THE MINIMAL GENUS PROBLEM

TERRY LAWSON

Abstract. This paper gives a survey of recent work on the problem of finding the minimal genus of an embedded surface which represents a two-dimensional homology class in a closed oriented smooth 4-manifold. During the last 5 years there have been major breakthoughs on this question, completely solving it in some cases. The most significant results have come as an application of gauge theory and the Seiberg-Witten invariants. We present the background from Seiberg-Witten theory and show how it has been applied it to this problem.

Contents

1. Introduction

In an earlier paper [L], the author gave a survey of results on the problem of representing a 2-dimensional homology class in an oriented 4-manifold by an embedded sphere. This

Date: January 29, 1997.
Research supported in part by NSF grant 9403533.

problem has a long history which involved the development of many of the important techniques used in 4-dimensional topology. It is part of a bigger question of characterizing the minimal genus of an embedded surface which represents a given homology class. Although some of the results presented in [L] pertained to this larger question, most did not.

In the last few years there have been some major advances in studying the minimal genus problem, and we want to discuss them here. We will report here only on work in the smooth category. The reader interested in results for topological locally flat embeddings should consult the series of papers by Lee and Wilczynski [LW1], [LW2], [LW3], [LW4], where they have essentially given a complete solution of the problem. The first major breakthroughs involved work of Kronheimer and Mrowka in a series of papers culminating in [KM2],[KM3]. They used a theory of connections with controlled singularities along the embedded surfaces and gauge theoretic techniques which they developed in this context. We will report on their major results here, but will approach them from a different viewpoint. After their initial work, a new technique of using the Seiberg-Witten equations instead of the anti-self-duality equations was introduced by Seiberg and Witten, and utilized by many mathematicians to reprove and extend results proved earlier by gauge theory. Since this is the technique of choice today, we will present most of the new results from the viewpoint of the Seiberg-Witten equations, even though Kronheimer and Mrowka and others had originally proven some of them by gauge theoretic techniques based on the anti-self-duality equations and Donaldson series.

An outline of the paper follows. In the next section we introduce the main features of the Seiberg-Witten equations, following a general outline of Morgan [M]. In the following section we then explain the main ways these equations have been used to study the minimal genus problem, concentrating first on the case when $b_2^+ > 1$. In the next section we will look at the case when $b_2^+ = 1$ and present the Kronheimer-Mrowka proof of the Thom conjecture. The following section describes further results when $b_2^+ = 1$. We first discuss Ruberman's extension of this to embedded surfaces in rational complex surfaces. In this section we also discuss parallel results which were obtained by Li and Li, as well as special aspects of the case when $b_2^+ = 1$ such as the form of the adjunction inequality and different varieties of the Seiberg-Witten invariant. All of the above applications relate to classes with nonnegative square. We then look at Fintushel and Stern's results for classes of possibly negative square and restrictions on the positive double points. The next section surveys the work of Morgan, Szabo, and Taubes which provides an alternate viewpoint on and extensions of these earlier results. We then discuss applications of Furuta's theorem and its generalizations toward minimal genus estimates. The following section provides many other recent results which were developed using other techniques. We then have a section featuring two illustrative examples coming from applying the earlier techniques. We close with a section of comments and directions for further work.

We are largely interpreting and presenting work of other researchers, with the goal of giving the reader some appreciation of both the main results and the methods which have been used. We have tried to give an original source where possible, but in fact many of the initial results were discovered about the same time shortly after Seiberg and Witten introduced their new invariants and independent proofs were given by multiple authors. Thus our references may not always cite all of the original independent sources of a result, but we do try to at least cite one source whenever possible.

Finally, we remark on our use of notation. We will usually denote an embedded surface as well as the homology class it represents by the same letter, such as Σ, but may use $[\Sigma]$ for the homology class if we feel it is necessary to avoid confusion. In particular, we denote by S_0 the class in $H_2(\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2})$ represented by the complex line $\mathbb{CP}^1$ with its standard orientation in the first $\mathbb{CP}^2$, and by S_i the class represented by a negatively oriented complex line in the ith copy of $\overline{\mathbb{CP}^2}$. These are sometimes denoted by H and $-E_i$, where E_i is called the ith exceptional class, by other authors. We will also use Poincaré duality to identify $H^2(X)$ and $H_2(X)$. In particular, we will sometimes indicate a line bundle L with first Chern class $c_1(L)$ by the homology class given by the Poincaré dual $PDc_1(L)$. In particular, for a complex surface or symplectic manifold with corresponding almost complex structure, we will identify the canonical line bundle $\Lambda^{2,0}(T^*_{\mathbb{C}}X)$ with $K = PDc_1(\Lambda^{2,0}(T^*_{\mathbb{C}}X)) = -PDc_1(T_{\mathbb{C}}X)$. In doing this the reader should note the relation $\alpha(b) = PD(\alpha) \cdot b$. We will sometimes write $\alpha \cdot b$ and $\alpha \cdot \beta$ as well, where we use $\cdot$ for evaluation and cup product as well as for the usual intersection product. All of these forms are related to intersection product using Poincaré duality to identify cohomology and homology classes. Finally, we will sometimes denote a *spin*^{c} structure on a 4-manifold with its determinant line bundle.

The author thanks Ron Fintushel for valuable discussions as well as his hospitality while visiting Michigan State University when a substantial portion of this work was done.

2. The Seiberg-Witten Equations

In this section we give an outline of the basic theory behind the Seiberg-Witten equations. We refer the reader to the book by Morgan [M] for more details – we will use most of the same notation that is used there and follow closely the main ideas of his exposition. The reader can also find a very nice, shorter treatment of Seiberg-Witten theory in the article by Akbulut [A] and a more advanced exposition with connections to earlier applications of gauge theory in the article by Donaldson [D2]. The book by Lawson and Michelson [LM] is an excellent source for background information on *spin*^{c} structures and the Dirac operator. We also recommend the book by Salamon [S] for more advanced topics in Seiberg-Witten theory.

Let X denote a smooth, connected, closed oriented 4-manifold. Any such manifold possesses a *spin*^{c} structure. The group *spin*$^c(4)$ can be viewed as a double covering of $SO(4) \times U(1)$ or as a S^1 bundle over $SO(4)$ by composing this double cover with the projection. The cover is given by extending the usual double cover $spin(4) \to SO(4)$ to $spin^c(4) = spin(4) \times U(1)/\{\pm 1\}$ by using the double cover $U(1) \to U(1)$ on the second factor. A *spin*^{c} structure on X is a principal $spin^c(4)$ bundle $\widetilde{P}(X)$ which double covers the $SO(4) \times U(1)$ bundle $P(X) \times L$. Here $P(X)$ is the principal frame bundle of X and L is an appropriately chosen line bundle, which is called the determinant line bundle of the *spin*^{c} structure. If $H^1(X, \mathbb{Z}_2) = 0$, then *spin*^{c} structures are classified by their associated determinant line bundles L, and we may identify them. When $spin^c(4)$ is viewed as an S^1 bundle over $SO(4)$, then given one $spin^c$ structure, the others arise by a natural construction of tensoring with any line bundle L', thus expressing the set of all $spin^c$ structures as an affine copy of $H^2(X)$, which classifies line bundles. The operation of tensoring with L' will change the determinant line bundle by tensoring with $(L')^2$, or, additively, by adding two times the classifying cohomology class. Thus those elements which don't change the determinant line bundle correspond to the 2-torsion in $H^2(X)$. The determinant line bundle L is characteristic– its first Chern class reduces mod 2 to the second Stiefel-Whitney class $w_2(X)$.

For any (almost) complex surface, the almost complex structure naturally determines a $spin^c$ structure using the map $U(2) = SU(2)\times U(1)/\{\pm 1\} \to SU(2)\times SU(2)\times U(1)/\{\pm 1\} = spin(4) \times U(1)/\{\pm 1\} = spin^c(4)$. The determinant line bundle of this $spin^c$ structure is just the determinant line bundle of the original almost complex structure. Given a $spin^c$ structure $\widetilde{P}$, there are associated complex 2-plane bundles of spinors $S^{\pm}_{\mathbb{C}}(\widetilde{P})$ – the operation of tensoring a $spin^c$ structure with a line bundle E tensors these spinor bundles with E.

A symplectic manifold has a canonical $spin^c$ structure with determinant line bundle given by the anti-canonical bundle $-K$. Other $spin^c$ structures come from tensoring with a line bundle E so that the resulting determinant line bundle is $-K+2E$ when written additively. In this situation the $spin^c$ structures are usually identified with the line bundles E instead of with the determinant line bundles. Taubes has proved remarkable theorems that connect the Seiberg-Witten invariants for the $spin^c$ determined by E to the Gromov invariants of E, which measure the pseudoholomorphic curves in the homology class of E. He has also shown that for the anticanonical bundle $-K$ itself the Seiberg-Witten invariant is nontrivial ([T1],[T2],[T3]).

Suppose we have a fixed $spin^c$ structure $\widetilde{P}$ with determinant line bundle L. There is an associated spinor bundle $S_{\mathbb{C}}(\widetilde{P})$ with irreducible subbundles $S^{\pm}_{\mathbb{C}}(\widetilde{P})$. A connection A on L together with the Levi-Civita connection on TX induces a spin connection $\tilde{\Delta}_A$, from which we can define the Dirac operator $\mathbb{D}_A : C^\infty(S_{\mathbb{C}}(\widetilde{P})) \to C^\infty(S_{\mathbb{C}}(\widetilde{P}))$ using Clifford multiplication. The Dirac operator is a formally self adjoint elliptic operator with symbol given by Clifford multiplication by i times a cotangent vector ξ. It interchanges the $\pm$ spinor bundles and we look at its restriction to sections of the plus spinor bundle

$$\mathbb{D}_A : C^\infty(S^+_{\mathbb{C}}(\widetilde{P})) \to C^\infty(S^-_{\mathbb{C}}(\widetilde{P})).$$

We consider pairs (A,ψ) where A is a unitary connection on L and $\psi \in C^\infty(S^+_{\mathbb{C}}(\widetilde{P}))$. The Seiberg-Witten equations are

$$F^+_A = q(\psi) = \psi \otimes \psi^* - \frac{|\psi|^2}{2} Id \tag{1}$$

$$\mathbb{D}_A(\psi) = 0 \tag{2}$$

In the first equation, which we call the curvature equation, we are identifying $q(\psi)$ as an element of $\mathrm{End}_{\mathbb{C}}(S^+(\widetilde{P}))$ and the traceless endomorphisms of $S^+(\widetilde{P})$ with sections of $\Lambda^2_+(X)\otimes$

$\mathbb{C}$. To put everything in the appropriate analytic framework we form a configuration space

$$\mathcal{C}(\widetilde{P}) = \mathcal{A}_{L^2_2}(L) \times L^2_2(S^+(\widetilde{P}))$$

where $\mathcal{A}_{L^2_2}(L)$ is the space of unitary L^2_2 connections on L. Although we use this Sobelev norm and stronger norms, the resulting moduli spaces we form will be independent of the norm and consist of C^∞ objects up to gauge equivalence. The Seiberg-Witten function

$$F : \mathcal{C}(\widetilde{P}) \to L^2_1((\Lambda^2_+ T^*X \otimes i\mathbb{R}) \oplus S^-(\widetilde{P}))$$

is given by

$$F(A,\psi) = (F^+_A - q(\psi), \mathbb{D}_A(\psi)).$$

The Seiberg-Witten equations then become

$$F(A,\psi) = 0.$$

The gauge group $\mathcal{G}(\widetilde{P})$ is taken as the L^2_3 automorphisms of $\widetilde{P}$ covering the identity of the frame bundle P, which can be identified with L^2_3 maps of X to the center S^1 of $spin^c(4)$. This is an infinite dimensional Lie group under pointwise multiplication with Lie algebra $L^2_3(X; i\mathbb{R})$. There is a smooth right action

$$\mathcal{C}(\widetilde{P}) \times \mathcal{G}(\widetilde{P}) \to \mathcal{C}(\widetilde{P})$$

given by

$$(A,\psi)\cdot\sigma = ((\det\sigma)^*A, S^+(\sigma^{-1})(\psi))$$

so that $F((A,\psi)\cdot\sigma) = F(A,\psi)\cdot\sigma$. This action has trivial stabilizer except at pairs where $\psi = 0$, in which case the stabilizer is given by constant maps from X to S^1, which we identify with S^1. Pairs (A,ψ) with $\psi \neq 0$ are called irreducible and those with $\psi = 0$ are called reducible. The quotient space

$$\mathcal{B}(\widetilde{P}) = \mathcal{C}(\widetilde{P})/\mathcal{G}(\widetilde{P})$$

is formed and shown to be a Hausdorff space with local slices for the action of $\mathcal{G}(\widetilde{P})$ on $\mathcal{C}(\widetilde{P})$. The complement of the equivalence classes of reducible configurations is denoted $\mathcal{B}^*(\widetilde{P})$. It has the structure of of a Hilbert manifold with tangent space at $[A,\psi]$ identified with $L^2_2((T^*X \otimes i\mathbb{R}) \oplus S^+(\widetilde{P}))/\mathrm{Image}(2d, -(\cdot)\psi)$. A neighborhood of a reducible equivalence class $[A,0]$ is homeomorphic to a quotient of $L^2_2((T^*X \otimes i\mathbb{R}) \oplus S^+(\widetilde{P}))/\mathrm{Image}(2d,0)$ by a linear semi-free action of $S^1 = \mathrm{Stab}(A,0)$. Inside $\mathcal{B}(\widetilde{P})$ is the set of equivalence classes determined by $F^{-1}(0)/\mathcal{G}(\widetilde{P})$, which we call the Seiberg-Witten moduli space and denote by $\mathcal{M}(\widetilde{P})$.

Associated to the Seiberg-Witten equations is an elliptic complex coming from the linearizations of the action of the gauge group and the Seiberg-Witten equations

$$0 \to L^2_3(X; i\mathbb{R}) \to L^2_2((T^*X \otimes \mathbb{R}) \otimes S^+(\widetilde{P})) \to L^2_1((\Lambda^2_1 T^*X \otimes i\mathbb{R}) \otimes S^-(\widetilde{P})) \to 0$$

where the first map is $(2d, -(\cdot)\psi)$ and the second map is given by the matrix

$$\begin{pmatrix} P_+d & -Dq_\psi \\ \frac{1}{2}\psi & \mathbb{D}_A \end{pmatrix}.$$

By homotoping this complex to the direct sum of the deRham complex

$$0 \to L^2_3(X; i\mathbb{R}) \xrightarrow{2d} L^2_2(T^*X \otimes i\mathbb{R}) \xrightarrow{P_+d} L^2_1(\Lambda^2_+ T^*X \otimes i\mathbb{R}) \to 0$$

and the Dirac complex

$$0 \to 0 \to L^2_2(S^+(\widetilde{P})) \xrightarrow{\mathbb{D}_A} L^2_1(S^-(\widetilde{P})) \to 0$$

its Euler characteristic is computed to be

$$-[(1 + b_2^+(X) - b_1) + 2\ \mathrm{index}_{\mathbb{C}}(\mathbb{D}_A)] = -[(1 + b_2^+(X) - b_1) + \frac{c_1(L)^2 - \sigma(X)}{4}] =$$

$$-\frac{1}{4}[c_1(L)^2 - (2\chi(X) + 3\sigma(X))]$$

where $\chi(X)$ is the Euler characteristic and $\sigma(X)$ is the signature.

If $[A, \psi]$ is an irreducible solution, the zeroth cohomology of the complex is trivial and the first cohomology is a finite dimensional linear space, called the Zariski tangent space. The second cohomology is called the obstruction space. An irreducible point $[A, \psi]$ of $\mathcal{M}(\widetilde{P})$ is called a smooth point of the moduli space when the obstruction space is trivial. The implicit function theorem then implies that a neighborhood of a smooth point $[A, \psi]$ is a smooth submanifold of $\mathcal{B}^*(\widetilde{P})$ with tangent space given by the Zariski tangent space, hence is of dimension

$$d = \frac{1}{4}[c_1(L)^2 - (2\chi(X) + 3\sigma(X))].$$

A key fact about the Seiberg-Witten equations is the Weitzenböch formula relating the solutions to the scalar curvature κ.

$$\mathbb{D}_A \circ \mathbb{D}_A(\psi) = \nabla_A^* \nabla_A(\psi) + \frac{\kappa}{4}\psi + \frac{F_A}{2} \cdot \psi \tag{3}$$

Using this one finds bounds on the L^∞ norm of ψ and F_A^+ and the L^2 norms of $\nabla_A(\psi)$, F_A^+, and F_A^- in terms of $\kappa_X^- = \max(-\kappa(x), 0)$. Of particular importance for the genus inequality is the bound

$$|F_A^+| \leq \frac{\kappa_X^-}{2\sqrt{2}} \tag{4}$$

since the genus inequality is derived from it. The bound

$$|\psi|^2 \leq \kappa_X^- \tag{5}$$

is used to show that there can only be reducible solutions to the Seiberg-Witten equations when there is a metric of positive scalar curvature. The other L^2 bounds are used in a bootstrapping argument together with the dependence of F_A^+ and ψ from the curvature equation to show compactness of the moduli space. The bounds on the curvature are used to show that there are only a finite number of *spin*^{c} structures for which the moduli space is nonempty.

