b_1, B_1, \delta, a, n)$ only.

In particular if $\mathbf{M}$ is a small compact perturbation of $\mathbf{R}^n$, $n \ge 3$, then (1.2) holds.

Moreover if $\mathbf{M}$ is a small compact perturbation of a $n(\ge 3)$ dimensional manifold with nonnegative Ricci curvature and maximum volume growth, then (1.2) holds.

Proof.

Since $\mathbf{M}$ satisfies the extra conditions in the volume of geodesic balls in the corollary, it is easy to see that

$$N(V) = \sup_{x\in\mathbf{M}} \int_0^\infty \int_{\mathbf{M}} \frac{e^{-d(x,y)^2/t}}{|B(x,\sqrt{t})|} V(y) dy dt \le c \sup_{x\in\mathbf{M}} \int_{\mathbf{M}} \frac{d(x,y)^2}{|B(x,d(x,y))|} V(y) dy.$$

Write

$$K(V) \equiv \sup_{x\in\mathbf{M}} \int_{\mathbf{M}} \frac{d(x,y)^2}{|B(x,d(x,y))|} V(y) dy. \tag{1.5}$$

By direct calculation, if $V(x) \le \frac{\epsilon}{1+d(x)^{2+\delta}}$, then

$$\begin{aligned} K(V) &\le \sup_{x\in\mathbf{M}} \int_{d(x,y)\ge d(y)/2} \frac{d(x,y)^2}{|B(x,d(x,y))|} V(y) dy + \sup_{x\in\mathbf{M}} \int_{d(x,y)\le d(y)/2} \frac{d(x,y)^2}{|B(x,d(x,y))|} V(y) dy \\ &\le C\epsilon. \end{aligned}$$

Next, given $\phi \in C_0^\infty(B(x_0, r))$

$$\int_{\mathbf{M}} V(x)\phi^2(x)dx \le \big(\int_{\mathbf{M}} V^{n/2}(x)dx\big)^{2/n} \ \big(\int_{\mathbf{M}} \phi^{2n/(n-2)}(x)dx\big)^{(n-2)/n}.$$

By the Sobolev inequality

$$\int_{\mathbf{M}} V(x)\phi^2(x)dx \le S_0\big(\int_{\mathbf{M}} V^{n/2}(x)dx\big)^{2/n}\big[\int_{\mathbf{M}} |\nabla\phi|^2(x)dx + \frac{k}{r^2}\int_{\mathbf{M}} \phi^2(x)dx\big].$$

Simple calculation then shows that

$$\int_{\mathbf{M}} V(x)\phi^2(x)dx \le C\epsilon S_0\big[\int_{\mathbf{M}} |\nabla\phi|^2(x)dx + \frac{k}{r^2}\int_{\mathbf{M}} \phi^2(x)dx\big].$$

Hence all the condition of Theorem 1.1 is satisfied when ϵ is sufficiently small.

There are plenty of examples of such manifolds due the stability of $G(x, t; y, 0)$ under perturbation of the metric. For instance, let $\mathbf{M}$ be $\mathbf{R}^n$ equipped with a metric coming from a small perturbation of the Euclidean metric. Here $n \ge 3$. Then by standard results B_1 and b_1 can be chosen to be close to $1/(2\sqrt{\pi})^n$ and $1/4$, the Euclidean constants. At the same time, the nonpositive part of the Ricci curvature can be arbitrarily small. Therefore the above quantity $N(V)$ can be arbitrarily small while ϵ_0, depending only on B_1, b_1 and the doubling constant, is bounded away from zero. Thus $N(V) < \epsilon_0$.

The last statement in the corollary is proved in the same manner. $\square$

Remark 1.2. The constant ϵ_0 and C_0 in Theorem 1.1 can be estimated explicitly, as indicated in the proof. The assumption $V \in L^\infty(\mathbf{M})$ is not necessary. But we will not seek the full generality.

It is not clear whether the current method can show the Li-Yau gradient estimate on caloric functions [9] is stable.

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2. Proof of Theorem 1.1

Let u be a positive solution of $\Delta u = 0$.

The proof is carried out in several steps.

Step 1. We follow [Y] and [CY] to derive an equation of $\log u$.

Set $f = \log u$, then $\Delta f = -|\nabla f|^2$. Define

$$F \equiv |\nabla f|^2.$$

Following [Y] and [CY], by Bochner's identity, one knows that F obeys

$$\Delta F \ge -2\nabla f \nabla F + \frac{2}{n}F^2 - 2VF. \tag{2.1}$$

Here $V = V(x)$ is the absolute value of the negative part of the lowest eigenvalue of $Ric(x)$.

Step 2. From here our proof is different from that in [Y] and [CY] where the maximum principle was used. In contrast, we will use certain integral estimates motivated by the De gorgi-Nash-Moser theory on linear elliptic and parabolic equations.

The idea is to convert (2.1) into a linear inequality and to prove that the heat kernel of the corresponding operator satisfies a global Gaussian upper bound, when $N(V)$ (defined in (1.4)) is sufficiently small. Then one can use the local representation formula for solutions and Hölder inequality to derive a L^∞ bound for F. However, it is not clear that the linear operator

$$-\Delta - 2\nabla f \nabla + \frac{2}{n} F - 2V$$

associated with (2.1) is positive definite. Hence there is no hope of proving a global Gaussian upper bound for the corresponding heat kernel.