In general, the moduli space will have non-smooth points. However, we can perturb the equations to avoid this. The perturbation involves replacing the curvature equation by the new equation

$$F_A^+ = q(\psi) + ih \tag{6}$$

The equivalence classes in the perturbed moduli space form a compact manifold of dimension d given as above by the index formula when $d \geq 0$ (and empty when $d < 0$), except near the reducible points. The moduli space depends on the metric g, and this dependence affects whether there are any reducible solutions. However, transversality arguments are used to show that if $b_2^+ > 0$, there are no reducible solutions for generic metrics, and if $b_2^+ > 1$, then reducible solutions can also be avoided in parametrized moduli spaces coming from connecting two metrics.

This perturbed moduli space is also orientable, with orientation coming from a choice of orientations of $H^1(X,\mathbb{R})$ and $H^2_+(X,\mathbb{R})$. As long as $b_2^+ > 1$, the oriented cobordism class of the perturbed moduli space $\mathcal{M}(g,h)$ generically doesn't depend on g,h. In the case $b_2^+ = 1$, it is still independent as long as there are never reducible solutions (e.g. if it is in dimension 0 and $2\chi + 3\sigma > 0$ since the positive class $c_1(L)$ can't be orthogonal to the positive class $[\omega_g]$ then and this is the required condition for a reducible solution). If $b_2^+ = 1$ and $H^1(X,\mathbb{R}) = 0$, two such moduli spaces $\mathcal{M}(g_0)$ and $\mathcal{M}(g_1)$ can be connected with an oriented bordism with at most one singularity of the form of the cone on $\mathbb{CP}^{d/2}$, where the singularity comes from the unique reducible equivalence in the parametrized moduli space.

We summarize the situation by restating some main theorems from [M].

Theorem 1 ([M, 6.1.1,6.5.1,6.6.4,6.9.4])**.** *Suppose X is a closed, smooth oriented 4-manifold with spin^c structure $\widetilde{P}$ and $b_2^+(X) > 0$.*

- *Fix a metric g. For a generic C^∞ self-dual real two-form h on X, the moduli space $\mathcal{M}(\widetilde{P},h) \subset \mathcal{B}(\widetilde{P})$ of gauge equivalence classes of pairs $[A,\psi]$ which are solutions of the perturbed Seiberg-Witten equations forms a smooth compact submanifold of $\mathcal{B}^*(P)$ of dimension*
$$d = \frac{c_1(L)^2 - (2\chi(X) + 3\sigma(X))}{4}$$
or is empty. In particular, the moduli space is empty if $d < 0$ or if X has a metric of positive scalar curvature.
- *The manifold $\mathcal{M}(\widetilde{P},g,h)$ is orientable, with orientation determined by a choice of orientation of $H^1(X,\mathbb{R})$ and $H^2_+(X,\mathbb{R})$.*
- *Suppose $b_2^+ > 1$, and g_0, g_1 are metrics on X and h_0, h_1 are self-dual two-forms so the above parts hold, and that orientations of H^2_+, H^1 are chosen to orient the moduli spaces. Let $\gamma = \gamma(t)$ be a smooth path of metrics connecting g_0 to g_1 and let $\eta = \eta(t)$*

be a generic path of C^∞-self-dual L^2_3 two-forms connecting h_0 and h_1. Define the parametrized moduli space $\mathcal{M}(\widetilde{P},\gamma,\eta)$ consisting of all

$$([A,\psi],t) \subset \mathcal{B}(\widetilde{P}) \times [0,1]$$

satisfying the equations

$$F_A^{+_t} = q(\psi) + ih(t)$$
$$\mathcal{D}_{A,g_t}(\psi) = 0$$

where $+_t$ means the self-dual projection with respect to g_t and $\mathcal{D}_{A,g_t}$ means the Dirac operator constructed using the Levi-Civita connection associated to g_t and the connection A on L. Then $\mathcal{M}(\widetilde{P},\gamma,\eta)$ consists only of irreducible points and is a smooth, oriented compact submanifold with boundary of $\mathcal{B}^*(\widetilde{P})\times[0,1]$ which forms an oriented cobordism between the oriented manifolds $\mathcal{M}(\widetilde{P},g_0,h_0)$ and $\mathcal{M}(\tilde{P},g_1,h_1)$.

- *Suppose $b_2^+ = 1, H^1(X,\mathbb{R}) = 0, d \geq 0$ is even and γ,η are as above except that γ,h are chosen to be transverse to the wall W_t determined by the condition $(2\pi c_1(L) + [h(t)])\cdot[\omega_{g_t}] = 0$ at a single point where it is hit. Form the parametrized moduli space as before. After possible additional perturbation near the reducible point, its neighborhood has the form of a cone on $\mathbb{CP}^{d/2}$. Removing this neighborhood of the reducible point gives an oriented cobordism with boundary*

$$\mathcal{M}(\widetilde{P},g_1,h_1) - \mathcal{M}(\widetilde{P},g_0,h_0) + \mathbb{CP}^{d/2}.$$

We can now use Theorem 1 to define the Seiberg-Witten invariant when $b_2^+ > 1$ for a given *spin*^{c} structure $\widetilde{P}$. Choose orientations for H^1, H^2_+, and a generic self-dual two-form h so that the perturbed moduli space $\mathcal{M}(\widetilde{P},g,h)$ is an oriented smooth submanifold of $\mathcal{B}^*(\widetilde{P})$. There is a principal S^1-bundle over $\mathcal{B}^*(\widetilde{P})$ with total space $\mathcal{A}(\widetilde{P})/\mathcal{G}^0(\widetilde{P})$, where $\mathcal{G}^0(\widetilde{P})$ is the based gauge group of automorphisms which are the identity at a fixed point x_0. Let μ be the first Chern class of this bundle. When $d \geq 0$ is even, we then define the Seiberg-Witten invariant

$$SW(\widetilde{P}) = \int_{\mathcal{M}(\widetilde{P},g,h)} \mu^{d/2}$$

The cobordism property from the theorem then implies this is well defined. Associated with one *spin*^{c} structure $\widetilde{P}$ with determinant line bundle L there is a conjugate structure $-\widetilde{P}$ whose determinant line bundle is the conjugate bundle $-L$. One has $SW(-\widetilde{P}) = (-1)^{(\chi+\sigma)/4}SW(\widetilde{P})$.

In the case when $b_2^+ = 1$, we can define $SW(\widetilde{P},g)$ by the same formula where the notation indicates that it is dependent on the metric. The allowable pairs (g,h) are divided into two classes by the equations $\pm(2\pi c_1(L) + [h])\cdot[\omega_g] > 0$ and one can define two Seiberg-Witten invariants $SW^\pm(\widetilde{P})$ depending on the value. By varying h, we can always find representatives satisfying each condition. However, applications to the genus inequality will only use small perturbations h. For this reason, we also define $SW^{0,+}, SW^{0,-}$ where we require arbitrarily small perturbations used to achieve smoothness of the moduli space. In this case, the inequality will depend primarily on g and $c_1(L)$. It sometimes happens that with small perturbations, only one form of the inequality can be satisfied. This happens, for example, when $b_2^- \leq 9$ since $2\chi+3\sigma \geq 0$ then and $d \geq 0$ implies $c_1(L)^2 \geq 0$, so it can't be orthogonal to ω_g [Sz]. Note that when there are allowable representatives, one has $SW^{\pm}(\widetilde{P}) = SW^{0,\pm}(\widetilde{P})$. However, the example of X_9 satisfies $SW^-(c_1(TX_9)) = \pm 1$, and $SW^{0,-}(c_1(TX_9)) = 0$ since there are no allowable representatives. Here $SW^{0,+}(c_1(TX_9)) = SW^+(c_1(TX_9)) = 0$.

There are other forms of the Seiberg-Witten invariants which are used when $b_2^+ = 1$ ([MST],[Sz],[FS3],[T3]). Let x be a fixed class with $x \cdot x \geq 0$. Then one may consider $SW^{x,\pm}(\widetilde{P})$ where one uses (g,h) satisfying the condition $\pm(p_+(2\pi c_1(L)) + h) \cdot x > 0$. If $x \cdot \omega_g > 0$, then $SW^{x,\pm}(\widetilde{P}) = SW^{\pm}(\widetilde{P})$, whereas if $x \cdot \omega_g < 0$, then $SW^{x,\pm}(\widetilde{P}) = SW^{\mp}(\widetilde{P})$. In [MST] $SW^{x,-}$ is denoted by SW^x, and in [FS3] $SW^{x,\pm}$ is denoted by $SW^{\pm}$ since they are only interested in a fixed $x = T$. In the context of symplectic manifolds, Taubes [T3] studies a particular $ih = F_{A_0} - ir\omega$, where $\omega = \omega_g$ is the symplectic form and $r >> 0$. The Seiberg-Witten invariant using the symplectic metric g and this h is just SW^- since $r >> 0$ makes the term $-r\omega \cdot \omega < 0$ dominate.

When a path of metrics and perturbations crosses the wall formed from when this product is zero in a transversal manner, there is a wall crossing formula.

Theorem 2 (Wall Crossing Formula, [M],[LLu2])**.** *Suppose $b_2^+(X) = 1$. In the situation of transversally crossing a wall, then*

$$SW^+(\widetilde{P}, g_1) - SW^-(\widetilde{P}, g_0) = w(L).$$

If $b_1(X) = 0$, then $w(L) = -(-1)^{d/2}$ If $b_1(X) \neq 0$, then $w(L)$ is computed using an an index evaluation that depends on the cup product structure of one dimensional classes and the class L. (see [LLu2]*).*

One interesting aspect of manifolds with $b_2^+ = 1$ is that

$$SW^+(-\widetilde{P}) = (-1)^{(\chi+\sigma)/4} SW^-(\widetilde{P}).$$

When this is combined with the wall crossing formula, we get the result (cf. [FS3, Lemma 5.6])

$$|SW^{\pm}(\widetilde{P}) \pm SW^{\pm}(-\widetilde{P})| = 1$$

when $b_1 = 0$. This implies that when one of these is zero the other is nonzero.

This has useful implications about Taubes-Gromov invariants which are identified with solutions of deformed equations using the symplectic form ω by Taubes [T3]. In particular, the wall crossing formula takes the form [MS]

$$|Gr(\beta) \pm Gr(K - \beta)| = \tilde{w}(\beta).$$

In the case when $b_1 = 0$, then $\tilde{w}(\beta) = 1$ and the sign is $+$ so the formula gives $Gr(K) = 0$ from $Gr(0) = 1$.

Most applications rely on some basic results concerning the Seiberg-Witten invariants.

Theorem 3 (Connected sum theorem, [D2],[W], [FS1]). *If $X = X_-\#X_+$ with $b_2^+(X_\pm) > 0$, then all Seiberg-Witten invariants for X vanish.*

This is a special case of a more general theorem concerning splittings along positive scalar curvature 3-manifolds.

Theorem 4 (Positive curvature splitting theorem,[D2],[W], [FS1]). *If $X = X_- \cup_Y X_+$ where Y is a 3-manifold with positive scalar curvature and $b_2^+(X_\pm) > 0$, then all Seiberg-Witten invariants for X vanish.*

In general, positive scalar curvature manifolds play a special role in gluing formulas for Seiberg-Witten invariants.

A key technique in the computations is blowing up by taking connected sum with $\overline{\mathbb{CP}^2}$. The primary result here is that the Seiberg-Witten invariants in the manifold and its blowup are closely related.

Theorem 5 (Blowup Theorem [D2],[FS1, Thm. 1.4]). *Suppose X is a smooth closed oriented 4-manifold with $b_2^+(X) > 1$. If $\dim \mathcal{M}_X(L) - r(r+1) \geq 0$, then*

$$SW_X(L) = SW_{X\#\overline{\mathbb{CP}^2}}(L \pm (2r+1)E).$$

Remark 1. In this theorem line bundles are identified with their first Chern classes and written additively. The exceptional class $E \in H^2(\overline{\mathbb{CP}^2})$ is the Poincaré dual of the exceptional divisor (i.e. standard complex line) in $\overline{\mathbb{CP}^2}$. We will mainly use the above theorem in the simpler case when $r = 0$. It is also known that the only non-zero Seiberg-Witten homology classes on the blowup arise in this fashion.

The simplest applications of the above results occur in the case $b_2^+ > 1$, which we assume for the moment. We say that K is a Seiberg-Witten homology class (or a basic class) if it is the Poincaré dual to the first Chern class of the determinant line bundle L of a *spin*^{c} structure $\widetilde{P}$ and $SW(\widetilde{P}) \neq 0$. Note that if K is a Seiberg-Witten homology class, then $-K$ is also one using natural isomorphisms between L and $-L$ and the corresponding *spin*^{c} bundles. Since there are only a finite number of *spin*^{c} structures with nonzero Seiberg-Witten invariants, there are only a finite number of Seiberg-Witten homology classes.

There is a conjecture which has much computational evidence that the Seiberg-Witten homology classes arising from moduli spaces of dimension 0 are the same as the basic classes which were introduced by Kronheimer and Mrowka [KM2],[KM3] for manifolds of simple type. In particular, Seiberg-Witten invariants have been computed for many algebraic surfaces and this conjecture has been verified. Some important computations are given in the following theorems. K_X denotes the canonical class which is given in terms a Kähler structure or a symplectic structure. We state the form due to Taubes for symplectic structures which generalized the earlier theorem of Witten for Kähler structures.

Theorem 6 ([D2],[W],[T1],[T2], [T3],[MST]). *Let X a closed symplectic 4-manifold with $b_2^+(X) > 1$, with symplectic form ω. Suppose that we have chosen a compatible almost complex* *structure and Hermitian metric, for which the symplectic form is self-dual. Then if K_X is the canonical class, $SW(K_X) = \pm 1$. If $[\omega]$ is the cohomology class of the symplectic form, then $K_X \cdot [\omega] \geq 0$ and any other class κ with non-zero Seiberg-Witten invariant satisfies*

$$|\kappa \cdot [\omega]| \leq K_X \cdot [\omega],$$

with equality iff $\kappa = \pm K_X$. The Seiberg-Witten invariant $SW(\kappa) \neq 0$ only when the formal dimension $d(\kappa) = 0$: i.e. X has Seiberg-Witten simple type. Moreover, there is an identification of the the Seiberg-Witten invariant of $-K + 2E$ with the Taubes-Gromov invariant $Gr(E)$ of E which counts the pseudoholomorphic curves (possibly disconnected) in the homology class of E.

When $b_2^+(X) = 1$, there are similar statements:

$$SW^-(-K_X) = \pm 1, -K_X \cdot [\omega] \leq \kappa \cdot [\omega] \text{ with equality iff } \kappa = -K_X.$$

Also, $SW^-(-K_X + 2E) = Gr(E)$ whenever $(-K + 2E) \cdot S \geq -1$ for all classes S with square -1 which are represented by symplectically embedded spheres.

Theorem 7 ([D2],[FS2],[FM])**.** *For an elliptic fibration X the Seiberg-Witten homology classes classes are rational multiples $rK, |r| \leq 1$, of the canonical class. The canonical divisor is a sum of fibers in this case.*

3. Applications of Seiberg-Witten invariants to the genus inequalities

Having completed the general overview of the Seiberg-Witten moduli spaces and corresponding invariants, we now look at their connection to genus inequalities following the ideas in [KM1, Proposition 8, Lemma 9]. We will look at other connections later. In this section we will concentrate mainly on the case $b_2^+ > 1$. Suppose X splits as $X = X_- \cup_Y X_+$ along an oriented 3-manifold Y, with product metric near Y. Let (X_R, g_R) be formed from (X, g) by cutting open along Y and inserting the cylinder $[R, R] \times Y$. Then Kronheimer and Mrowka show that any solution to the Seiberg-Witten equations will lead to special solutions on this tube about Y.

Theorem 8 ([KM1, Proposition 8])**.** *Suppose the moduli space $\mathcal{M}(\widetilde{P}, g_R)$ is non-empty for all sufficiently large R. Then there exists a solution of the equations on the cylinder $\mathbb{R} \times Y$ which is translation invariant in a temporal gauge.*

We assume that Σ is an embedded oriented surface in X of genus $g \geq 1$ with trivial normal bundle. Then the normal bundle of Σ is $D^2 \times \Sigma = X_-$ and allows us to decompose $X = X_- \cup_Y X_+$ with $Y = S^1 \times \Sigma$. We assume that the metric near Y is a Riemannian product $\mathbb{R} \times S^1 \times \Sigma$, and Σ has constant scalar curvature, with the metric normalized so that Σ is of unit area and the scalar curvature is $-2\pi(4g - 4)$. We assume that Theorem 8 allows us to get a solution of the Seiberg-Witten equations which is translation invariant in a temporal gauge. We start with the bound

$$|F_A^+| \leq \frac{2\pi(2g - 2)}{\sqrt{2}}$$

from (4) on the norm of the self-dual component of the curvature. For this product neighborhood and translation invariant solution, the norms of the self-dual and anti-self-dual components of the curvature are equal, so there is a bound

$$|F_A| \leq 2\pi(2g-2)$$

on the curvature. The inequality follows from the fact that $\dfrac{i}{2\pi}F_A$ represents $c_1(L)$ and integrating:

$$\begin{aligned}
|c_1(L)[\Sigma]| &= \left| \frac{i}{2\pi} \int_\Sigma F_A \right| \\
&\leq \frac{1}{2\pi} \sup |F_A| \ \mathrm{Area}(\Sigma) \\
&\leq 2g-2.
\end{aligned}$$

Lemma 9 ([KM1, Lemma 9])**.** *If $g = g(\Sigma) \geq 1$ and there is a solution to the Seiberg-Witten equations on $\mathbb{R} \times S^1 \times \Sigma$ which is translation invariant in the temporal gauge, then*

$$|c_1(L)(\Sigma)| \leq 2g-2.$$

Putting this together gives the following result.

Theorem 10 ([KM1])**.** *Suppose X is a smooth, oriented closed 4-manifold with $b_2^+ \geq 1$. Suppose $\widetilde{P}$ is a spin^c structure on X with determinant line bundle L, and Σ is an embedded oriented surface of genus $g \geq 1$ with trivial normal bundle. Suppose that when we stretch the tube about $Y = S^1 \times \Sigma$ the Seiberg-Witten invariants $SW(\widetilde{P}, g_R)$ are non-zero for arbitrarily large R. Then*

$$|c_1(L)(\Sigma)| \leq 2g-2.$$

Remark 2. As given, the argument applies to the non-perturbed equations with $h = 0$. However, it is easily modified to handle small perturbations h. When $b_2^+ X > 1$, the invariant $SW(\widetilde{P}, g_R)$ is independent of the metric and so the hypothesis just becomes $SW(\widetilde{P}) \neq 0$. When $b_2^+ = 1$, we need $SW^{0,+}(\widetilde{P}) \neq 0$ if $2\pi c_1(L) \cdot \omega_R > 0$ for large R and $SW^{0,-}(\widetilde{P}) \neq 0$ if $2\pi c_1(L) \cdot \omega_R < 0$ for large R.

The following theorem is the Seiberg-Witten analog of the adjunction inequality from [KM2],[KM3].

Theorem 11 (Generalized Adjunction Inequality [KM1],[MST],[FS1],[K])**.** *Let X be a smooth oriented closed 4-manifold with $b_2^+ > 1$. Suppose Σ is an embedded surface of genus g representing a homology class (still denoted Σ) with $\Sigma \cdot \Sigma \geq 0$. Suppose K is a Seiberg-Witten homology class for X. If $[\Sigma]$ is not a torsion class, then $g \geq 1$. If $g \geq 1$, then*

$$2g-2 \geq |K \cdot \Sigma| + \Sigma \cdot \Sigma \tag{7}$$

Sketch of proof ([KM1],[MST],[FS1],[K]. By possibly replacing Σ with $-\Sigma$, we may assume $K \cdot \Sigma = |K \cdot \Sigma|$. The first part of the conclusion of the theorem is that the existence of a nontrivial Seiberg-Witten invariant implies that Σ is not a sphere, or, otherwise stated, if there were a rationally non-trivial embedded sphere with nonnegative self intersection the Seiberg-Witten invariant must vanish.

First consider the case when $g \geq 1$. Let $\Sigma \cdot \Sigma = n \geq 0$. The key fact needed is Theorem 5 on what happens to the Seiberg-Witten invariant when we blow up X to form $X \# n\overline{\mathbb{CP}^2}$. Denote by S_i the homology class of the negatively oriented complex line in the i^{th} copy of $\overline{\mathbb{CP}^2}, i = 1, \cdots, n$. If we replace K by $\widetilde{K} = K - (S_1 + \cdots + S_n)$ (and L by $\widetilde{L} = PD\widetilde{K}$), then the Seiberg-Witten invariant for $\widetilde{K}$ is the same as for K and so is non-zero. Moreover, if $\widetilde{\Sigma} = \Sigma + S_1 + \cdots + S_n$, then the genus $g(\widetilde{\Sigma}) = g(\Sigma) = g$ and

$$\widetilde{K} \cdot \widetilde{\Sigma} = K \cdot \Sigma + \Sigma \cdot \Sigma. \tag{8}$$

Using $c_1(\tilde{L})(\widetilde{\Sigma}) = \widetilde{K} \cdot \widetilde{\Sigma}$ and Theorem 10 gives the result when $g \geq 1$.

Now consider the case when $g = 0$. The simplest case is when $\Sigma \cdot \Sigma > 0$. Then a tubular neighborhood of Σ has normal bundle N with boundary a lens space. Since ∂N has a metric with positive scalar curvature, and N is positive definite (since $\Sigma \cdot \Sigma > 0$), Theorem 4 gives that the Seiberg-Witten invariants must all vanish. One could also stabilize by blowing up enough times to bring the self intersection to 1 and then use the Connected Sum Theorem.

The case when $\Sigma \cdot \Sigma = 0$ is more difficult and uses a contradiction to the finiteness of the number of Seiberg-Witten homology classes. We give an argument from [FS1, Theorem 4]. There is a parallel argument given in [K]. Let K be the homology class Poincaré dual to the determinant line bundle with non-zero Seiberg-Witten invariant. Stabilize by taking connected sum with $\overline{\mathbb{CP}^2}$ and add on the exceptional divisor E to K, which will still have nonzero Seiberg-Witten invariant by Theorem 5. Add copies of Σ to E to form $E_n = E + n\Sigma$. The class E_n is represented by a sphere with square -1, allowing us to rewrite $\overline{X} = X \# \overline{\mathbb{CP}^2} = Y_n \# \overline{\mathbb{CP}^2}$, where the exceptional fiber in the right decomposition comes from E_n. The Blowup theorem shows $SW_{\overline{X}}(K + E) \neq 0$. Rewriting $K + E = K_n + E_n$, another application of the Blowup theorem implies $SW_{Y_n}(K_n) \neq 0$. Applying the Blowup theorem again gives $SW_{\overline{X}}(K_n - E_n) \neq 0$. But $K_n - E_n = K + E - 2E_n = K - E - 2n\Sigma$, and the assumption that Σ is rationally nontrivial means that these are all distinct homology classes. This contradicts the finiteness of the number of Seiberg-Witten homology classes. □

Remark 3. This formula was first proved for embedded surfaces in the $K3$ surface and generalized to a wide variety of 4-manifolds to give bounds on the genus in terms of basic classes related to non-vanishing Donaldson polynomials [KM2],[KM3], but the proof given then did not apply to $\mathbb{CP}^2$. It is related to the adjunction formula for an algebraic curve C (or more generally, pseudo-holomorphic curve in a symplectic manifold), which is

$$2g - 2 = K \cdot C + C \cdot C \tag{9}$$

where K is the canonical class. Taubes [T1] has shown that $\pm K$ are Seiberg-Witten classes for any symplectic manifold with $b_2^+ > 1$. The content of the theorem is then that the algebraic (or pseudo-holomorphic) curves minimize the genus of an embedded surface representing a given homology class in a Kähler surface or symplectic manifold. This last statement is known as the generalized Thom conjecture. (7) is called the generalized adjunction inequality. Mikhalkin [M2] has shown that this need not be true if we just assume that X possesses an almost complex structure. He gives the example of $3\mathbb{CP}^2$. It has an almost complex structure with canonical class $K = -(3, 3, 1)$. The class $(4, 0, 0)$ has a pseudo-holomorphic representative with genus 3 given by the adjunction formula, but it can be represented by a smooth torus.

There are some special situations of the adjunction inequality of note. We have been thinking of the Seiberg-Witten invariants as giving a means to provide restrictions through the adjunction inequality on the minimal genus of an embedded surface within a homology class. Knowledge of this minimal genus for certain surfaces in conjunction with the generalized adjunction inequality can also be used to find out information about Seiberg-Witten invariants. The simplest situation is when there is a surface of square too high be be allowed by the adjunction inequality. For example, an embedded torus of square 1 would violate the adjunction inequality, and so there can be no basic classes.

A good example where the adjunction inequality determines all of the basic classes is the $K3$ surface [KM3]. This is an elliptic fibration over S^2 which has a section of square -2 which intersects the fiber in one positive intersection point. By taking parallel copies of the fiber and tubing them together with the section, we can find surfaces Σ of genus g with self intersection $\Sigma \cdot \Sigma = 2g - 2$ for each $g \geq 0$. For these surfaces, the adjunction inequality will imply that $K \cdot \Sigma = 0$ for any basic class K. But the diffeomorphism group of the $K3$ surface acts transitively on ordinary (i.e. non-characteristic) homology classes of a given square, and so the same argument says that one has $K \cdot \Sigma = 0$ for all ordinary homology classes of square ≥ -2. Since any characteristic class in $K3$ is an even multiple of an ordinary class, this implies $K = 0$ is the only possible basic class. That $K = 0$ is a basic class follows easily. Note also that the transitivity of elements of a given square under the diffeomorphism group together with the examples above says that for any homology class ξ of square greater or equal to -2, the minimal genus of an embedded surface representing it is given by the formula $2g - 2 = \Sigma \cdot \Sigma$.

Another important class of examples comes from tori of square 0. When there is a torus Σ of square 0, then the adjunction inequality implies that $K \cdot \Sigma = 0$ for all basic classes. There are parallel statements for other surfaces of positive genus. For example, if there is a surface Σ of genus g and square $2g - 2$, then the adjunction inequality implies that $K \cdot \Sigma = 0$ for all basic classes. We will give a version of this for spheres of square -2 in a later section. In particular, if we can find a large collection of embedded surfaces Σ of genus g with $\Sigma \cdot \Sigma = 2g - 2$, any basic class must be orthogonal to all of them and this can be used to restrict what the basic classes must be. In general, one expects that the adjunction inequality should determine the basic classes if one knew the minimal genus of a embedded surface representing each homology class of nonnegative square.

Another situation arising in applications is a double torus D of square 1. The adjunction inequality says $2 \geq |K \cdot D| + 1$, which implies that $|K \cdot D| = 1$ since K is characteristic. Now suppose we have two tori T_1, T_2 of squares $-1, 0$ which intersect in a single point in a positive intersection. Then we can tube them together at this point to form a double torus D of square 1. Then $K \cdot D = \pm 1$ and $K \cdot T_2 = 0$ implies that $K \cdot T_1 = \pm 1$ for all basic classes.

4. Kronheimer-Mrowka proof of the Thom conjecture

Kronheimer and Mrowka [KM1] use Theorem 10 to prove the Thom conjecture, which says that the genus g of an embedded surface Σ representing a homology class dS in the complex projective plane $\mathbb{CP}^2$ is greater than or equal as that of an algebraic curve:

Theorem 12 (Thom Conjecture [KM1])**.** *If $g(\Sigma) = g$ is the genus of a smooth surface Σ representing the homology class dS in $\mathbb{CP}^2$, where S is the class represented by the complex line $\mathbb{CP}^1$ and $d > 0$, then*

$$g \geq \frac{(d-1)(d-2)}{2} \tag{10}$$

This formula may also be expressed in terms of the canonical class $K = -3S$ of $\mathbb{CP}^2$

$$2g - 2 \geq K \cdot \Sigma + \Sigma \cdot \Sigma = -3d + d^2 \tag{11}$$

Note that this is of the same form as in Theorem 11. We are abusing notation in these formulas and identifying the canonical class with its Poincaré dual.

In order to prove this theorem, Kronheimer and Mrowka use a mod 2 version of the Seiberg-Witten invariant. They look at the blowup $X = \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ with $n = d^2$. This is chosen so that the connected sum of the original surface with negatively oriented complex lines S_i in each of the added complex projective planes will now have self intersection 0. That is, Σ is replaced by $\widetilde{\Sigma} = \Sigma + S_1 + \cdots + S_n$. The canonical class is replaced by $\widetilde{K} = K - (S_1 + \cdots + S_n)$ and so the right hand side of the adjunction inequality as well as the left hand side given in terms of the genus does not change. The desired inequality becomes

$$2g - 2 \geq \widetilde{K} \cdot \widetilde{\Sigma}.$$

In looking at the Seiberg-Witten equations they use the *spin*^{c} structure coming from the complex structure on X. For this *spin*^{c} structure we will have $PDc_1(L) = -\widetilde{K}$ and our desired inequality takes the form

$$|c_1(L)(\widetilde{\Sigma})| \leq 2g - 2.$$

The self intersection $\widetilde{\Sigma} \cdot \widetilde{\Sigma}$ being 0 now means that the boundary of a neighborhood of the embedded $\widetilde{\Sigma}$ will be $S^1 \times \widetilde{\Sigma} = Y$. A crucial role is played by X possessing a Kähler metric with positive scalar curvature, hence there are no irreducible solutions. A nearby generic metric will still have positive scalar curvature and will have no reducible solutions as well since $b_2^+ = 1$. They then prove a wall-crossing formula which says how the invariant changes when one takes a family of metrics which pass transversally through the wall in the space of harmonic 2-forms given by being orthogonal to $c_1(L)-$ this is the condition for a reducible solution of the equations. Using the Kuranishi model of a neighborhood of the reducible connection, they show that the invariant changes parity when one passes through the wall determined by $c_1(L) \cdot [\omega_g] = 0$. This is just a mod 2 version of the wall crossing formula discussed earlier. Since there is no solution for the Kähler metric with $c_1(L) \cdot [\omega_g] > 0$, this means that for a metric satisfying $c_1(L) \cdot [\omega_g] < 0$ the Seiberg-Witten invariant will be nonzero and so there will be a solution to the Seiberg-Witten equations. Here ω_g is a normalized self dual harmonic form in the same component as the Poincaré dual of S.

The argument then just uses the existence of this solution. They split X open along Y to give $X = X_- \cup_Y X_+$, and then investigate what happens as one stretches the metric in a neighborhood $[-R, R] \times Y$. As R gets large one gets $c_1(L) \cdot [\omega_{g(R)}] < 0$, and thus there is a solution to the Seiberg-Witten equations. This last part of the argument uses the assumption $d > 3$; it is straightforward to check the theorem when $d \leq 3$. Then Theorem 10 gives

$$|c_1(L)(\widetilde{\Sigma})| \leq 2g - 2 \tag{12}$$

5. The minimal genus problem for $b_2^+ = 1$

In independent papers, Ruberman [R] and Li and Li [LL1] each extended the results of Kronheimer and Mrowka to give bounds on the minimal genus for an embedded surface of non-negative square in a rational surface $X_m = \mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$. We will present Ruberman's argument in more detail here. Afterwards, we will comment on the areas where Li and Li get somewhat stronger results as well as on their methods.

Ruberman's argument is a modification of that given above. As a consequence of this computation, Ruberman also gives the minimal genus of embedded surfaces in $S^2 \times S^2$. The main result is

Theorem 13 (Ruberman [R])**.** *Let X_m be a rational surface with canonical class K_X, and let Σ be an embedded surface with $\Sigma \cdot \Sigma \geq 0$. Then the genus $g = g(\Sigma)$ satisfies*

$$2g - 2 \geq K_X \cdot |\Sigma| + \Sigma \cdot \Sigma \tag{13}$$

In this context $\Sigma = a_0 S_0 + \sum_i^n a_i S_i$, with S_i as above and $S_0 = S$. Forming $|\Sigma|$ replaces all a_i by $|a_i|$. Since there are self-diffeomorphisms changing the signs here, we can assume without loss of generality that $\Sigma = |\Sigma|$.