To overcome the difficulty, we consider the function

$$w = F^m, \qquad m = 5n. \tag{2.2}$$

By direct calculation, one easily finds, via (2.1), that

$$\Delta w + 2\nabla f \nabla w - 10Fw + 10nVw \geq 0. \tag{2.3}$$

We define the operators L_1 and L_2 by

$$L_1 = \Delta + 2\nabla f \nabla - 10F \tag{2.4}$$

and

$$L_2 = \Delta + 2\nabla f \nabla - 10F + 10nV; \tag{2.5}$$

and their corresponding heat kernel by G_1 and G_2 respectively. We will eventually prove that G_2 has a global Gaussian upper bound when $N(V)$ is small. More importantly, the coefficients in the Gaussian upper bound should be independent of f which is not a fixed function. This is achieved by exploiting the special structure of the operator L_1.

First, we have to show that G_1 satisfies a global Gaussian upper bound first.

Step 3. global Gaussian upper bound for $G_1 = e^{-L_1 t}$.

This step is divided into two sub-steps.

Step 3.1. a L^2 mean value inequality for positive solutions of

$$L_3 w - w_t \equiv \Delta w + 2\nabla f \nabla w - 8Fw - w_t \geq 0. \tag{2.6}$$

More precisely, we prove, for any positive solutions of (2.6),

$$\sup_{Q_{r/2}(x,t)} w^2 \leq \frac{C_7}{B(x,r) r^2} \|w\|^2_{L^2(Q_r(x,t))}. \tag{2.6'}$$

Here and later $Q_r(x,t)$ or simply Q_r, stands for $B(x,r) \times [t - r^2, t]$.

Notice that the zero order term in L_3 is $-8Fw$ instead of $-10Fw$ in L_1. This makes L_3 a 'bigger' operator than L_1.

Choose $\psi = \phi(y)\eta(s)$ to be a cut-off function satisfying, for $\sigma > 0$,

$$supp\ \eta \subset (t - (\sigma r)^2, t); \quad \eta(s) = 1, \quad s \in [t - r^2, t]; \quad |\eta'| \leq 2/((\sigma - 1)r)^2; \quad 0 \leq \eta \leq 1;$$

$$supp\ \phi \subset B(x, \sigma r); \quad \phi(y) = 1, \quad y \in B(x,r); \quad 0 \leq \phi \leq 1;$$

$$|\nabla \phi| \leq \frac{A}{(\sigma - 1)r}, \qquad A > 0.$$

Using $w\psi^2$ as a test function on (2.6), one obtains

$$\int_{Q_{\sigma r}} (\Delta w - 2\nabla f \nabla w - 8Fw - \partial_s w) w \psi^2 dyds \geq 0.$$

Using integration by parts, one deduces

$$
\begin{aligned}
&\int_{Q_{\sigma r}} \nabla(w\psi^2)\nabla w dyds \\
&\qquad \le \int_{Q_{\sigma r}} 2\nabla f \nabla w (w\psi^2) dyds - \int_{Q_{\sigma r}} 8Fw^2 dyds - \int_{Q_{\sigma r}} (\partial_s w) w\psi^2 dyds.
\end{aligned} \tag{2.7}
$$

By direct calculation,

$$
\begin{aligned}
&\int_{Q_{\sigma r}} \nabla(w\psi^2)\nabla w dyds = \int_{Q_{\sigma r}} \nabla[(w\psi)\psi]\nabla w dyds \\
&= \int_{Q_{\sigma r}} [\ \nabla(w\psi)(\ \nabla(w\psi) - (\nabla\psi)w) + w\psi\nabla\psi\nabla w] dyds \\
&= \int_{Q_{\sigma r}} [\ |\nabla(w\psi)|^2 - |\nabla\psi|^2 w^2\ ] dyds.
\end{aligned}
$$

Substituting this to (2.7), we obtain

$$
\begin{aligned}
&\int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dyds \\
&\qquad \le \int_{Q_{\sigma r}} 2\nabla f \nabla w (w\psi^2) dyds - \int_{Q_{\sigma r}} 8Fw^2 dyds \\
&\qquad\qquad - \int_{Q_{\sigma r}} (\partial_s w) w\psi^2 dyds + \int_{Q_{\sigma r}} |\nabla\psi|^2 w^2 dyds.
\end{aligned} \tag{2.8}
$$

Next, notice that

$$
\begin{aligned}
&\int_{Q_{\sigma r}} (\partial_s w) w\psi^2 dyds = \frac{1}{2}\int_{Q_{\sigma r}} (\partial_s w^2)\psi^2 dyds \\
&\qquad = -\int_{Q_{\sigma r}} w^2\phi^2\eta\partial_s\eta dyds + \frac{1}{2}\int_{B(x,\sigma r)} w^2(y,t)\phi^2(y) dy.
\end{aligned}
$$

Combining this with (2.8), we see that

$$
\begin{aligned}
&\int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dyds + \frac{1}{2}\int_{B(x,\sigma r)} w^2(y,t)\phi^2(y) dy \\
&\le \int_{Q_{\sigma r}} (|\nabla\psi|^2 + \eta\partial_s\eta)\ w^2 dyds \\
&\qquad + \int_{Q_{\sigma r}} 2\nabla f \nabla w (w\psi^2) dyds - \int_{Q_{\sigma r}} 8Fw^2 dyds.
\end{aligned} \tag{2.9}
$$

The first term on the righthand side of (2.9) is already in good shape. So let us estimate the second term as follows.

$$
\begin{aligned}
&\int_{Q_{\sigma r}} 2\nabla f(\nabla w)(w\psi^2)dyds\\
&\quad= 2\int_{Q_{\sigma r}} \nabla f[\nabla(w\psi) - w\nabla\psi]w\psi dyds\\
&\quad= 2\int_{Q_{\sigma r}} \nabla f\nabla(w\psi)w\psi dyds - 2\int_{Q_{\sigma r}} (w\psi\nabla f)w\nabla\psi dyds\\
&\quad\le \frac{1}{2}\int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dyds + 4\int_{Q_{\sigma r}} |\nabla f|^2 (w\psi)^2 dyds\\
&\qquad+ \int_{Q_{\sigma r}} |\nabla f|^2 (w\psi)^2 dyds + \int_{Q_{\sigma r}} w^2|\nabla\psi|^2 dyds.
\end{aligned}
$$