The conclusion of Theorem 13 can be restated as the following inequality:

$$g \geq \binom{|a_0| - 1}{2} - \sum_{i=1}^{n} \binom{|a_i|}{2} \tag{14}$$

In order to prove Theorem 13 Ruberman uses the same general strategy as Kronheimer and Mrowka. Given an embedded surface Σ with nonnegative square, we stabilize by adding copies of negatively oriented complex lines in $\overline{\mathbb{CP}^2}$s to bring the square to 0, and try to find metrics where $c_1(L) \cdot [\omega_g] < 0$. This part of the argument is different, however. First, we can assume as before that all of the $a_i \geq 0$. The argument works by contradiction to the assumption of a counterexample of the original inequality. This uses the following lemma of independent interest:

Lemma 14 (Kronheimer-Mrowka [KM3])**.** *Let Y be an closed, connected, oriented 4-manifold. Let $a(\Sigma) = 2g(\Sigma) - 2 - \Sigma \cdot \Sigma$. If $h \in H_2(Y, \mathbb{Z})$ is a homology class with $h \cdot h \geq 0$ and Σ_h is a surface of genus g representing h and $g \geq 1$ when $\Sigma_h \cdot \Sigma_h = 0$, then for all $r > 0$, the class rh can be represented by an embedded surface Σ_{rh} with*

$$a(\Sigma_{rh}) = ra(\Sigma_h).$$

Proof. We look at a tubular neighborhood of Σ_h, and first assume that $p = \Sigma_h \cdot \Sigma_h > 0$. Choose r sections in general position where each section intersects the other in p positive intersection points, for a total of $p\binom{r}{2}$ positive intersection points. At each self intersection point we will replace a pair of intersecting disks by an annulus connecting their boundary circles to remove the intersection point. Replacing $r-1$ of these intersections by annuli makes a connected immersed surface of genus $rg(\Sigma_h)$ and then replacing the remaining intersection points by annuli increases the genus to $rg(\Sigma_h) + p\binom{r}{2} - (r-1)$. The self intersection number of this new surface representing rh is r^2p. Thus the difference

$$a(\Sigma_{rh}) = 2(rg(\Sigma_h) + p\binom{r}{2} - (r-1)) - 2 - r^2p = r(2g(\Sigma_h) - 2) - p) = ra(\Sigma_h).$$

For the case when $\Sigma_h \cdot \Sigma_h = 0$, the normal bundle is trivial. The assumption that the genus is positive is used to choose a surjective homomorphism $\psi : \pi_1(\Sigma_h) \to \mathbb{Z}_r$. Letting P be the principal $\mathbb{Z}_r$ bundle coming from the cover induced by ψ, note that the associated S^1 bundle is trivial since it is classified by the mod r Bockstein of a class in $H^1(X, \mathbb{Z}_r)$ which comes from an integral class. Hence P may be embedded in $S^1 \times \Sigma_h$ and so inside of N. Its image is a connected surface with self intersection number 0 which represents $r[\Sigma_h]$, giving the formula. □

Remark 4. By the same argument as above we also have $b(\Sigma_{rh}) = rb(\Sigma_h)$ where $b(\Sigma) = a(\Sigma) - K \cdot \Sigma$ since the last term is linear in $[\Sigma]$. This is what we use to say that a counterexample for $[\Sigma]$ with positive genus gives one for $[r\Sigma]$.

Proof of Theorem 13([R]*).* We first prove the inequality (13) holds with the additional assumption that $g(\Sigma) = g \geq 1$. Suppose Σ is a counterexample which has been stabilized to have self intersection 0 in X_n. We stretch the tube as before, denoting the metrics as g_R corresponding to the cylinder $[-R, R] \times S^1 \times \Sigma$. We normalize the corresponding self-dual harmonic form $[\omega_R]$ so that $[\omega_R] \cup PD(S_0) = 1$ – i.e. $PD[\omega_R] = S_0 - \sum_i x_i S_i$. Ruberman extends Lemma 14 to show that if either $\Sigma \cdot \Sigma > 0$ or $\Sigma \cdot \Sigma = 0$ and $K_X \cdot \Sigma > 0$ for a counterexample, then there is a counterexample Σ' in the class $r[\Sigma], (r \geq 1)$ so that in applying the stabilization construction to Σ', then for R sufficiently large we have $c_1(L) \cup [\omega_R] < 0$. This gives a contradiction since it implies Σ' satisfies the inequality for which it is supposed to be a counterexample. The logic of the argument is that an assumed counterexample leads to another counterexample which satisfies conditions leading to a contradiction.

To see this, note that $[\omega_R]$ having positive square implies $\sum_{i=1}^n x_i^2 < 1$. Then

$$PDc_1(L) = \widetilde{\Sigma} + (3 - a_0)S_0 + \sum_{i=1}^n (1 - a_i)S_i$$

implies

$$[\omega_R] \cup c_1(L) = [\omega_R] \cup \widetilde{\Sigma} + (3 - a_0) - \sum_{i=1}^n (a_i - 1)x_i.$$

Lemma 10 of [KM1] shows that the term $[\omega_R] \cdot \widetilde{\Sigma} \to 0$ as $R \to \infty$. In the original argument in [KM1] this was sufficient to show that we got $[\omega_R] \cup c_1(L) < 0$ in the limit since it was assumed that $a_0 > 3$ and $a_i = 1$ by the construction. Here an additional argument is needed.

Unfortunately, it is not always the case that the conditions $\Sigma \cdot \Sigma = a_0^2 - \sum_i^n a_i^2 \geq 0$ and $\sum_i^n x_i^2 < 1$ (from $[\omega_R]^2 > 0$) imply $(3 - a_0) - \sum_i^n x_i(a_i - 1) < 0$. We use the technique of Lagrange multipliers to find the maximum value of $f(x) = (3 - a_0) - \sum_i^n x_i(a_i - 1)$ subject to the constraint $\sum x_i^2 = \epsilon^2 < 1$. The maximum will occur when

$$x = \frac{-\epsilon}{\sqrt{\sum_1^n (a_i - 1)^2}}(a_1 - 1, \cdots, a_n - 1),$$

and has maximal value $(3 - a_0) + \epsilon\sqrt{\sum_{i=1}^n (a_i - 1)^2}$. Since $0 < \epsilon < 1$, the maximum value is always less than $(3 - a_0) + \sqrt{\sum_{i=1}^n (a_i - 1)^2}$. This being nonpositive is equivalent to

$$a_0 - 3 \geq \sqrt{\sum_i^n (a_i - 1)^2}$$

or, after squaring,

$$(a_0^2 - \sum_1^n a_i^2) - 6a_0 + 2\sum_1^n a_i + (9 - n) \geq 0.$$

This last inequality may be expressed in terms of the classes Σ, K_X as

$$\Sigma \cdot \Sigma + 2(K_X \cdot \Sigma) + K_X \cdot K_X = \Sigma \cdot \Sigma + 2(K_X \cdot \Sigma) + (9 - n) \geq 0.$$

As we multiply a by r, the left hand side changes to $r^2(a_0^2 - \sum_1^n a_i^2) + r(-6a_0 + 2\sum_1^n a_i) + (9 - n)$. As r gets larger, the term $r^2(a_0^2 - \sum_1^n a_i^2) = r^2 \Sigma \cdot \Sigma$ dominates the expression, and will get arbitrarily large if $\Sigma \cdot \Sigma > 0$. If it is 0, then the term $r(-6a_0 + 2\sum_1^n a_i) = 2r(K_X \cdot \Sigma)$ will dominate. We are thus led to a contradiction for $g \geq 1$ except when both $\Sigma \cdot \Sigma = 0$ and $K_X \cdot \Sigma = 0$. But the inequality holds trivially in this case.

We now deal with the case when $g = 0$. If there were a counterexample with $\Sigma \cdot \Sigma > 0$, we could first stabilize to get a counterexample with $\Sigma \cdot \Sigma = 0$. This would imply that

$K_X \cdot \Sigma \geq 0$. If $K_X \cdot \Sigma > 0$, we could just add a handle as in the proof of Theorem 11 to increase the genus to 1 and still have a counterexample, contradicting the case we have already proved. Thus we can reduce to the case when $g(\Sigma) = 0$ and $K_X \cdot \Sigma = 0 = \Sigma \cdot \Sigma$. Here we get a contradiction by a different argument. Then the condition $\Sigma \cdot \Sigma = 0$ says that a neighborhood of Σ is $\Sigma \times D^2$. Thus we can do surgery on Σ; let us call the result X'. Doing the surgery will decrease the rank of X' by 2 to $n - 1$, and X' will now be a negative definite manifold, since the effect of the surgery on the real intersection form will be to form the quotient by the subspace spanned by the class represented by Σ and a dual class Σ' with $\Sigma \cdot \Sigma' = 1$. The condition $K_X \cdot \Sigma = 0$ implies that the class K_X will live as a characteristic class in X' with square $9 - n$. This means that the absolute value of the square of this characteristic class is less than the rank of X'. This contradicts Donaldson's theorem that the intersection form of a definite smooth 4-manifold is standard. Note that we don't know that X' is simply connected so we need the stronger version of Donaldson's theorem from [D1, p. 397].

□

One consequence of these results is that we may specify for any homology class in $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ or $S^2 \times S^2$ the minimal genus of an embedded surface which represents the class.

Theorem 15 ([R])**.** *The minimal genus of a surface in $S^2 \times S^2$ representing the class (a, b) with respect to the basis $S^2 \times q, p \times S^2$ with $ab \neq 0$ is $(|a|-1)(|b|-1)$. The classes $(a, 0), (0, b)$ are represented by embedded spheres. The minimal genus of a surface in $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ which represents $a_0 S_0 + a_1 S_1$ is*

$$\binom{|a_0| - 1}{2} - \binom{|a_1|}{2}$$

if $|a_0| > |a_1|$. If $|a_0| < |a_1|$, the roles of a_0 and a_1 are reversed in the formula. If $|a_0| = |a_1|$ then the class is represented by an embedded sphere.

Proof. Since these manifolds possess an orientation reversing diffeomorphism which interchanges the classes of negative square with classes of positive square, we can reduce to the case of nonnegative square. Also, there are diffeomorphisms which change orientation on copies of the S^2-factors in $S^2 \times S^2$ and in $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ so we may assume coefficients of these classes are positive. We express homology classes in terms of their coefficient with respect to standard bases given by the two spheres in $S^2 \times S^2$ and by S_i in $\mathbb{CP}^2$. There is a diffeomorphism from $S^2 \times S^2 \# \overline{\mathbb{CP}^2}$ to $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ which sends the class $(a, b, 0) \in H_2(S^2 \times S^2 \# \overline{\mathbb{CP}^2})$ to the class $(a+b, -a, -b) \in H_2(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2})$. Applying Theorem 13 to the class $(a+b, -a, -b)$ then gives a genus bound for the original class $(a, b) \in H_2(S^2 \times S^2)$. The diffeomorphism used can be constructed using Kirby calculus. Finally, we can construct embeddings which realize the minimal genus as follows. For $S^2 \times S^2$, take $a > 0$ parallel copies of $S^2 \times q_i, i = 1, \ldots, a$, and $b > 0$ parallel copies of $p_j \times S^2, j = 1, \ldots, b$. These intersect in ab points. Forming connected sum along $a+b-1$ of these will give us an immersed sphere. Removing the remaining $ab - a - b + 1 = (a-1)(b-1)$ double points by adding handles gives us an embedded surface of genus $(a-1)(b-1)$ which represents the class (a,b). To represent the classes $(a,0),(0,b)$ by spheres, just take the connected sum of parallel copies of the component spheres.

To realize the class $(a_0,a_1), a_0 > a_1 \geq 0$, in $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ by a surface of the predicted minimal genus we can use the fact that the class (a_1,a_1) is represented by a_1 parallel spheres. This follows from writing $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ as the nontrivial S^2-bundle over S^2, with $(1,1)$ corresponding to the fiber. By tubing these together we could find an embedded sphere which represents the class (a_1,a_1). A section of this bundle (which represents $(1,0)$) will intersect each of these fibers once. Forming the connected sum then gives an embedded sphere S which represents (a_1+1,a_1). Now add in a_0-a_1-1 other sections. For the first one, it will intersect S in a_1+1 positive intersection points. Using one of these to form a connected sum and removing the others by adding handles gives a surface of genus a_1 which represents (a_1+2,a_1). Adding the next section creates a_1+2 intersections, and leads to a connected surface of genus $a_1+(a_1+1)$ representing (a_1+3,a_1). Inductively, we can get a surface of genus $a_1+\cdots+a_0-2=(1+\cdots a_0-2)-(1+\cdots+a_1-1)=\binom{a_0-1}{2}-\binom{a_1}{2}$ which represents (a_0,a_1). □

Li and Li [LL1] give a different approach to finding the minimal genus in rational surfaces, showing that for low m the bound given in Theorem 13 is strict. Their work relies on work of Li and Liu [LLu1] where they proved a version of the generalized adjunction inequality for symplectic manifolds:

Theorem 16 ([LLu1])**.** *Suppose X is a symplectic four-manifold with $b_2^+(X)=1$ and ω a symplectic form. Let C be a smooth, connected, embedded surface with nonnegative self-intersection. If $C\cdot\omega>0$, then $2g(C)-2\geq K_X\cdot C+C\cdot C$. In particular, the adjunction equality for symplectic embedded surfaces says that a symplectic surface always minimizes the genus in its homology class.*

The hypothesis $C\cdot\omega>0$ plays a key role here. First note that it is automatically satisfied when C is a symplectic surface. Note also that in the conclusion we are not showing $2g-2\geq|K_X\cdot C|+C\cdot C$ as one would have when $b_2^+\geq 2$. In fact, that form is not true as can be seen with the example of $\mathbb{CP}^2$ with C representing $\mathbb{CP}^1$.

Sketch of proof. The first step in the proof is to stabilize to reduce to the case $C\cdot C=0$. The proof then breaks into three cases. The first case is when $K_X\cdot C<0$ (this case applies to rational surfaces $X_m, m\leq 9$ unless C is a multiple of K_X and $m=9$). Since K_X is a characteristic class, then means that $K_X\cdot C\leq -2$ and the inequality is trivially satisfied. The next case is when $K_X\cdot C=0$: here it is shown that C can't be represented by a smoothly embedded sphere, so we must have $g\geq 1$ and the inequality is satisfied. The final case is when $K_X\cdot C>0$. Here it is shown that when one stretches the neck on the boundary of a tubular neighborhood of C, the corresponding 2-form ω_R tends to a positive multiple of C. The hypothesis $K_X\cdot C>0$ then implies that $-K_X\cdot\omega_R<0$, so solutions to the Seiberg-Witten equations with small perturbations of g_R metric equations correspond to $SW^{0,-}(-K_X)$. The $b_2^+=1$ version of Theorem 6 gives $SW^-(-K_X)=\pm 1$, so we have $SW^{0,-}(-K_X)=\pm 1$. If C were a sphere, the irreducible solution over the neck would be a contradiction to the existence of a positive scalar curvature metric there. Thus we can assume $g(C) \geq 1$, and apply Theorem 10 to $-K_X$. The result now follows since $K_X \cdot C = |-K_X \cdot C|$. □

Note that in this form of the theorem the only class which enters is the anti-canonical class, unlike what happens in the case $b_2^+ > 1$ where any nonzero Seiberg-Witten class occurs. We now show that for an oriented smooth closed manifold with $b_2^+ = 1$, the arguments of Li and Liu lead to a form

$$2g(C) - 2 \geq -\kappa \cdot C + C \cdot C$$

of the adjunction inequality involving other classes κ than the anti-canonical class. These other inequalities also follow directly from the work of Morgan, Szabo, and Taubes we discuss later. We will assume we have blown up so that $C \cdot C = 0$ and $SW^{C,-}(\kappa) \neq 0$. If $-\kappa \cdot C < 0$, then the inequality holds trivially. If $-\kappa \cdot C > 0$, then $\kappa \cdot C < 0$ and we get a nontrivial solution for (g, h) with $p_+(2\pi\kappa + h) \cdot C < 0$. As above, stretching the neck gives a metric g_R which so that ω_R is close to a positive multiple of C, so we get $p_+(2\pi\kappa + h) \cdot \omega_R = (2\pi\kappa + h) \cdot \omega_R < 0$ for small h (i.e. $SW^{0,-}(\kappa) \neq 0$). This allows us to complete the argument as before in this case. The middle case of the argument is now handled by [MST, Lemma X.1.1], which will require an additional hypothesis in the general (non-symplectic) setting. We sketch their argument below in our proof of Lemma 33. We thus get:

Theorem 17. *Suppose X is a oriented smooth closed four-manifold with $b_2^+ = 1$. Let C be a smooth, connected, embedded surface with $C \cdot C \geq 0, SW^{C,-}(\kappa) \neq 0$. If $g(C) = 0$, assume also that $[C]$ has infinite order. Then*

$$2g(C) - 2 \geq -\kappa \cdot C + C \cdot C.$$

Remark 5. In the symplectic case the assumption $SW^{C,-}(\kappa) \neq 0$ can be restated as $C \cdot \omega > 0, SW^-(\kappa) \neq 0$ since we then would have $SW^-(\kappa) = SW^{\omega,-}(\kappa) = SW^{C,-}(\kappa)$. We don't know of any examples of symplectic 4-manifolds where these other inequalities give any stronger results than the result for the canonical class.

Remark 6. To illustrate some of the subtleties involving the different forms of Seiberg-Witten invariants when $b_2^+ = 1$, we consider a symplectic 4-manifold such as a Dolgachev surface where $b_2^+ = 1, b_1 = 0, K_X \cdot K_X = 0$. Suppose E satisfies $E \cdot E > 0, E \cdot \omega > 0$. Then McDuff and Salamon [MS] show that $SW^-(-K_X + 2nE) = Gr(nE) \neq 0$ for all large n. Now suppose $C \cdot C \geq 0, C \cdot \omega > 0$ as in the hypotheses of Theorem 16. Even though there are an infinite number of n with $SW^-(-K_X + 2nE) \neq 0$, there are still only a finite number of n where the required condition $(-K_X + 2nE) \cdot \omega_R < 0$ is satisfied to get an adjunction inequality for $-K_X + 2nE$ since $\omega_R \sim \lambda C, \lambda > 0$, and $E \cdot C \geq 0$ by the light cone lemma. Moreover, when $(-K_X + 2nE) \cdot \omega_R > 0$, the wall crossing lemma will imply $SW^+(-K_X + 2nE) = 0$ so we can't get an adjunction inequality then.

A more complex example comes in the recent work of Szabo and, independently, Fintushel and Stern to produce examples of irreducible symplectic 4-manifolds with $b_2^+ = 1$.

In [Sz], Szabo constructs a minimal nonsymplectic 4-manifold M which is homeomorphic to the rational surface $E(1) = X_9$. Here the sign of $\kappa \cdot \omega_g$ is the same for each metric, and so there is no wall crossing involving small deformations. He thus will have a Seiberg-Witten invariant SW^0 coming from choosing a small deformation h. In our earlier notation $SW^0(\kappa) = SW^{0,+}(\kappa)$ if $\kappa \cdot \omega_g > 0$ and $SW^0(\kappa) = SW^{0,-}(\kappa)$ if $\kappa \cdot \omega_g < 0$.

This manifold M involves a log transform of order k. It contains a torus T, and Szabo shows that the only classes with nonzero SW^0 are $\pm T$. That M is not symplectic follows from his calculation that $SW^0(T) = -k, SW^0(-T) = k$ together with Taubes theorem that for the anticanonical class $SW(-K) = \pm 1$. This translates in our terminology to $SW^{0,+}(T) = -k, SW^{0,-}(-T) = k$.

In [FS3] Fintushel and Stern construct families of examples that correspond to a construction involving a knot K. Szabo's example corresponds to performing their construction on $E(1)$ using a k-twist knot K_k. When their result is rephrased in our terminology, it says $SW^{T,-}(-T) = k, SW^{T,-}(T) = -k-1, SW^{T,-}(mT) = -1, m \geq 3, m$ odd, and $SW^{T,+}(T) = -k, SW^{-T,+}(-T) = k+1, SW^{T,+}(-mT) = 1, m \geq 3, m$ odd. Note the last three follow from the first three by the relation between $SW^{x,-}(\kappa)$ and $SW^{x,+}(-\kappa)$. Since $T \cdot \omega_g > 0$, we have $SW^{T,-}(-T) = SW^{0,-}(-T) = k$. Similarly, $SW^{T,+}(T) = SW^{0,+}(T) = -k$. However, the calculations $SW^{T,-}(T) = -k-1, SW^{T,-}(mT) = -1, m \geq 3, m$ odd, only imply something about $SW^-(m'T), m \geq 1$ and not about $SW^{0,-}(m'T)$ as the necessary condition $m'T \cdot \omega_g < 0$ is not satisfied. A similar comment applies to the calculations $SW^{-T,+}(-T) = k+1, SW^{T,+}(-mT) = 1, m \geq 3, m$ odd.

Li and Li [LL1] apply Theorem 16 to the rational surface X_m, which they show has a unique symplectic structure for $m \leq 9$ up to diffeomorphism and deformation. A portion of their work is the careful analysis of the J-effective curve cone $NE^J(V)$ for $m \leq 6$ which allows them to find ω with $C \cdot \omega > 0$ as well as find pseudo-holomorphic representatives satisfying the adjunction formula to show that the genus bound is strict. They also describe geometric constructions which will give minimal genus representatives of a homology class in many cases. Once they are able to show the hypotheses apply to use Theorem 16 it is easy to deduce the minimal genus bounds. In particular, this can be done directly for $S^2 \times S^2$ instead of indirectly as Ruberman did. For the canonical class $K = (-2, -2), C = (a, b), a > 0, b > 0$, we get $2g - 2 \geq -2a - 2b + 2ab$, or $g \geq ab - a - b + 1 = (a-1)(b-1)$. The results for the other rational surfaces are similarly deduced from the adjunction inequality. By using symplectic geometry combined with Mori's cone theorem for algebraic varieties and Wall's results on the relation of the automorphism and diffeomorphism groups, they are successful in finding appropriate pseudo-holomorphic representatives of homology classes in some symplectic structure to show that the bound given in Theorem 13 is the minimal one. To properly state their result we have to introduce the notion of a reduced class. The class

$\xi = aS_0 + \sum_{i=1}^m b_i S_i$ is called reduced if $a \geq 0, b_1 \geq b_2 \geq \cdots \geq b_m \geq 0$, and

$$a \geq \begin{cases} b_1, & \text{if } m = 1 \\ b_1 + b_2, & \text{if } m = 2 \\ b_1 + b_2 + b_3, & \text{if } m \geq 3 \end{cases}$$

Using automorphisms generated by reflections through homology classes represented by embedded spheres of square -1 or -2, they show that for $m \leq 9$, any ξ with nonnegative square is sent to a reduced class by a diffeomorphism. Note that classes in the same orbit of the diffeomorphism group will have the same minimal genus, but the bound given by the formula will be different. The reduced class turns out to give the class in an orbit with the best bound, and the strictness of the bound can be shown for reduced classes for $m \leq 6$ by finding holomorphic representatives. They also use the definition of a S-class. A class $\eta \in H^2(X, \mathbb{R})$ is called an S-class if there exists a symplectic form ω such that $\eta = [\omega]$, where $[\omega]$ denotes the cohomology class of ω. Let K_η be the canonical line bundle of ω, which depends up to isomorphism only on $[\omega]$. For $\xi \in H_2(M)$, let

$$S_\xi = \{\eta | \eta \text{ is a S-class and } \eta \cdot \xi > 0\}.$$

Let

$$g_\xi = \max_{\eta \in S_\xi} \{1 + \frac{1}{2}(K_\eta \cdot \xi + \xi \cdot \xi)\}.$$

Theorem 18 ([LL1])**.** *Let $\xi \in H_2(X_m)$ with $\xi \cdot \xi > 0, m \leq 6$. Then g_ξ is the minimal genus for ξ. For a reduced class,*

$$g_\xi = \binom{a-1}{2} - \sum_{i=1}^m \binom{b_i}{2}.$$

Moreover, the minimal genus is realized by a holomorphic curve for some complex structure with the standard orientation.

Remark 7. In the proof they show a reduced class is represented by an algebraic curve which satisfies the adjunction formula by verifying that it satisfies the criteria $\xi \cdot \xi > 0, \xi \cdot L \geq 0$ for all -1 lines L. As an example of the role of reduced classes, note that the genus bound given by the formula for the class $(6, 4, 3)$ is $g \geq 1$, but it is in the orbit of the reduced class $(4, 2, 1)$ with genus bound $g \geq 2$, so its minimal genus is 2.

Although Li and Li rely on algebraic or symplectic geometry to find their minimal genus surfaces for classes of positive square for $m \leq 6$, they do show that this is sometimes not possible for the case of square 0. For example, in X_1 they show that the only non-zero class of square 0 which is represented by a J-holomorphic sphere in some symplectic structure is $\pm S_0 \pm S_1$, whereas any class with $|a| = |b_1|$ is represented by an smoothly embedded sphere. When a and b are positive such a sphere is formed by taking connected sum of a copies of the fiber in the structure as a S^2-bundle over S^2.

This theorem doesn't apply to classes of square 0, but it was already proved by Li[Li] that for $m \leq 8$ classes of square 0 are represented by smoothly embedded spheres. The argument depends on results of Wall which allow us to realize automorphisms of the intersection form by diffeomorphisms.

Li and Li also get some results on the minimal genus for the cases $m = 7, 8$ by using the results for $m \leq 6$ and other geometric constructions. As a simple case of these types of arguments, we now use one of their arguments for realizing certain types of reduced classes.

Theorem 19. *Suppose ξ is a reduced class in $H_2(X_m)$ which also satisfies $a \geq b_1 + \cdots + b_m$. Then there is a smoothly embedded surface $\Sigma \subset X_m$ of genus $g = \binom{a-1}{2} - \sum_{i=1}^m \binom{b_i}{2}$ which represents ξ.*

Proof. Choose m distinct points x_i in $\mathbb{CP}^2$ and corresponding points y_i in $\overline{\mathbb{CP}^2}_i$ where we ultimately will be forming our connected sums. In $\mathbb{CP}^2$ take a configuration of b_i lines which intersect transversally in x_i and the same configuration at y_i in the ith copy of $\overline{\mathbb{CP}^2}$. Add $a - (b_1 + \cdots + b_m)$ lines in $\mathbb{CP}^2$ so that any two intersect each other or a previous line in distinct positive intersection point different from the x_i. When we form the connected sums at x_i, y_i we will remove the intersections at x_i and leave a spheres with $\binom{a}{2} - \sum_{i=1}^m \binom{b_i}{2}$ intersection points. At each of these replace a pair of intersecting disks with an annulus. The first $a - 1$ of these will form an immersed sphere and the remainder will increase the genus to g. □

In related work, Li and Li[LL2] have solved the problem of finding the minimal genus of an embedded surface in an S^2-bundle over an orientable surface. They use the symplectic structure on these 4-manifolds and the generalized adjunction inequality for symplectic manifolds, Theorem 16. Let M_g be a connected closed orientable surface of genus g. For the trivial bundle $S^2 \times M_g$, the second homology is $\mathbb{Z} \oplus \mathbb{Z}$, with generators x_1, x_2 coming from S^2, M_g, respectively. Li and Li show:

Theorem 20 ([LL2]). *Let $\xi = a_1 x_1 + a_2 x_2 \in H_2(S^2 \times M_g), a_1 \geq 0, a_2 \geq 0$, and g_ξ the minimal genus of ξ. Then*

$$g_\xi = \begin{cases} a_2 g + (a_1 - 1)(a_2 - 1), & \textit{if } (a_1 + g)a_2 \neq 0 \\ 0 & \textit{otherwise} \end{cases}$$

The general case follows from the statement given above since there are diffeomorphisms which reverse orientation on each factor. That the minimal genus is greater than or equal to the number given follows easily from the Theorem 16. The standard Kähler structure has canonical class $K = (2g - 2)x_1 - 2x_2$ and an application of the generalized adjunction inequality implies that the minimal genus is greater than or equal to g_ξ as given above. To realize a surface of this genus, we can just connect up a_1 parallel copies of $S^2 \times q_j$ and a_2 copies of $p_i \times M_g$.

Li and Li also give the minimal genus for a nontrivial S^2 bundle N over a surface. Since the case when $g = 0$ is already taken care of, we assume $g > 0$. Then N can be regarded as a Kähler surface which is a geometric ruled surface with a holomorphic section with homology class x with $x^2 = 1$ and holomorphic fiber with homology class y.

Theorem 21 ([LL2]). *For $\xi = ax + by \in H_2(N)$, where N is the nontrivial S^2-bundle over a surface M_g with genus $g > 0$, let g_ξ be the minimal genus of ξ. When $a \neq 0$, let $a' = |a|, b' = \dfrac{a}{|a}b$. Then*

$$g_\xi = \begin{cases} a'g + \frac{1}{2}a'(a'-1) + (a'-1)(b'-1), & \text{if } a \neq 0, a' + 2b' \geq 0 \\ a'g + \frac{1}{2}a'(a'-1) - (a'-1)(a'+b'+1), & \text{if } a \neq 0, a' + 2b' \leq 0 \\ 0, & \text{if } a = 0 \end{cases}$$

The authors use orientation reversing diffeomophisms f, g of N which extend orientation reversing diffeomorphisms of the fiber and base, respectively, to send a class $ax + by$ to one where $a \geq 0, a + 2b \geq 0$. Since the case $a = 0$ is easily handled, one can then assume $a > 0$. For the symplectic form $\hat{x} + \hat{y}$, the symplectic canonical class is $K = -2x + (2g-1)y$. Then the upper bound for the minimal genus is again given by using the adjunction inequality. Realizing this upper bound is relatively easy when $a > 0, b \geq 0$, but requires a much more sophicated construction when $a > 0, b < 0$.

6. Positive double points of immersed spheres

Instead of looking at embedded surfaces in the four manifold, we might instead look at immersed spheres. We will look at this for the rational surface $X_m = \mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$. There are parallel results available, but with two differences. First, we will get bounds on the number of positive double points in place of the genus. Second, the results will apply to surfaces Σ with any value of $\Sigma \cdot \Sigma$, not just nonnegative values. In fact, just as we stabilized in the arguments above to get to a class of square zero, we will now stabilize to get to a class of negative self intersection. A tubular neighborhood N of such a class will have boundary a lens space $L(s, -1)$, which is a rational homology sphere. Thus the manifold X_N in which it is embedded will decompose as $X_N = Z \cup N$, where N and its boundary have metrics with positive scalar curvature. This will have as a consequence that the Seiberg-Witten moduli space has a simple decomposition. The results we present are due to R. Fintushel and R. Stern [FS1]. The main result is the following:

Theorem 22 ([FS1]). *Let $\alpha = dS_0 - \sum_{i=1}^m a_i S_i \in H_2(X_m), d, a_i \geq 0$. Then if $d \geq 2$ and*

$$d^2 - 3d - \sum_1^m (a_i^2 - a_i) - 2p \geq 0$$

the class α can't be represented by an immersed 2-sphere with p positive double points.

Remark 8. Another way to say this is that if α is immersed with p positive double points, then

$$2p \geq (d-1)(d-2) - \sum_{i=1}^m a_i(a_i - 1)$$

or

$$p \geq \binom{d-1}{2} - \sum_{i=1}^{m} \binom{a_i}{2}.$$

Note that the inequality can be rewritten in the same form as in Ruberman's theorem, with g replaced by p. The result above will follow from Ruberman's result when the square is non-negative. For if $(a_0, a_1, \ldots, a_n)$ is represented in $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ by an immersed sphere with p positive and n negative double points, then we will show below that we can stabilize to remove all of the negative double points in X_{m+n} without changing the homology class (using $H_2(X_m) \subset H_2(X_{m+n})$). Removing the positive double points by adding handles leads to a surface of genus p with in X_{m+n} which represents $(a_0, a_1, \ldots, a_m, 0, \ldots, 0)$. Theorem 13 then gives the result.

There are two basic constructions which we will use. Suppose there is an immersion $\Sigma \subset X$ with p positive double points and n negative double points. We show how to remove each type of double point by forming a new surface in $X \# \overline{\mathbb{CP}^2}$. For the positive double point x, consider two complex lines in $\overline{\mathbb{CP}^2}$ which intersect in one negative intersection point x'. Then choose an orientation reversing diffeomorphism of a neighborhood of $x \in X$ to a neighborhood of $x' \in \overline{\mathbb{CP}^2}$, and use it to form a connected sum of the pair. The result will replace the original two intersecting disks in X at p by the two nonintersecting complementary disks from the two complex lines in $\overline{\mathbb{CP}^2}$. The surface $\Sigma \subset X$ will be replaced by a new surface $\Sigma' \subset X \# \overline{\mathbb{CP}^2}$ which has one fewer positive intersection point and represents the homology class which is the sum of $[\Sigma]$ and $2[S]$ where S is the standard complex line with the opposite orientation. If there were a negative intersection point, then we could do a similar construction to remove it. Now we would have to create a positive intersection point in $\overline{\mathbb{CP}^2}$. The way to achieve this is to take two complex lines, but orient one in the standard fashion and the other with the opposite orientation. In terms of homology the surface Σ' will represent the class $[\Sigma]$ now considered as a homology class in $X \# \overline{\mathbb{CP}^2}$.