Recall that $|\nabla f|^2 = F$. Hence the above becomes

$$
\int_{Q_{\sigma r}} 2\nabla f(\nabla w)(w\psi^2)dyds \le \frac{1}{2}\int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dyds + 5\int_{Q_{\sigma r}} F(w\psi)^2 dyds + \int_{Q_{\sigma r}} w^2|\nabla\psi|^2 dyds.
$$

Substituting the above to the right hand side of (2.9), we deduce

$$
\int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dyds + \frac{1}{2}\int_{B(x,\sigma r)} w^2(y,t)\phi^2(y)dy \le 2\int_{Q_{\sigma r}} (2|\nabla\psi|^2 + \eta\partial_s\eta)\ w^2 dyds. \tag{2.10}
$$

Observe that the terms containing F drop out. For later use let us remark that if w satisfies, for a function h,

$$
L_3 w - w_t + h \ge 0,
$$

then follow exactly the same calculation, one has

$$
\begin{aligned}
&\int_{Q_{\sigma r}} |\nabla(w\psi)|^2 dyds + \frac{1}{2}\int_{B(x,\sigma r)} w^2(y,t)\phi^2(y)dy\\
&\qquad\le 2\int_{Q_{\sigma r}} (2|\nabla\psi|^2 + \eta\partial_s\eta)\ w^2 dyds + \int_{Q_{\sigma r}} hw\psi^2 dyds.
\end{aligned} \tag{2.10$'$}
$$

By direct calculation it is easy to see that for any $p > 1$,

$$
\begin{aligned}
L_3 w^p &\equiv \Delta w^p + 2\nabla f\nabla w^p - 8Fw^p - (w^p)_t\\
&\ge p(p-1)|\nabla w|^2 w^{p-2} + 8F(p-1)w^p \ge 0.
\end{aligned}
$$

Hence by repeating the above argument, we obtain, for any $p > 1$,

$$
\int_{Q_{\sigma r}} |\nabla(w^p\psi)|^2 dyds + \frac{1}{2}\int_{B(x,\sigma r)} (w^p(y,t))^2\phi^2(y)dy \le 2\int_{Q_{\sigma r}} (2|\nabla\psi|^2 + \eta\partial_s\eta)\ (w^p)^2 dyds.
$$

Therefore

$$
\int_{Q_{\sigma r}} |\nabla(w^p\psi)|^2 dyds + \frac{1}{2}\int_{B(x,\sigma r)} (w^p(y,t))^2\phi^2(y)dy \le \frac{C}{r^2}\int_{Q_{\sigma r}} (w^p)^2 dyds. \tag{2.11}
$$

It is well known that (2.11) and the Sobolev inequality leads to the following mean value inequality via Moser's iteration.

$$
\sup_{Q_r} w^2 \le \frac{B}{|Q_r|}\int_{Q_{2r}} w^2 dyds. \tag{2.12}
$$

By keeping track of the constants in the above computation, we know that the constant B is independent of f or F. For the sake of completeness we give a sketch of the proof.

The rest of the proof is standard. By Hölder's inequality,

$$\int\int (w^p\psi)^{2(1+(2/\alpha))} dyds \le \int \Big(\int (w^p\psi)^{2\alpha/(\alpha-2)} dy\Big)^{(\alpha-2)/\alpha} \Big(\int (w^p\psi)^2\Big)^{2/\alpha} ds.$$

Using the Sobolev inequality (Assumption (B')), one obtains

$$\int\int (w^p\psi)^{2(1+(2/\alpha))} dyds$$
$$\le S_0 |B(x,r)|^{-2/\alpha} \sup_{s\in[t-\sigma r^2, t]} \Big(\int (w^p\psi)^2 dy\Big)^{2/\alpha} \int\int \big(r^2 |\nabla (w^p\psi)|^2 + (w^p\psi)^2 dyds\big).$$

The last inequality, together with (2.11) implies

$$\int_{Q_{\sigma' r}(x,t)} w^{2p\theta} \le \big(C_5 S_0 \lambda(r)^{-1} \int_{Q_{\sigma r}(x,t)} w^{2p}\big)^\theta, \tag{2.13}$$

where $\theta = 1 + (2/\alpha)$, $\tau = \sigma - \sigma'$ and

$$\lambda(r) = |B(x,r)|^{2/(2+\alpha)} (r\tau)^{4/(2+\alpha)}.$$

We now set

$$\tau_i = 2^{-i-1}, \quad \sigma_0 = 1, \quad \sigma_i = \sigma_{i-1} - \tau_i = 1 - \Sigma_1^i \tau_j, \quad p = \theta^i.$$

Inequality (2.13) then yields

$$\int_{Q_{\sigma_{i+1}}(x,t)} w^{2\theta^{i+1}} \le C \big(C_6^{i+1} \lambda(r)^{-1} \int_{Q_{\sigma_i r}(x,t)} w^{2\theta^i}\big)^\theta.$$

After iterations the above implies

$$\Big(\int_{Q_{\sigma_{i+1}}(x,t)} w^{2\theta^{i+1}}\Big)^{\theta^{-i-1}} \le C^{\Sigma\theta^{-j-1}} C_6^{-\Sigma(j+1)\theta^{-j-1}} (\lambda(r)^{-1})^{\Sigma\theta^{-j}} \int_{Q_r(x,t)} w^2.$$

Here j goes from 0 to i. Letting $i \to \infty$ and noticing that $\Sigma_{j=0}^\infty \theta^{-j} = (\alpha+2)/2$ we arrive at

$$\sup_{Q_{r/2}(x,t)} w^2 \le \frac{C_7}{B(x,r) r^2} \|w\|^2_{L^2(Q_r(x,t))}.$$

This proves the mean value inequality for w satisfying $L_3 w - w_t \ge 0$.