Suppose we have an immersed sphere Σ in the rational surface $X_m = \mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ with p positive double points and n negative double points which represents the class $dS_0 + \sum_{i=1}^{m} a_i S_i$. Then we may blow up p times to remove the p positive double points, n times to remove the n negative double points, and q times to reduce the self intersection as in our earlier arguments so it is negative. The new surface $\widetilde{\Sigma}$ we obtain will represent the homology class

$$[\widetilde{\Sigma}] = dS_0 + \sum_{i=1}^{m} a_i S_i + 2\sum_{j=1}^{p} S_j + \sum_{\ell=1}^{q} S_\ell$$

and the new canonical class will be

$$\widetilde{K} = -3S_0 - \sum_{i=1}^{n} S_i - \sum_{j=1}^{p} S_i - \sum_{k=1}^{n} S_k - \sum_{\ell=1}^{q} S_\ell.$$

The notation here emphasizes the three types of blowups used with subscripts j, k, ℓ. These all have the same effect on $\widetilde{K}$ but different effects on $\widetilde{\Sigma}$, adding 2,0,1 copies of S_i in the three cases. An important aspect of this construction is that the intersection

$$(\widetilde{K}+\widetilde{\Sigma})\cdot\widetilde{\Sigma}=(d^2-3d)-\sum_{i=1}^{n}(a_i^2-a_i)-2p.$$

Note also that if we have an immersion we can always increase the number of positive and negative double points by introducing a pair of cancelling double points. Thus the way to restate the theorem is that after stabilization so that $(d^2-3d)-\sum_{i=1}^{n}(a_i^2-a_i)-2p=0$, the assumption of having an immersed sphere with p positive double points will lead to a contradiction.

The method of the proof is to compute the Seiberg-Witten invariant in two different ways to get contradictory results. The first *spin*^{c} structure we use has determinant line bundle $\widetilde{L}=-\widetilde{K}$ as in earlier applications. Then $c_1(\widetilde{L})=c_1(T\widetilde{X})$ means that the dimension of the Seiberg-Witten moduli space will be 0. When we look at $\widetilde{L}-2\widetilde{\Sigma}$ instead, the dimension computation adds the term

$$4(\widetilde{\Sigma}-\widetilde{L})\cdot\Sigma=4(\widetilde{K}+\widetilde{\Sigma})\cdot\widetilde{\Sigma}=0$$

by our assumption on p. Moreover, these two moduli spaces will be empty for the positive scalar curvature metric g, and nontrivial for a metric with the opposite sign of $c_1(\widetilde{L})\cdot[\omega_g]$. By pinching the metric along the connected sums, we see the positive definite form $[\omega_g]$ tends to the Poincaré dual of S_0. But $(\widetilde{L}-2\widetilde{\Sigma})\cdot S_0=3-2d$ has opposite sign from $\widetilde{L}\cdot S_0$ when $d\geq 2$. We thus conclude that for any metric g' where $\widetilde{L}\cdot[\omega_{g'}]$ and $(\widetilde{L}-2\widetilde{\Sigma})\cdot[\omega_{g'}]$ have the same sign, the two moduli spaces $\mathcal{M}(\widetilde{L},g')$ and $\mathcal{M}(\widetilde{L}-2\widetilde{\Sigma},g')$ will have one empty and the other consisting of a single point.

We get a contradiction by using $\widetilde{\Sigma}$ to find a metric g' where these two moduli spaces are the same. The key observation here is that the embedding of $\widetilde{\Sigma}\subset\widetilde{X}$ leads to a decomposition $\widetilde{X}=Z\cup N$, where N is a tubular neighborhood of $\widetilde{\Sigma}$. We can use a metric with positive scalar curvature on N. Since $\Sigma\cdot\Sigma<0$, we still have $b_2^+(Z)=1$ and so we can avoid reducibles with a generic metric on Z. For each *spin*^{c} structure stretching the metric on $[-R,R]\times\partial N$ as before it leads in the limit to moduli spaces on Z_+ and N_+ with the following properties. First, the two moduli spaces over N_+ consist of a single reducible solution since the metric has positive scalar curvature there. Second, the moduli spaces also agree on Z_+ since the *spin*^{c} structures agree there. Gluing theory over the positive scalar curvature ∂N then implies that with a metric g' corresponding to high R these two moduli spaces $\mathcal{M}(\widetilde{L},g')$ and $\mathcal{M}(\widetilde{L}-2\widetilde{\Sigma},g')$ can be identified. But the fact that N_+ is negative definite means $\widetilde{\Sigma}\cdot[\omega_{g'}]=0$. Hence $\widetilde{L}\cdot[\omega_{g'}]$ and $(\widetilde{L}-2\widetilde{\Sigma})\cdot[\omega_{g'}]$ will have the same sign, which gives the contradiction to our earlier computation that the Seiberg-Witten invariants should differ.

The arguments used above can be isolated to prove some more general results. What we are doing is showing that the homology class $(a_0,a_1,\cdots,a_m,2,\cdots,2,0,\cdots,0,1,\cdots,1)\in H_2(X_{m+p+n+q})$ can't be represented by an embedded sphere. This follows from two lemmas whose proofs are contained in the argument above. They have a common hypothesis

(H) X is a closed oriented smooth 4-manifold with $b_2^+ = 1$, $H_1(X;\mathbb{R}) = 0$, L is a characteristic line bundle which is the determinant line bundle of a *spin*^{c} structure, still denoted by L, and Σ is an embedded surface. Suppose $d(L) = \dim \mathcal{M}_X(L) \geq 0$ and $(L - \Sigma) \cdot \Sigma = 0$, where we are identifying L with its Poincaré dual.

Lemma 23. *Let (X, L, Σ) satisfy (H). Suppose X has a positive scalar curvature metric g. Denote by ω_g the self dual harmonic 2-form of norm 1 in a chosen orientation of $H^2_+(X;\mathbb{R})$ used to define the Seiberg-Witten invariant. Suppose $L\cdot[\omega_g]$ and $(L-2\Sigma)\cdot[\omega_g]$ have opposite signs. Then for any other metric g' with $\Sigma \cdot [\omega_{g'}]$ close enough to zero so that $(L-2\Sigma)\cdot[\omega_g]$ and $L\cdot[\omega_g]$ have the same sign, we have $SW(L-2\Sigma, g') \neq SW(L, g')$.*

Lemma 24. *Let (X, L, Σ) satisfy (H). Suppose Σ is an embedded 2-sphere and $\Sigma \cdot \Sigma < 0$. Then there is a metric g' formed by stretching the neck around $\partial N(\Sigma)$ starting from a positive scalar curvature metric about on the tubular neighborhood $N(\Sigma)$ so that $SW(L - 2\Sigma, g') = SW(L, g')$.*

We close this section by stating a generalization of the adjunction inequality for $b_2^+ > 1$ which applies to immersed spheres.

Theorem 25 (Generalized Adjunction Formula for Immersed Spheres [FS1])**.** *Suppose X is a smooth 4-manifold with $b_2^+ > 1$ and L is a characteristic line bundle with $SW_X(L) \neq 0$ dim $\mathcal{M}_X(L) = \sum_{i=1}^r \ell_i(\ell_i + 1)$ with each integer $\ell_i \geq 0$. If $x \neq 0 \in H^2(X)$ is represented by an immersed sphere with p positive double points, then either*

$$2p - 2 \geq x^2 + |x \cdot L| + 4\sum_{i=1}^{r} \ell_i, p \geq r$$

$$2p - 2 \geq x^2 + |x \cdot L| + 4\sum_{i=1}^{p} \ell_i + 2\sum_{i=p+1}^{r} \ell_i, p < r$$

or

$$SW_X(L) = \begin{cases} SW_X(L + 2x) & \text{if } x \cdot L \geq 0 \\ SW_X(L - 2x) & \text{if } x \cdot L \leq 0 \end{cases}$$

This is proved using ideas of this section to reduce to a result on embedded spheres, which is proved in a similar manner as the blowup formula by replacing $\overline{\mathbb{CP}^2}$ by a neighborhood of an embedded sphere with negative self intersection. The result for embedded spheres is

Proposition 26 ([FS1])**.** *Suppose X is a smooth 4-manifold with $b_2^+(X) > 1$ and S is an embedded sphere with self-intersection $-r < 0$. Let L be a characteristic line bundle with $SW_X(L) \neq 0$ and write*

$$|S \cdot L| = kr + R$$

with $0 \leq R \leq r - 1$. If $k > 0$, then

$$SW_X(L) = \begin{cases} SW_X(L + 2S) & \text{if } L \cdot S > 0 \\ SW_X(L - 2S) & \text{if } L \cdot S < 0 \end{cases}$$

One special case of interest in the last proposition is when $S \cdot S = -2$. Since $L \cdot S \equiv S \cdot S$ mod 2, the proposition implies that either $L \cdot S = 0$ or one of $SW_X(L+2S)$ or $SW_X(L-2S)$ is nontrivial. This is usually applied in situations where these last two conclusions can be ruled out, and so the conclusion becomes $L \cdot S = 0$ for all basic classes. This then says that for spheres of square -2 the adjunction formula holds. This provides an extension of Remark 3 concerning tori.

If the manifold X only has non-zero Seiberg-Witten invariants when the formal dimension of the moduli space $d = 0$, it is said to be of Seiberg-Witten simple type. Symplectic manifolds with $b_2^+ > 1$ are known to be of Seiberg-Witten simple type [T3]. When $|S \cdot L| > r$ above, the dimension of the moduli space would be increased for $L \pm 2S$, which would give a contradiction to an assumption of simple type. Thus there is the following corollary.

Corollary 27. *Suppose X is a smooth 4-manifold with $b_2^+(X) > 1$ with an embedded sphere S with self-intersection $-r < 0$ which is of Seiberg-Witten simple type. If $SW_X(L) \neq 0$, then*

$$|L \cdot S| \leq r.$$

Versions of the above proposition hold when $b_2^+ = 1$, but are somewhat more complicated to state. Here is one version.

Proposition 28. *Suppose X is a smooth 4-manifold with $b_2^+(X) = 1$ and S is an embedded sphere with self-intersection $-r < 0$. If L is a characteristic line bundle with $SW_X^{S,+}(L) \neq 0, (L + 2S) \cdot S > 0$, then $SW_X^{S,+}(L) = SW_X(L + 2S)$. If $SW_X^{S,-}(L) \neq 0, (L - 2S) \cdot S < 0$, then $SW_X^{S,-}(X) = SW_X^{S,-}(L - 2S)$.*

There is also a version of the corollary, but unfortunately the hypothesis of simple type doesn't usually hold when $b_2^+ = 1$.

7. Generalized Thom Conjecture

In this section we want to discuss the solution to the Generalized Thom Conjecture by Morgan, Szabo, and Taubes [MST] for symplectic manifolds and some of the general results relating to the minimal genus problem which are part of it. The bulk of the paper establishes a gluing theorem for moduli spaces and deduces a product formula for Seiberg-Witten invariants. We will restrict our attention to describing how these are used in applications to the minimal genus problem. Their results include versions of Theorem 11 and Theorem 16.

Theorem 29 (Generalized Thom Conjecture [MST])**.** *Let X be a compact symplectic 4-manifold and let $C \subset X$ be a smooth symplectic curve with $C \cdot C \geq 0$. Let $C' \subset X$ be a C^∞ embedding of a Riemann surface representing the same homology class as C. Then $g(C') \geq g(C)$.*

This theorem follows easily from the existence of an adjunction inequality involving the canonical class as before since a symplectic class satisfies the adjunction formula. The proof involves application of a product formula for computing Seiberg-Witten invariants when there is a splitting $M = X_1 \cup_N X_2$, where $N = S^1 \times C$, with C a Riemann surface. This formula is proved for a special setup which is keyed to the desired adjunction inequality. The product formula has the following important corollary which is used in our applications. Suppose $M = X\#_C Y$ is formed from X, Y by removing product neighborhoods $D^2 \times C$ of $C \subset X, Y$ and gluing their boundaries.

Theorem 30 ([MST, Corollary to Product Formula]). *Let $g = g(C) > 1$ and $M = X\#_C Y$. If there are characteristic classes $\ell_1 \in H^2(X), \ell_2 \in H^2(Y)$ with $\ell_1 \cdot C = \ell_2 \cdot C = 2g-2$ and with $SW_X(\ell_1) \neq 0, SW_Y(\ell_2) \neq 0$, then there is a characteristic class $k \in H^2(M)$ with $k|_N = \rho^* k_0$ for $k_0 \in H^2(C)$ satisfying $k_0 \cdot C = 2g-2$ for which $SW_M(k) \neq 0$. (We use SW^{C^*} whenever $b_2^+ = 1$.)*

An important special case of the product formula is when we assume an embedding of C into X with $C \cdot C = 0$ (so C has a neighborhood $N = D^2 \times C$) and we can choose $X_1 = X \setminus \operatorname{int} N, X_2 = N$. In order to prove the product formula, there is a careful analysis of manifolds with cylindrical ends, and the appropriate modifications of the Seiberg-Witten equations needed to get compact moduli spaces and gluing theorems. Their arguments break into two types, handing $g(C) > 1$ and showing that a symplectic torus can't have its homology class represented by a sphere.

For the moment we assume $g = g(C) > 1$. The splittings being studied are closely tied to the adjunction inequality in that they only look at $spin^c$ structures so that the determinant line bundle k satisfies $k \cdot C = 2g(C) - 2$. In the case when k is the canonical bundle K for a symplectic structure, this means that the adjunction formula holds for C, which would be true if C has a symplectic representative. Their arguments work in the case $b_2^+ = 1$ where they look at forms of the Seiberg-Witten invariants well adapted to their situation. For a real cohomology class x with $x \cdot x \geq 0$, they define the x-negative Seiberg-Witten invariant $SW^x(k)$ to be the Seiberg-Witten invariant where the metric g and perturbation h are chosen so that $p_+(2\pi k + h) \cdot x < 0$. Here p_+ is the projection onto the self-dual forms determined by the metric g – the projection will be a non-zero multiple of the form ω_g. For the application here, we are interested in $SW^{C^*}(k)$ where C^* is the Poincare dual of C. In the case of a symplectic manifold, Taubes has shown that using the symplectic metric compatible with the symplectic form ω, then for the canonical bundle K we have $SW^\omega(-K) = \pm 1$, and this motivates the choice of sign here. Part of the argument comes in identifying $SW^{C^*}(-K)$ and $SW^\omega(-K)$. This is related to the condition $C \cdot \omega > 0$, which is precisely the hypothesis that Li and Liu used in proving their form Theorem 16 of the adjunction inequality. Note that in our earlier notation, their $SW^{C^*}(k)$ is our $SW^{C,-}(k)$.

The arguments are applied in the symplectic case, but they do prove more general results.

Theorem 31 ([MST, Prop. IV.1.1]). *Let X be a closed oriented 4-manifold with $b_2^+(X) + b_1(X)$ odd, and let $C \subset X$ be a smooth surface of genus $g > 1$ and square 0. Suppose that there is a characteristic class $k \in H^2(X)$ such that $k \cdot C = 2g-2$ and suppose that the Seiberg-Witten invariant $SW(k) \neq 0$, (this means $SW^{C^*}(k) \neq 0$ in the case $b_2^+ = 1$). Then any C^∞ surface in the same homology class as C has genus at least as large as that of C.*

Using stabilization and a blowup formula for the Seiberg Witten invariants, there is the general form for classes of non-negative square:

Theorem 32 ([MST, Proposition IV.1.2]). *Let X be as above and suppose $C \subset X$ is a smoothly embedded Riemann surface of genus $g > 1$ with $C \cdot C \geq 0$. Suppose there is a characteristic class k such that $k \cdot C + C \cdot C = 2g - 2$ and $SW(k) \neq 0$ ($SW^{C^*}(k) \neq 0$ when $b_2^+ = 1$). Then any C^∞ curve in the same homology class as C has genus at least as large as C.*

Remark 9. Note that since the class C satisfies the generalized adjunction formula by hypothesis for k in place of K, any other representative will have to satisfy the adjunction inequality as given in Theorem 11. Moreover, this result may be used to deduce Theorem 11 for $g \geq 1$ by starting out with an assumed counterexample C'. This would mean that $2g(C') - 2 < k \cdot C' + C' \cdot C'$. Then we can add trivial handles to C' to form C with $2g(C) - 2 = k \cdot C, C \cdot C = 0$. But the fact that $g(C') < g(C)$ contradicts the above theorem. Moreover, the result also implies the form of the generalized adjunction formula we discussed in the last section when $b_2^+ = 1$.

We now want to outline the argument for Theorem 31. There is a separate argument using cohomological calculations that says that a class minimizes the genus if $H_1(C) \to H_1(X)$ is injective, so we can assume that it has non-trivial kernel. Then we remove a tubular neighborhood $D^2 \times C$ of C and glue two copies of the complement together along the boundary to form $M = X\#_C X$. Since there is a nontrivial kernel and $b_2^+(X) \geq 1$, we get $b_2^+(M) \geq 2$. The Product Formula is then used to compute that $SW_M(k') \neq 0$ for some class k' on M which restricts to $C \subset S^1 \times C$ to evaluate as $2g-2$. If this surface did not minimize the genus within its homology class, then by adding trivial handles to a smaller genus representative, we can find a representative with the same genus containing a trivial handle to which to apply this argument and get a nonzero Seiberg-Witten invariant. However, the existence of this trivial handle can be used to show the Seiberg-Witten invariant must vanish. One way of showing this is to show that M contains a $S^2 \times S^2$ summand.

We now want to sketch how to handle the case when C is a torus, and indicate why the class can't be represented by an embedded sphere. Using Taubes' nonvanishing result for the Seiberg-Witten invariant $SW^\omega(K)$ on the canonical class and the identification of $SW^{C^*}(K)$ with $SW^\omega(K)$, the minimality of genus for a symplectic torus reduces to the following lemma.

Lemma 33 ([MST, Lemma X.1.1]). *Let X be a closed, oriented 4-manifold, and suppose $S \subset X$ is a smoothly embedded sphere of square 0 representing a homology class of infinite order in $H_2(X)$. If $b_2^+ > 1$, then the Seiberg-Witten invariant SW_X vanishes identically. If $b_2^+ = 1$, letting S^* denote the cohomology class Poincaré dual to S, the Seiberg-Witten invariant $SW_X^{S^*}$ vanishes on any characteristic class k with the property that $k \cdot S = 0$.*

Remark 10. Note that the case $b_2^+ > 1$ is contained in Theorem 11 and the special case for the rational surfaces when $b_2^+ = 1$ is contained in the proof of Theorem 13.

We sketch the idea of the argument. For $b_2^+ > 1$, look at the manifold Z with cylindrical end $X_0 \cup [0,\infty) \times S^1 \times S, X_0 = X \setminus N(C)$. Finite energy solutions on Z with bound e on the energy satisfy (up to gauge) exponential decay conditions on the spinor field, tend to a flat connection A_0 on $S^1 \times S$ with similar exponential order bounds, and have compact moduli spaces $\mathcal{M}_e(\widetilde{P}, g_Z, h)$ for any *spin*^{c} structure. There is a smooth map $\partial : \mathcal{M}_e(\widetilde{P}, g_Z, h) \to S^1$ which assigns to each solution the limiting flat connection at infinity, which is transverse to -1 for generic h. When the *spin*^{c} structure comes from one on X, the gluing theorem identifies the moduli space $\mathcal{M}(\widetilde{P}, g_R, h_R)$ associated to stretching the neck around $S^1 \times C$ with the codimension 1 submanifold $\partial^{-1}(-1)$. Moreover, this submanifold is the Poincaré dual of a class $\mu(S^1)$ and allows us to evaluate the Seiberg-Witten invariant as an integral over $\mathcal{M}_e(\widetilde{P}, g_Z, h)$ of $\mu^d \cup \mu(S^1)$. The hypothesis that S is of infinite order then implies that in Z the class S^1 is of finite order (if $S \cdot T = n$ with transversal intersection, then nS^1 bounds the surface T_0 obtained from T by removing neighborhoods of the intersection points). This then implies that the integral is 0.

When $b_2^+ = 1$, it is first shown that if ω_R^+ denotes the g_R-self-dual form on X of norm 1 with positive integral over S, then as $R \to \infty$ the forms ω_R^+ converge to zero on $(X_0 \amalg D^2 \times S) \subset X$. This uses the fact that the corresponding cylindrical end manifolds $Z \amalg T$ have $b_2^+ = 0$ and so have no non-zero self-dual L^2-forms. In particular, this is used to show that for $k \cdot S = 0$, then k is represented by a closed 2-form λ on X with support in X_0 and $\int_X \omega_R^+ \wedge \lambda = 0$. They look at the Seiberg-Witten moduli space using the curvature equation

$$F_A^+ = q(\psi) - ir\omega_R^+$$

which for $r >> 0$ may be used to compute the S^*-negative Seiberg-Witten invariant. An analytic argument using the Weitzenböch formula and $\int_X \omega_R^+ \wedge \lambda = 0$ is used to show that for $R >> 0, r >> R$, there are no solutions to these equations whenever the determinant line bundle evaluates 0 on S.

8. Applications of Furuta's theorem

The 11/8 conjecture is a long-standing conjecture which states that for an indefinite spin 4-manifold with nonzero signature, the ratio of the second Betti number to the absolute value of the signature must satisfy $\dfrac{b_2}{|\sigma|} \geq \dfrac{11}{8}$. The $K3$ surface and, more generally, the connected sum of n copies of homotopy $K3$s, give examples where there is equality. For a smooth spin 4-manifold with indefinite intersection form, the intersection form is $Q = \pm 2pE_8 \oplus qH, q \neq 0$. The 11/8 conjecture can be rephrased in these terms as $q \geq 3p$. Note also $q = \min(b_2^+, b_2^-)$ in this decomposition. By using the compactness of the Seiberg-Witten moduli space and K-theory constructions, Furuta [Fu] has proved a slightly weaker form of this conjecture.

Theorem 34 ([Fu])**.** *Suppose X is a closed connected smooth spin 4-manifold with indefinite intersection form $\pm 2pE_8 \oplus qH, q \neq 0$. Then $q \geq 2p + 1$. Equivalently,*

$$\min(b_2^+(X), b_2^-(X)) \geq \frac{|\sigma(X)|}{8} + 1.$$

Note that this result contains within it the earlier theorem of Donaldson which says that if $q = 1, 2$ then $p = 0$. There had been applications made of this more restricted result to restrictions on characteristic spheres earlier, and it had been noted by many authors that the 11/8 conjecture would allow also one to make statements about the square of an embedded sphere representing a characteristic homology class.

Yasuhara [Y] provides results about characteristic homology classes of any genus by relating them to characteristic embedded spheres through a connecting lemma.

Lemma 35 ([Y, Connecting lemma III])**.** *Let X be a closed simply connected 4-manifold and F an embedded, closed, orientable surface in X that represents a characteristic homology class. If Arf(F) $= 0$, i.e. $[F] \cdot [F] \equiv \sigma(X) \mod 16$, then there exists an embedded, closed orientable surface F_1 in $M \# S^2 \times S^2$ such that $[F_1]$ is a characteristic homology class, Arf($[F]$) $= 0, [F_1] \cdot [F_1] = [F] \cdot [F]$ and genus(F_1) $=$ genus (F) $- 1$.*

Using this lemma, Yasuhara is able to take questions about a surface of genus g and reduce them to questions about an embedded sphere. One of these results uses an assumed 11/8 conjecture. Using Furuta's theorem, then the proof of Yasuhara's Theorem 1.2 can be modified to give:

Theorem 36 ([Y])**.** *Let X be a closed simply connected 4-manifold with $M = \max(b_2^+, b_2^-)$. Suppose ξ a characteristic homology class in $H_2(X)$ with $\xi \cdot \xi \equiv \sigma(X) \mod 16$, and ξ is represented by an embedded closed oriented surface with genus g. Then $\xi \cdot \xi = \sigma(X)$ or, if not,*

$$|\xi \cdot \xi - \sigma(X)| \leq 8(M + g - 2)$$

There are similar results available when $\xi \cdot \xi \equiv \sigma(X) + 8 \mod 16$ obtained by taking a connected sum with a torus in $S^2 \times S^2$ with $T \cdot T = 8$ and then applying the above result. The argument modifies the embedding to get to the case of a characteristic sphere with square 1, then removes a tubular neighborhood and adds a disk to get a spin manifold on which to apply Furuta's theorem.

One can get much stronger results by first using an orbifold version of Furuta's theorem. The idea for using an orbifold version of Furuta's theorem to study characteristic spheres of negative square was communicated to me by Ron Fintushel. Danny Acosta [Ac] has since worked out the details of the proof of the orbifold version of Furuta's theorem and provided refinements of Fintushel's suggestions and further applications.

Theorem 37 ([Fu],[F],[Ac])**.** *Suppose Y is a 4-dimensional spin orbifold with indefinite intersection form with a finite number of non-manifold singularities which are cones on lens spaces. Let $m = \min(b_2^+(Y), b_2^-(Y))$. Then*

$$m \geq |ind\ (D)| + 1$$

where D is the Dirac operator of the orbifold.

When ξ is a characteristic homology class in a manifold X with $\xi\cdot\xi\neq 0$ which is represented by a smoothly embedded 2-sphere, we can form an orbifold by removing a neighborhood of the sphere and replacing it by a cone on the boundary lens space. If the manifold is spin or $\xi\cdot\xi=0$, we first must stabilize by adding a $\mathbb{CP}^1\subset\overline{\mathbb{CP}^2}$ or $\mathbb{CP}^1\subset\mathbb{CP}^2$. The index of the Dirac operator is computed to be $\dfrac{\xi\cdot\xi-\sigma(X)}{8}$. We then get the following theorem.

Theorem 38 ([Fi],[Ac])**.** *Suppose ξ is a characteristic homology class in an indefinite smooth oriented 4-manifold X which is represented by a smoothly embedded 2-sphere. Let $m=\min(b_2^+(X),b_2^-(X))$. Then either $\xi\cdot\xi=\sigma(X)$ or, if not,*

- *If $\xi\cdot\xi=0$ or $\xi\cdot\xi,\sigma(X)$ have the same sign, $|\xi\cdot\xi-\sigma(X)|\leq 8(m-1)$.*
- *If $\sigma(X)=0$ or $\xi\cdot\xi,\sigma(X)$ have opposite signs, $|\xi\cdot\xi-\sigma(X)|\leq 8(m-2)$.*

We can apply Yasuhara's connecting lemma to this to get the following result.

Theorem 39 ([Ac])**.** *Suppose ξ is a characteristic homology class in an indefinite smooth oriented 4-manifold X which is represented by an embedded surface of genus g. Let $m=\min(b_2^+(X),b_2^-(X))$.*

- *If $\xi\cdot\xi\equiv\sigma(X) \mod 16$, then either $\xi\cdot\xi=\sigma(X)$ or, if not,*
  (a) *If $\xi\cdot\xi=0$ or $\xi\cdot\xi,\sigma(X)$ have the same sign, $|\xi\cdot\xi-\sigma(X)|\leq 8(m+g-1)$.*
  (b) *If $\sigma(X)=0$ or $\xi\cdot\xi,\sigma(X)$ have opposite signs*
  $$|\xi\cdot\xi-\sigma(X)|\leq 8(m+g-2)$$
- *If $\xi\cdot\xi\equiv\sigma(X)+8 \mod 16$, then*
  (a) *If $\xi\cdot\xi=-8$ or $\xi\cdot\xi+8,\sigma(X)$ have the same sign,*
  $$|\xi\cdot\xi+8-\sigma(X)|\leq 8(m+g+1)$$
  (b) *If $\sigma(X)=0$ or $\xi\cdot\xi+8,\sigma(X)$ have opposite signs,*
  $$|\xi\cdot\xi+8-\sigma(X)|\leq 8(m+g)$$

Note that these results just apply to characteristic homology classes, but they apply very generally where we may know nothing about Seiberg-Witten homology classes. Using an analogous application of Theorem 37 in the case when the original manifold is spin, Acosta [Ac] has also provided another proof of the following result, which was first proved by Ruberman [Rub] (with a few restrictions that have been removed by the new proof) using orbifold methods in gauge theory.

Theorem 40 ([Rub],[Ac])**.** *Let ξ be a 2-dimensional homology class in a smooth spin 4-manifold M with $b_2^+(M)=3, b_2\geq 8$. If $\xi\cdot\xi\geq 0$, then ξ can't be represented by an embedded sphere.*

In the proof of this theorem the spin hypothesis leads to a spin orbifold, and Theorem 37 now leads to an equation $b_2^+(M)\geq|\sigma(M)/8|+2$ whenever there is a class of non-negative square represented by an embedded sphere. Since $\sigma(M)/8$ is even this would lead to a contradiction here. More generally, the method gives the following theorem.

Theorem 41 ([Ac])**.** *Let M be a 4-dimensional indefinite smooth spin manifold. Suppose* $3 \leq \min(b_2^+, b_2^-) = \dfrac{|\sigma(M)|}{8} + 1,$ *Then if ξ is a 2 dimensional homology class with $\xi \cdot \xi \geq 0$, then ξ can't be represented by a smoothly embedded sphere.*

Remark 11. Although this theorem seems to be much more general than the previous result, the only known cases of where the hypotheses apply is when $\min(b_2^+, b_2^-) = 3$.

D. Kotshick and G. Matic [KoM] use gauge theory combined with branched covering constructions generalizing those used in the proof of Rochlin's genus theorem to prove the following result.

Theorem 42 ([KoM])**.** *Let X be a smooth closed oriented 4-manifold with $\pi_1(X) = 1$. Assume that X is not spin and that Σ is an embedded surface representing twice the Poincaré dual of a lift of $w_2(X)$ to integral coefficients. If $\frac{1}{4}\Sigma \cdot \Sigma \neq \sigma(X)$, then*

$$g(\Sigma) \geq \left| \frac{1}{4}\Sigma \cdot \Sigma - \sigma(X) \right| - b_2(X) + 3.$$

Although the main applications given in [KoM] are to cases of the Thom conjecture which have now been proven more generally using Seiberg-Witten methods, the results themselves are more general since they have no hypothesis relating to the Seiberg-Witten invariants. They are based ultimately on showing that the two-fold branched cover of X over Σ is a spin manifold under the given hypotheses and then applying Donaldson's Theorem C which says that a spin manifold with no 2-torsion in H_1 with nontrivial signature must have the minimum of b_2^+, b_2^- greater than or equal to 3. They indicate how to use the 11/8 conjecture to get other results. If we apply Furuta's Theorem 34 to the cover Y we get corresponding results. J. Bryan [B1] has done this in the context of q fold covers for $q = p^r, p$ prime. To guarantee that the cover is a spin manifold to which one can apply Furuta's theorem there is an appropriate hypothesis on the class involved or the original 4-manifold.

Theorem 43 ([B1])**.** *Let X be a smooth closed oriented simply connected 4-manifold, and $q = p^r, p$ prime. Suppose Σ is an embedded surface in X of genus g and $[\Sigma]$ is divisible by q. If q is even, assume that $\frac{1}{q}[\Sigma]$ is characteristic; if q is odd, assume that X is spin – these assumptions guarantee that the q-fold cover is spin. Then*

$$g \geq \frac{1}{q-1}\left[\frac{5}{4} \left| \frac{q^2-1}{6q}\Sigma \cdot \Sigma - \frac{q}{2}\sigma(X) \right| + 1 - \frac{q}{2} b_2(X) \right].$$

Proof. Take Y to be the q-fold branched cover of X over Σ. Since $w_2(Y) = \pi^*(w_2(X) - \frac{q-1}{q}PD[\Sigma]_2)$, our hypotheses guarantee that Y is spin. As in [KoM], we start with

$$b_2(Y) = b_2^+(Y) + b_2^-(Y) = qb_2(X) + 2(q-1)g$$
$$\sigma(Y) = b_2^+(Y) - b_2^-(Y) = q\sigma(X) - \frac{q^2-1}{3q}\Sigma\cdot\Sigma$$

Subtracting and adding the two equations gives:

$$b_2^+(Y) = \frac{q}{2}b_2(X) + (q-1)g - (\frac{q^2-1}{6q}\Sigma\cdot\Sigma - \frac{q}{2}\sigma(X))$$
$$b_2^-(Y) = \frac{q}{2}b_2(X) + (q-1)g + (\frac{q^2-1}{6q}\Sigma\cdot\Sigma - \frac{q}{2}\sigma(X))$$

Thus

$$\min(b_2^+(Y), b_2^-(Y)) = \frac{q}{2}b_2(X) + (q-1)g - \left|\frac{q^2-1}{6q}\Sigma\cdot\Sigma - \frac{q}{2}\sigma(X)\right|.$$

Now applying Furuta's Theorem 34 gives

$$\frac{q}{2}b_2(X) + (q-1)g - \left|\frac{q^2-1}{6q}\Sigma\cdot\Sigma - \frac{q}{2}\sigma(X)\right| \geq \frac{1}{4}\left|\frac{q^2-1}{6q}\Sigma\cdot\Sigma - \frac{q}{2}\sigma(X)\right| + 1,$$

giving

$$g \geq \frac{1}{q-1}\left[\frac{5}{4}\left|\frac{q^2-1}{6q}\Sigma\cdot\Sigma - \frac{q}{2}\sigma(X)\right| + 1 - \frac{q}{2}b_2(X)\right].$$

□

One should contrast this final result with the result coming from Rochlin's genus theorem, which says that

$$g \geq \left|\frac{q^2-1}{4q^2}\Sigma\cdot\Sigma - \frac{1}{2}\sigma(X)\right| - \frac{1}{2}b_2(X), q \neq 2,$$
$$g \geq \left|\frac{1}{4}\Sigma\cdot\Sigma - \frac{1}{2}\sigma(X)\right| - \frac{1}{2}b_2(X), q = 2^p.$$