Step 3.2. Gaussian upper bound for G_1.

The proof of the upper bound is done by modifying the standard method due to E. B. Davies [4]. In order to prove a bound that is independent of f or F, we have to use the special structure of the operator L_1.

For a fixed $\lambda \in \mathbf{R}$ and a fixed bounded function ψ such that $|\nabla\psi| \le 1$, we write

$$q(y) = e^{\lambda\psi(y)} \int G_1(y, s; z, 0) e^{-\lambda\psi(z)} h(z) dz.$$

Here h is a smooth compactly supported function. Then

$$
\begin{aligned}
\partial_s \|q\|_2^2 &= 2\int_{\mathbf{M}} q(y,s)\partial_s q(y,s) \\
&= 2\int_{\mathbf{M}} e^{\lambda\psi(y)} q(y,s)\int_{\mathbf{M}} \partial_s G_1(y,s;z,0)e^{-\lambda\psi(z)}h(z)dzdy \\
&= 2\int_{\mathbf{M}} e^{\lambda\psi(y)} q(y,s)\int_{\mathbf{M}} [\Delta_y G_1 + 2\nabla_y f\nabla_y G_1 - 10FG_1]e^{-\lambda\psi(z)}h(z)dzdy \\
&= 2\int_{\mathbf{M}}\int_{\mathbf{M}} e^{\lambda\psi(y)} q(y,s)\Delta_y G_1 e^{-\lambda\psi(z)}h(z)dzdy \\
&\quad + 2\int_{\mathbf{M}}\int_{\mathbf{M}} e^{\lambda\psi(y)} q(y,s)\nabla_y f\nabla_y G_1 e^{-\lambda\psi(z)}h(z)dzdy \\
&\quad - 10\int_{\mathbf{M}}\int_{\mathbf{M}} e^{\lambda\psi(y)} q(y,s) F G_1 e^{-\lambda\psi(z)}h(z)dzdy \\
&\equiv I_1 + I_2 + I_3.
\end{aligned} \tag{2.14}
$$

Using integration by parts, one has, by standard arguments,

$$
I_1 \le -2\int_{\mathbf{M}} |\nabla q(y,s)|^2 dy + 2c\lambda^2\int_{\mathbf{M}} q^2(y,s)dy. \tag{2.15}
$$

Next observe that

$$
\begin{aligned}
I_2 &= 2\int_{\mathbf{M}} e^{\lambda\psi(y)} q(y,s)\nabla_y f\int_{\mathbf{M}} \nabla_y G_1 e^{-\lambda\psi(z)}h(z)dzdy \\
&= 2\int_{\mathbf{M}} e^{\lambda\psi(y)} q(y,s)\nabla_y f\nabla_y\int_{\mathbf{M}} G_1(y,s;z,0) e^{-\lambda\psi(z)}h(z)dzdy \\
&= 2\int_{\mathbf{M}} e^{\lambda\psi(y)} q(y,s)\nabla_y f\nabla_y (e^{-\lambda\psi(y)}q(y,s))dy \\
&= \int_{\mathbf{M}} e^{2\lambda\psi(y)}\nabla_y f\nabla_y[(e^{-\lambda\psi(y)}q(y,s))^2]dy \\
&= -2\lambda\int_{\mathbf{M}} q^2(y,s)\nabla_y\psi\nabla_y f dy - \int_{\mathbf{M}} q^2(y,s)\Delta_y f dy.
\end{aligned}
$$

Since $|\nabla\psi| \le 1$ and $\Delta_y f = -|\nabla f|^2 = -F$, it follows that

$$
I_2 \le \lambda^2\int_{\mathbf{M}} q^2(y,s)dy + \int_{\mathbf{M}} q^2(y,s)|\nabla_y f|^2 dy + \int_{\mathbf{M}} q^2(y,s)|\Delta_y f|dy;
$$

i.e.

$$
I_2 \le \lambda^2\int_{\mathbf{M}} q^2(y,s)dy + 2\int_{\mathbf{M}} q^2(y,s)F dy. \tag{2.16}
$$

Notice also that

$$
I_3 = -10\int_{\mathbf{M}}\int_{\mathbf{M}} e^{\lambda\psi(y)} q(y,s) F G_1 e^{-\lambda\psi(z)}h(z)dzdy = -10\int_{\mathbf{M}} q^2(y,s)F dy. \tag{2.17}
$$

Substituting (2.15)-(2.17) to (2.14), we see that the terms containing F are negative. Hence

$$
\partial_s\|q(\cdot,s)\|_2^2 \le c_0\lambda^2\|q(\cdot,s)\|_2^2,
$$

which implies

$$\|q(\cdot, s)\|_2^2 \le e^{c_0\lambda^2 s}||h||_2^2. \tag{2.18}$$

Now consider the function

$$u(y, s) = e^{-\lambda\psi(y)}q(y, s)$$

which is a solution to $L_1 u - u_s = 0$ in $\mathbf{M} \times (\mathbf{0}, \infty)$. Hence

$$L_3 u - u_s = L_1 u - u_s + 2Fu \ge 0.$$

Here L_3 is defined in (2.6). By the mean value inequality of (2.12) with $Q_{\sqrt{t}/2}(x, t) = B(x, \sqrt{t}/2) \times (3t/4, t)$, we obtain

$$u(x, t)^2 \le \frac{C}{|Q_{\sqrt{t}/2}(x, t)|} \int_{3t/4}^t \int_{B(x,\sqrt{t}/2)} u^2$$

It follows that

$$\begin{aligned} e^{2\lambda\psi(x)}u(x, t)^2 &\le e^{2\lambda\psi(x)} \frac{C}{|Q_{\sqrt{t}/2}(x, t)|} \int_{3t/4}^t \int_{B(x,\sqrt{t}/2)} u^2 \\ &= \frac{C}{|Q_{\sqrt{t}/2}(x, t)|} \int_{3t/4}^t \int_{B(x,\sqrt{t}/2)} e^{2\lambda[\psi(x)-\psi(z)]}q^2 \\ &\le e^{2\lambda\sqrt{t}} \frac{C}{|B(x, \sqrt{t})|} e^{c_0\lambda^2 t}\|h\|_2^2. \end{aligned}$$

Taking the supremum over all $h \in L^2(B(y, \sqrt{t}))$ with $\|h\| = 1$, we find that

$$e^{2\lambda[\psi(x)-\psi(y)]} \int_{B(y,\sqrt{t}/2)} G_1(x, t; z, 0)^2 dz \le Ce^{4\lambda\sqrt{t}+c_0\lambda^2 t} \frac{1}{|B(x, \sqrt{t})|}.$$

Using the mean value inequality on the second entries of the heat kernel G_1 in the backward cubed

$$B(y, \sqrt{t}/2) \times [0, t/4],$$

we have

$$\begin{aligned} G_1(x, t; y, 0)^2 &\le \frac{C}{|Q_{\sqrt{t}/2}(y, t)|} \int_0^{t/4} \int_{B(y,\sqrt{t}/2)} G_1(x, t; z, s)^2 dzds \\ &\le \frac{C}{|B(x, \sqrt{t})|\ |B(y, \sqrt{t})|} e^{4\lambda\sqrt{t}+c_0\lambda^2 t-2\lambda[\psi(x)-\psi(y)]} \end{aligned}$$

Here we remark that the second entries of G_1 satisfies the conjugate equation of $L_1 u - u_s = 0$. That is, if $v(z, s) = G_1(x, t; z, s)$ then

$$\Delta v - 2\nabla f \nabla v - 2\Delta f v - 10Fv + v_s = 0.$$

Recall that $\Delta f = -F$. Hence v satisfies

$$L_3 v + v_s = \Delta v - 2\nabla f \nabla v - 8Fv + v_s = 0.$$

From step 3.1 it is clear that the mean value inequality still holds on the backward parabolic cube.

Choosing $\lambda = d(x, y)/(c_0 t)$ and ψ such that $\psi(x) - \psi(y) = d(x, y)$, we reach

$$G_1(x, t; y, 0)^2 \le \frac{C}{|B(x, \sqrt{t})|\ |B(y, \sqrt{t})|} e^{-d(x,y)^2/(2c_0 t)}. \tag{2.19}$$

This proves the Gaussian upper bound for G_1.

Step 5. Gaussian upper bound of G_2 by perturbation.

Using the bound for G_1 and a perturbation argument in [Z], we will prove a Gaussian upper bound for G_2 when $N(V)$ is small.

By Duhamel's formula

$$G_2(x,t;y,0) = G_1(x,t;y,0) + 10n\int_0^t \int_{\mathbf{M}} G_1(x,t;z,\tau)V(z)G_2(z,\tau;y,0)dzd\tau.$$

By step 3, for some $c_1, C_1 > 0$,

$$G_1(x,t;z,\tau) \le \frac{C_1}{|B(x,\sqrt{t-\tau})|} e^{-c_1 d(x,z)^2/(t-\tau)}.$$

Also, by standard perturbation argument (since V is bounded), there is $M_\tau > 0$ such that

$$G_2(z,\tau;y,0) \le \frac{M_\tau}{|B(z,\sqrt{\tau})|} e^{-c_1 d(z,y)^2/\tau} \le \frac{M_\tau}{|B(z,\sqrt{\tau})|} e^{-c_1 d(z,y)^2/(2\tau)}.$$

We need to prove that M_τ can be chosen independent of time.

Therefore

$$\begin{aligned} &G_2(x,t;y,0)\\ &\le \frac{C_1 e^{-c_1 d(x,y)^2/t}}{|B(x,\sqrt{t})|} + C_1 \int_0^t \int_{\mathbf{M}} \frac{e^{-c_1 d(x,z)^2/(t-\tau)}}{|B(x,\sqrt{t-\tau})|} V(z) \frac{M_\tau e^{-c_1 d(z,y)^2/(2\tau)}}{|B(z,\sqrt{\tau})|} dzd\tau. \end{aligned}$$

Let m_t be the minimum of the constants M_τ such that

$$G_2(z,\tau;y,0) \le \frac{m_\tau}{|B(z,\sqrt{\tau})|} e^{-c_1 d(z,y)^2/(2\tau)}$$

holds for all $\tau \in (0,t]$ and $y, z \in \mathbf{M}$. Then

$$\begin{aligned} &G_2(x,t;y,0)\\ &\le \frac{C_1 e^{-c_1 d(x,y)^2/t}}{|B(x,\sqrt{t})|} + C_1 m_t \int_0^t \int_{\mathbf{M}} \frac{e^{-c_1 d(x,z)^2/(t-\tau)}}{|B(x,\sqrt{t-\tau})|} V(z) \frac{e^{-c_1 d(z,y)^2/(2\tau)}}{|B(z,\sqrt{s})|} dzd\tau. \end{aligned}$$

By Lemma 4.1 on p1003 of [Z], there exists a constant c_5, depending only on the doubling constant ν such that

$$\int_0^t \int_{\mathbf{M}} \frac{e^{-c_1 d(x,z)^2/(t-\tau)}}{|B(x,\sqrt{t-\tau})|} V(z) \frac{e^{-c_1 d(z,y)^2/(2\tau)}}{|B(z,\sqrt{\tau})|} dzd\tau \le c_5 M(V) \frac{1}{|B(x,\sqrt{t})|} e^{-c_1 d(x,y)^2/(2t)}, \tag{2.20}$$

where

$$M(V) \equiv \sup_{x\in\mathbf{M}} \int_0^\infty \int_{\mathbf{M}} \frac{e^{-c_1 d(x,z)^2/(2t)}}{|B(x,\sqrt{t})|} V(z) dzdt.$$

Let us remark here that in the lemma quoted above, the constant in the exponential term of $M(V)$ was not given explicitly. However, by tracking the proof, one immediately concludes that the coefficient $-c_1/2$ above works. We caution that it is not clear that one can choose the original constant $-c_1$, except in the Euclidean case. We mention that (2.20) is the parabolic counter part of the basic inequality, for $n \ge 3$,

$$\int_{\mathbf{R}^n} \frac{1}{|x-z|^{n-2}} |V(z)| \frac{1}{|z-y|^{n-2}} dz \le C \sup_w \int_{\mathbf{R}^n} \frac{|V(y)|}{|y-w|^{n-2}} dy \ \frac{1}{|x-y|^{n-2}},$$

which can be found in many places, including [10] e.g.

Scaling the time variable suitably and use the volume doubling property, we see that

$$M(V) \le cN(V) = c \sup_{x \in \mathbf{M}} \int_0^\infty \int_{\mathbf{M}} \frac{e^{-d(x,z)^2/t}}{|B(x, \sqrt{t})|} V(z) dz dt.$$

Hence

$$G_2(x, t; y, 0) \le \frac{C_1 e^{-c_1 d(x,y)^2/t}}{|B(x, \sqrt{t})|} + C_1 c_5 m_t N(V) \frac{1}{|B(x, \sqrt{t})|} e^{-c_1 d(x,y)^2/(2t)}.$$

Now it follows that

$$G_2(x, t; y, 0) \le (C_1 + C_1 c_5 m_t N(V)) \frac{1}{|B(x, \sqrt{t})|} e^{-c_1 d(x,y)^2/(2t)}.$$

By the definition of m_t we have

$$m_t \le C_1 + C_1 c_5 m_t N(V).$$

Hence, if $N(V) < 1/(C_1 c_5)$, then

$$m_t \le C_1/[1 - C_1 c_5 N(V)],$$

for all $t > 0$. Therefore

$$G_2(x, t; y, 0) \le \frac{C_1}{1 - C_1 c_5 N(V)} \frac{1}{|B(z, \sqrt{\tau})|} e^{-c_1 d(x,y)^2/(2t)}$$

This proves the global upper bound for $G_2(x, t; y, 0)$.

Step 5. L^2 estimate of $F = |\nabla \log u|^2$.

Let $\phi = \phi(x)$ be a smooth cut-off function defined in $B(x, 2r)$ such that $0 \le \phi \le 1$, $\phi(y) = 1$ in $B(x, r)$ and $\phi(y) = 0$ in $B(x, 2r)^c$. Then

$$\begin{aligned}
\int_{B(x,2r)} F \phi^2 dy &= \int_{B(x,2r)} \frac{\nabla u \nabla u}{u^2} \phi^2 dy \\
&= - \int_{B(x,2r)} u \ div \big(\frac{\nabla u}{u^2} \phi^2 \big) dy \\
&= - \int_{B(x,2r)} u \frac{\Delta u}{u^2} \phi^2 dy - \int_{B(x,2r)} u \nabla u \nabla \big(\frac{\phi^2}{u^2} \big) dy \\
&= 2 \int_{B(x,2r)} \frac{u \nabla u \nabla u}{u^3} \phi^2 dy - 2 \int_{B(x,2r)} u \nabla u \frac{\phi \nabla \phi}{u^2} dy \\
&= 2 \int_{B(x,2r)} F \phi^2 dy - 2 \int_{B(x,2r)} u \nabla u \frac{\phi \nabla \phi}{u^2} dy.
\end{aligned}$$

Therefore

$$\int_{B(x,2r)} \frac{|\nabla u|^2}{u^2} \phi^2 dy \le 2 \int_{B(x,2r)} \frac{|\nabla u|}{u} \phi |\nabla \phi| dy.$$

Hence

$$\int_{B(x,2r)} \frac{|\nabla u|^2}{u^2} \phi^2 dy \le 4 \int_{B(x,2r)} |\nabla \phi|^2 dy.$$

This implies that

$$\int_{B(x,r)} F(y)dy = \int_{B(x,r)} \frac{|\nabla u|^2}{u^2} dy \le 4\frac{|B(x,2r)|}{r^2}. \tag{2.21}$$

Step 6. Mean value inequality for solutions of $L_2 w - w_t \ge 0$ (see (2.5) for definition of L_2).

Let ψ be the smooth cut-off function defined right below (2.6') with $\sigma = 2$ and take r there to be $r/2$. Since ψ is supported in $Q_r(x,t)$, we know that

$$\begin{aligned} &\Delta(w\psi) + 2\nabla f \nabla(w\psi) - 10F(w\psi) + 10nV(w\psi) - (w\psi)_t \\ &\ge 2(\nabla f \nabla \psi)w + (\Delta\psi)w - w\psi_t + 2\nabla\psi\nabla w. \end{aligned} \tag{2.22}$$

Since G_2 is the fundamental solution of the left hand side of the (2.22), we have

$$\begin{aligned} w(x,t) \le &-2\int_{Q_r(x,t)} G_2(x,t;y,s)(\nabla f \nabla\psi) w dyds \\ &- \int_{Q_r(x,t)} G_2(x,t;y,s)[(\Delta\psi)w - w\psi_t]dyds \\ &- \int_{Q_r(x,t)} G_2(x,t;y,s) 2\nabla\psi\nabla w dyds. \end{aligned}$$

After integration by parts, this becomes,

$$\begin{aligned} w(x,t) \le &-2\int_{Q_r(x,t)} G_2(x,t;y,s)(\nabla f \nabla\psi) w dyds \\ &+ \int_{Q_r(x,t)} G_2(x,t;y,s) w\psi_t dyds \\ &+ \int_{Q_r(x,t)} \nabla_y G_2(x,t;y,s)\nabla\psi w dyds \\ &- \int_{Q_r(x,t)} G_2(x,t;y,s)\nabla\psi\nabla w dyds \\ \equiv\ & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{2.23}$$

Let us estimate I_j terms separately.

Since (x,t) is bounded away from the support of $\nabla\psi$, and ψ_t by a parabolic distance of r, we have

$$G_2(x,t;y,s) \le \frac{C}{|B(x,r)|} \tag{2.24}$$

in all the integrals on the righthand side of (2.23). Hence (2.23) implies

$$\begin{aligned} |I_1 + I_2| \le &\frac{C}{|B(x,r)|}\Big(\int_{Q_r(x,t)} |\nabla f|^2 dyds\Big)^{1/2}\Big(\int_{Q_r(x,t)} w^2 dyds\Big)^{1/2} \\ &+ \frac{C}{r^2|B(x,r)|}\int_{Q_r(x,t)} w dyds. \end{aligned}$$

Using $|\nabla f|^2 = F = |\nabla u|^2/u^2$, by (2.21), we deduce

$$w(x,t) \le \Big(\frac{C}{r^2|B(x,r)|}\int_{Q_r(x,t)} w^2 dyds\Big)^{1/2} + \frac{C}{r^2|B(x,r)|}\int_{Q_r(x,t)} w dyds + |I_3| + |I_4|.$$

This shows, by Hölder's inequality,

$$w(x,t) \le \Big(\frac{C}{r^2|B(x,r)|}\int_{Q_r(x,t)} w^2 dyds\Big)^{1/2} + |I_3| + |I_4|. \tag{2.25}$$

In the next two steps we will find a bound for the last integrals in (2.25).

step 7. controlling the term

$$I_3 = \int_{Q_r(x,t)} \nabla_y G_2(x,t;y,s)\nabla\psi w dyds. \tag{2.26}$$

Using (2.24) and Hölder's inequality, we reduce (2.26) to

$$|I_3| \le \frac{C}{r}\Big(\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla_y G_2(x,t;y,s)|^2 dyds\Big)^{1/2}\Big(\int_{Q_r(x,t)} w^2 dyds\Big)^{1/2}. \tag{2.27}$$

What is remaining is to estimate the first term on the right hand side of (2.27).

Write

$$v = v(y,s) = G_2(x,t;y,s).$$

Since G_2 is the heat kernel of L_2, or, in another word the fundamental solution of the operator

$$\Delta + 2\nabla f\nabla - 10F + 10nV - \partial_s,$$

we know that v is a solution of the conjugate of $L_2 - \partial_s$, except at (x,t). i.e.

$$\Delta v - 2\nabla f\nabla v - 2\Delta f v - 10Fv + 10nVv + v_s = 0.$$

Since $\Delta f = -F$, the above becomes

$$\Delta v - 2\nabla f\nabla v - 8Fv + 10nVv + v_s = 0. \tag{2.28}$$

i.e.

$$L_3 v + v_s + 10nVv = 0.$$

Take a suitable cut-off function ψ_1 and use $\psi_1^2 v$ as a test function on (2.28) and $h = 10nVv$ in (2.10'). We can follow the argument between (2.6) and (2.10') verbatim to obtain

$$\begin{aligned}&\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla_y v|^2 dyds\\ &\le \frac{C}{r^2}\int_{Q_{2r}(x,t)-Q_{r/4}(x,t)} v^2 dyds + 10n\int_{Q_{4r}(x,t)-Q_{r/4}(x,t)} Vv^2 dyds.\end{aligned}$$

i.e.

$$\begin{aligned}&\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla_y G_2(x,t;y,s)|^2 dyds\\ &\le \frac{C}{r^2}\int_{Q_{2r}(x,t)-Q_{r/4}(x,t)} G_2(x,t;y,s)^2 dyds + 10n\int_{Q_{4r}(x,t)-Q_{r/4}(x,t)} VG_2(x,t;y,s)^2 dyds.\end{aligned}$$

It follows that

$$\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla_y G_2(x,t;y,s)|^2 dyds$$
$$\le \frac{C}{r^2} \int_{Q_{2r}(x,t)-Q_{r/4}(x,t)} G_2(x,t;y,s)^2 dyds$$
$$+ 10n \sup_{(y,s)\in Q_{4r}(x,t)-Q_{r/4}(x,t)} G_2(x,t;y,s) \int_{Q_{4r}(x,t)-Q_{r/4}(x,t)} VG_2(x,t;y,s)dyds.$$

Using the Gaussian bound on G_2 and the assumption on V there holds

$$\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla_y G_2(x,t;y,s)|^2 dyds \le C \frac{1+N(V)}{|B(x,r)|}.$$

Here we just used the inequality

$$\int_{Q_{4r}(x,t)-Q_{r/4}(x,t)} VG_2(x,t;y,s)dyds \le N(V),$$

which comes from the Gaussian upper bound of G_2 and rescaling in time (see (1.4)). Inserting the L^2 estimate on the gradient of G_2 to (2.27) we obtain

$$|I_3| \le \Big(\frac{C}{r^2|B(x,r)|} \int_{Q_{4r}(x,t)} w^2 dyds\Big)^{1/2}. \tag{2.29}$$

step 8. controlling the term

$$|I_4| \equiv |\int_{Q_r(x,t)} G_2(x,t;y,s)\nabla\psi\nabla w dyds|.$$

Using (2.24) and Hölder's inequality we reach

$$\begin{aligned} |I_4| &\le \frac{C}{r}\Big(\int_{Q_r(x,t)-Q_{r/2}(x,t)} G_2(x,t;y,s)^2 dyds\Big)^{1/2}\Big(\int_{Q_r(x,t)} |\nabla w|^2 dyds\Big)^{1/2} \\ &\le C\Big(\frac{1}{|B(x,r)|}\int_{Q_r(x,t)} |\nabla w|^2 dyds\Big)^{1/2}. \end{aligned} \tag{2.30}$$

Recall that

$$L_2 w - w_s \ge 0,$$

and hence

$$L_3 w - w_s + 10nVw \ge 0.$$

Here L_3 is defined by (2.6).

Take $h = 10nVw$ and $\sigma = 2$ in (2.10'), we obtain

$$\int_{Q_{2r}(x,t)} |\nabla(\psi_2 w)|^2 dyds \le \frac{C}{r^2}\int_{Q_{2r}} w^2 dyds + 10n \int_{Q_{\sigma r}} V(w\psi_2)^2 dyds.$$

Here ψ_2 is the cut-off function in (2.10') with $\sigma = 2$. By Condition (1.4) for V we have, for a constant C',

$$\int_{Q_r(x,t)} |\nabla w|^2 dyds \le \int_{Q_{2r}(x,t)} |\nabla(\psi_2 w)|^2 dyds \le \frac{C'}{r^2}\int_{Q_{2r}} w^2 dyds. \tag{2.31}$$

By (2.31) and (2.30), we have

$$I_4 \le C\big(\frac{1}{r^2|B(x,r)|}\int_{Q_{2r}(x,t)} w^2 dyds\big)^{1/2}. \tag{2.32}$$

Substituting (2.32) and (2.29) to (2.25), we reach

$$w(x,t) \le \big(\frac{C}{r^2|B(x,r)|}\int_{Q_{4r}(x,t)} w^2 dyds\big)^{1/2}. \tag{2.32}$$

step 9. completion of the proof.

Recall that $w = F^{5n}$ and w is independent of time. Hence (2.29) becomes

$$F(x) \le \big(\frac{C}{|B(x,r)|}\int_{B(x,2r)} F^{10n} dy\big)^{1/(10n)} \tag{2.33}$$

By the well known trick of Li-Schoen [7], inequality (2.33) implies

$$F(x) \le \frac{C}{|B(x,r)|}\int_{B(x,2r)} F dy \tag{2.34}$$

Let us mention that in the paper [7], it was shown that a L^2 mean value inequality implies a L^1 mean value inequality. However, applying the same method, one can also deduces (2.34) from (2.33) without any difficulty.

Combining (2.34) with (2.21), we have

$$\frac{|\nabla u|^2}{u^2} = F(x) \le \frac{C}{|B(x,r)|}\int_{B(x,2r)} F dy \le \frac{C}{r^2}.$$

This finishes the proof of the global gradient bound. □

References

[1] Carron, G. *Inègalitś de Hardy sur les variètès riemanniennes non-compactes.* (French) [Hardy inequalities on noncompact Riemannian manifolds] J. Math. Pures Appl. (9) 76 (1997), no. 10, 883–891.

[2] Coulhon, Thierry; Saloff-Coste, Laurent, *Vari et es riemanniennes isom etriques à l'infini. (French) [Isometric Riemannian manifolds at infinity]* Rev. Mat. Iberoamericana 11 (1995), no. 3, 687–726.

[3] Cheng, S. Y.; Yau, S. T. *Differential equations on Riemannian manifolds and their geometric applications.* Comm. Pure Appl. Math. 28 (1975), no. 3, 333–354.

[4] Davies, E. B., *Heat kernels and spectral theory.* Cambridge Tracts in Mathematics, 92. Cambridge University Press, Cambridge, 1989.

[5] Grigoryan, A., The heat equation on noncompact Riemannian manifolds. (Russian) *Mat. Sb.* 182 (1991), no. 1, 55–87; (English translation) *Math. USSR Sb.* 72 (1992), 47–77.

[6] Grigor yan, Alexander, *Estimates of heat kernels on Riemannian manifolds.* Spectral theory and geometry (Edinburgh, 1998), 140–225, London Math. Soc. Lecture Note Ser., 273, Cambridge Univ. Press, Cambridge, 1999.

[7] Li, Peter; Schoen, Richard, *L^p and mean value properties of subharmonic functions on Riemannian manifolds.* Acta Math. 153 (1984), no. 3-4, 279–301.

[8] Li, Peter; Wang, Jiaping, *Mean value inequalities.* Indiana Univ. Math. J. 48 (1999), no. 4, 1257–1283.

[9] Li, P.; Yau, S.T., On the parabolic kernel of the Schrödinger operator, *Acta Math.*, 156 (1986) 153-201.

[10] Simon, B., Schrödinger semigroups, *Bull. AMS*, **7** (1982), 447-526.

[11] Saloff-Coste, L. *A note on Poincaré, Sobolev, and Harnack inequalities.* Internat. Math. Res. Notices 1992, no. 2, 27–38.

[12] Saloff-Coste, Laurent, *Aspects of Sobolev-type inequalities.* London Mathematical Society Lecture Note Series, 289. Cambridge University Press, Cambridge, 2002.

[13] Yau, Shing Tung, *Harmonic functions on complete Riemannian manifolds.* Comm. Pure Appl. Math. 28 (1975), 201–228.

[14] Zhang, Qi S. *On a parabolic equation with a singular lower order term. II. The Gaussian bounds.* Indiana Univ. Math. J. 46 (1997), no. 3, 989–1020.

e-mail: qizhang@math.ucr.edu

Department of Mathematics, University of California, Riverside, CA 92521, USA