In [B1], Bryan outlines work in progress to get better results by applying the method of Furuta's proof to the cover, but using the full $\mathrm{Pin}(2)\tilde{\times}\mathbb{Z}_{2q}$ symmetry that the Seiberg-Witten moduli space possesses. In a talk [B2] he has announced results generalizing Furuta's theorem for spin actions on spin manifolds. Suppose that $\tau : X \to X$ is an orientation preserving isometry generating a $\mathbb{Z}_q$ action. τ is said to be a spin action if τ_* preserves the spin structure. Spin actions of $\mathbb{Z}_q$ fall into two categories depending on whether the lifts of τ to the spin bundle have order $\mathbb{Z}_q$ or $\mathbb{Z}_{2q}$ (called the *even* and *odd* cases repectively); these two cases are easily distinguished using a lemma of Atiyah and Bott [At-Bo]. Bryan's results are:

Theorem 44 ([B2]). *Suppose $q = 2^r$ and $\tau : Y \to Y$ generates an odd spin action of $\mathbb{Z}_q$ on a spin manifold Y of negative signature. Let $k = -\sigma_Y/16$ and Y_i denote $Y/\mathbb{Z}_{2^i}$ for $i = 1, \dots, r$. Assume that $b_2^+(Y) - 2k \neq b_2^+(Y_1)$ and $b_2^+(Y_1) \neq b_2^+(Y_2) \neq \cdots \neq b_2^+(Y_r)$. Then*

$$b_2^+(Y) \geq 2k + 1 + r.$$

In the case of even spin actions the best result is for $\mathbb{Z}_2$ actions:

Theorem 45 ([B2]). *Suppose $\tau : Y \to Y$ generates an even involution on a spin manifold Y of negative signature. Let $k = -\sigma_Y/16$ and assume that $b_2^+(Y) \neq b_2^+(Y/\tau)$. Then*

$$b_2^+(Y) \geq 2k + 2.$$

Of course the analogous theorems in the case of positive signature are obtained by replacing b_2^+ with b_2^-. Note that these results improve the estimate $m \geq 2k + 1$ from Furuta's theorem by the addition of r on right hand side. Spin actions arising from a branched cover are always odd actions and when Theorem 44 is applied to the minimal genus problem, it leads to the following improvement of Theorem 43:

Theorem 46 ([B2]). *Let X be a smooth closed oriented simply connected 4-manifold, and $q = 2^r$. Suppose Σ is an embedded surface in X of genus g and $[\Sigma]$ is divisible by q. Assume that $\dfrac{1}{q}[\Sigma]$ is characteristic and that $b_2^+(X) > r$. Then*

$$g \geq \frac{1}{q-1}\left[\frac{5}{4}\left(\frac{q^2-1}{6q}\Sigma\cdot\Sigma - \frac{q}{2}\sigma(X)\right) + 1 + r - \frac{q}{2}b_2(X)\right].$$

One can get results concerning characteristic homology classes themselves by combining Bryan's results and the bounds on representing multiple classes in terms of the genus of a representative of a characteristic class.

We first assume that there is a characteristic class $\xi = [\Sigma]$ and $g = g(\Sigma), n = |\Sigma \cdot \Sigma|$ in a manifold X with $b_2^+(X) > 1$. Then the class 2ξ is represented by an embedded surface Σ' of genus $2g + n - 1$. Theorem 46 gives the estimate

$$g(\Sigma') \geq \frac{5}{4}\,|\Sigma\cdot\Sigma - \sigma(X)| + 2 - b_2X.$$

Substituting in our value of $g(\Sigma')$ in terms of $g = g(\Sigma)$ gives

$$2g + |\Sigma\cdot\Sigma| - 1 \geq \frac{5}{4}\,|\Sigma\cdot\Sigma - \sigma(X)| + 2 - b_2X,$$

$$g \geq \frac{1}{2}\left[\frac{5}{4}\,|\Sigma\cdot\Sigma - \sigma(X)| - |\Sigma\cdot\Sigma|\right] + 3/2 - \frac{b_2X}{2}.$$

This inequality may be expressed as follows. As before, let $M = \max(b_2^+(X), b_2^-(X)), m = \min(b_2^+(X), b_2^-(X))$. Then

$$g \geq \begin{cases} \dfrac{|\Sigma \cdot \Sigma - \sigma(X)|}{8} + 2 - M & \text{if } \Sigma \cdot \Sigma \leq \sigma(X) \leq 0 \text{ or } 0 \leq \sigma(X) \leq \Sigma \cdot \Sigma \\ \dfrac{9|\Sigma \cdot \Sigma - \sigma(X)|}{8} + 2 - M & \text{if } \sigma(X) \leq \Sigma \cdot \Sigma \leq 0 \text{ or } 0 \leq \Sigma \cdot \Sigma \leq \sigma(X) \\ \dfrac{|\Sigma \cdot \Sigma - \sigma(X)|}{8} + 2 - m & \text{if } \sigma \leq 0 \leq \Sigma \cdot \Sigma \text{ or } \Sigma \cdot 0 \leq \sigma(X) \end{cases}$$

This result lies somewhere in between the results of Yasuhara and those of Fintushel and Acosta discussed earlier. One can try to use the same method for classes of odd divisibility, but unfortunately no new results can come from it without a better estimate of the genus of the multiple class in terms of the genus of the original class.

9. A survey of other recent results

In this section we survey a number of other recent results which feature other methods than those already discussed.

9.1. **Results on representations by spheres and tori.** In the last few years, a number of authors [Ga], [GG], [K1], [K2] have made contributions to the problem of determining which homology classes can be represented by an embedded 2-sphere, particularly to rational surfaces. Using Donaldson's theorem on smooth definite 4-manifolds, Kikuchi [K2] provided a general characterization of when a homology class with positive square could be represented by an embedded sphere in an almost definite 4-manifold.

Theorem 47 ([K2])**.** *Let X be a closed oriented smooth 4-manifold with $b_2^+ = 1, b_2^- = m \geq 1$, and ξ a class in $H_2(X)$ with $\xi \cdot \xi = s > 0$. If ξ is represented by a 2-sphere, then either of the following holds:*

1. *There exist $\zeta_1, \cdots, \zeta_m$ in $Fr(H_2(X)$ such that*
$$Fr(H_2(X)) = < 1 > \oplus m < -1 >$$
with respect to the basis $< \eta; \zeta_1, \cdots, \zeta_m >$, where $Fr(\xi) = 2\eta$;
2. *There exist $\eta, \zeta_1, \cdots, \zeta_{m-1}$ in $Fr(H_2(X))$ such that*
$$Fr(H_2(X)) = \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix} \oplus (m-1) < -1 >$$
with respect to the basis $< Fr(\xi), \eta; \zeta_1, \cdots, \zeta_{m-1} >$.

This can then be used to characterize explicitly which homology classes with positive square can be represented by an embedded sphere in a rational surface. Recall that a class is in reduced form if $a \geq b_1 + b_2 + b_3, a \geq 0, b_1 \geq \cdots \geq b_n \geq 0$ and that any class of positive square is in the orbit under the action of the orthogonal group of an element in reduced form. When $n \leq 9$, all of these automorphisms are realizable by diffeomorphisms. Kikuchi [K2] shows that any class with positive square representable by an embedded sphere must be in the orbits of one of the reduced classes $(2; 0, \cdots, 0), (k+1, k, 0, \cdots, 0), (k+1, k, 1, 0, \cdots, 0)$ under the action of the orthogonal group, with the converse holding when $n \leq 9$ for $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$.

Note that there are an infinite number of such orbits. This could also be deduced from the results of Ruberman or Li and Li discussed above. Li and Li [LL1] extended this last result to minimal genus 1 and 2.

Theorem 48 ([LL1, Theorem 4]). *Denote by $(a; b_1, \cdots, b_m)$ the class $\xi = aS_0 + \sum_i^m b_i S_i$ in reduced form in $H_2(\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2})$ with $\xi \cdot \xi > 0, m \leq 8$. Then*

(1) *The set of ξ with minimal genus 1 is*

$$(3; 0, \cdots, 0), (4; 2, 2, 0, \cdots, 0), (3; 1; b_2, \cdots, b_n)$$

(2) *The set of ξ with minimal genus 2 is*

$$(6; 2, \cdots, 2), n = 8; (5; 3, 2, 0, \cdots, 0), (4; 2, 1, b_3, \cdots, b_n)$$

9.2. **Minimal genus in a disk bundle.** In [KM3] Kronheimer and Mrowka introduced the problem of finding the minimal genus of a class $m\Sigma_g, m > 0$ in a disk bundle of Euler class $n > 0$ over a surface of genus g. An upper bound for this minimal genus is given by a geometric construction starting with m sections of the bundle and then connecting them up and removing intersections to give a representative of genus

$$g(m\Sigma_g) = \frac{1}{2}(m^2 n + m(2g - 2 - n)) + 1.$$

They conjecture that this is in fact the minimal genus, and prove this is true when $2g-2-n \geq 0$.

Conjecture 1 (Thom-Kronheimer-Mrowka Conjecture). *The minimal genus of a surface F representing $m\Sigma$ in a disk bundle over a surface g with Euler number n is given by*

$$g(F) = \frac{1}{2}(m^2 n + m(2g - 2 - n)) + 1.$$

In [M1] Mikhalkin proves the special case of the conjecture when $m = 2$ using branched covering arguments similar to those used by Rochlin and Hsiang-Szczarba in proving the Rochlin genus estimates. In this case the Rochlin genus estimate would be

$$g(m\Sigma_g) \geq k^2 n - 1, m = 2k$$

and Mikhalkin verifies the conjecture by proving

$$g(m\Sigma_g) \geq 2g + k^2 n - 1.$$

Note that the Thom conjecture itself corresponds to proving the estimate for the case $g = 0, n = 1$. By using the generalized Thom conjecture, we can see that the conjecture holds in other cases as well. First note the following principle. If Σ_g denotes the surface of genus g embedded in a 4-manifold X and $n = \Sigma_g \cdot \Sigma_g$, then the disk neighborhood $N(\Sigma_g) \subset X$. Then the minimal genus of a surface representing $m\Sigma_g$ in X will be less than or equal to the minimal genus of $m\Sigma_g$ in the disk neighborhood. Thus if we can find a 4-manifold X so that the minimal genus is given by the estimate above, it means that the minimal genus in the neighborhood is also given by this estimate. Secondly, anytime the minimal genus is governed by a generalized adjunction inequality

$$2g - 2 \geq \kappa \cdot \xi + \xi \cdot \xi$$

for all classes ξ and we have equality for a class Σ, then the conjecture will hold for $n = \Sigma \cdot \Sigma$ and for all m :

$$2g(m\Sigma) - 2 \geq \kappa \cdot m\Sigma + m^2\Sigma \cdot \Sigma = m(2g - 2 - n) + m^2 n.$$

The verification of the conjecture for $2g - 2 - n \geq 0$ due to Kronheimer and Mrowka just follows from the fact that for the $K3$ surface and its blow-ups we can find embedded surfaces of any genus where we have equality.

Note that the classes $(k+1, k, 0), (k+1, k, 1) \in \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ are represented by embedded spheres and are reduced, and they have square given by $2k+1, 2k$. Hence any square may occur. Now Li and Li [LL1] show that for $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$, for any reduced class ξ the minimal genus is governed by a formula

$$g(\xi) = \max_{\eta \in S_\xi} 1 + \frac{1}{2}(K_\eta \cdot \xi + \xi \cdot \xi).$$

Note that $S_\xi = S_{m\xi}$ and ξ is reduced exactly when $m\xi$ is. From this formula it follows that the minimal genus for $m\xi$ obeys the conjecture in $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$, hence in the disk neighborhood. For $\xi = (k+1, k, 0), (k+1, k, 1)$ we will have

$$2g(m\xi) - 2 = K \cdot (m\xi) + m^2(\xi \cdot \xi) = m(2g - 2 - \xi \cdot \xi) + m^2(\xi \cdot \xi)$$

which is precisely what the conjecture says when $g = 0$. Hence the conjecture holds for $g = 0$. One can use the results of [LL1] to verify the conjecture in other cases as well. The above argument will work to verify the conjecture for all m for those n which occur as squares of a class of genus g in $\mathbb{CP}^2 \# p\overline{\mathbb{CP}^2}, p \leq 8$ where [LL1, Theorem 3] can be applied. In particular, if one uses [LL1, Theorem 4] to see precisely which squares occur for classes of minimal genus 1 or 2, we see that the conjecture will hold for all m when $g = 1$ as long as $n \leq 9$ and for $g = 2$ as long as $4 \leq n \leq 12$. One can find examples for other genuses by this method. In particular, whenever you have an algebraic surface where the class ξ is realized by an algebraic curve of genus g and square n, then the adjunction formula for ξ and the adjunction inequality for $m\xi$ using the canonical class will prove the conjecture holds for g, n and all m. Using the results of [LL2] on the minimal genus of the nontrivial S^2 bundle over M_g, the case of $n = 1$ is also seen to hold.

10. Applications of the adjunction inequality

10.1. **Determining basic classes.** We now give an example to illustrate the power of the adjunction inequality in determining the basic classes. Our example is a small piece of one step of Szabo's construction of 4-manifolds X so that neither X nor $\overline{X}$ can have a symplectic structure [Sz]. As a first step in his construction, Szabo constructs a manifold M with $b_1 = 3, b_2^+ = 2$ for which he identifies the Seiberg-Witten classes using the adjunction inequality. He starts with the rational surface $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2} = E(1)$ which is an elliptic fibration over S^2. In this elliptic fibration the fiber F is a holomorphic torus which represents the class $3H-E_1-\cdots-E_9$. There is a decomposition of $E(1) = M_8 \cup_{\Sigma(2,3,5)} N(1)$. Here M_8 is a negative definite piece (the Milnor fiber) which is the plumbing on the $-E_8$ configuration of 8 copies of the cotangent bundle of S^2, and $N(1)$ is a Gompf nucleus [G] which is the neighborhood of a cusp fiber and a section which has square -1 . Note that M_8 contains eight spheres of square -2, which we will denote $S_1, \cdots, S_8$. Since $E(1)$ has a Kähler metric of positive scalar curvature, the Seiberg-Witten invariants computed with this metric will be 0. Since $2e + 3\sigma = 0$ means that any possible basic class for another metric will have $K \cdot K \geq 0$, there are no walls involved since $K \cdot \omega_g$ is never zero (the orthogonal complement of ω_g has negative square). Thus the Seiberg-Witten invariant is independent of the metric and must vanish. One interesting aspect of this is that Taubes' theorem gives $SW^-(-K) \neq 0$. However, $-K \cdot \omega > 0$ and so it will be the case that $-K \cdot \omega_g > 0$ for all metrics.

The other piece used in constructing M is a manifold which was introduced by Thurston [Th] as an example of a symplectic manifold which is not Kähler. We will denote it by Th. Here one starts with a diffeomorphism ϕ of a torus which induces the map on $\pi_1 = H_1$ sending a_1 to $a_1 + a_2$ and a_2 to a_2 and has a fixed point. Then Th is a torus bundle over a torus which is just a product $T_\phi \times S^1$, where T_ϕ denotes the mapping torus. This has fundamental group given by $< a_1, a_2, \mu, \tau : [a_1, a_2] = 1, \mu^{-1} a_i \mu = \phi_\# a_i, [x, \tau] = 1 >$. The generators a_i come from the torus fiber T, the generator μ comes from the section of T_ϕ using the fixed point, and the generator τ comes from the last S^1 factor in the base. Thurston shows that this has a symplectic structure so that the section $T_1 = \mu \times S^1$ is symplectically embedded. Computing $H_1(Th)$ shows that $a_1 = a_1 + a_2$, so $a_2 = 0$. In fact the loop given by a_2 bounds a punctured torus in the mapping torus. Thus the first homology group is $3\mathbb{Z}$ with generators a_1, μ, τ. Since Kähler surfaces have even first Betti number, Th is an example of a symplectic manifold which is not Kähler. The second homology group can then be computed to be $4\mathbb{Z}$ with two factors coming from $T_2 = a_1 \times S^1, T_1 = \mu \times S^1$ and the other two coming from the fiber T of T_ϕ (which is also the fiber of the fibration over the torus) and one more class which is homologically $\mu \times a_2$ and is given by the torus T_3 formed by restriction of the mapping torus T_ϕ to the circle $(a_2)_\phi$ after making a slight adjustment. Since all the generators are represented by tori of square 0, the Seiberg-Witten invariants of Th must also vanish by Remark 3. But Taubes has shown that $\pm K$, where K is the canonical class, must have nonvanishing Seiberg-Witten invariant, so we conclude that the canonical class for the Thurston manifold is the zero class.

We form the symplectic connected sum $E(1)\#_T Th$ by removing neighborhoods $D^2 \times T$ of the fiber $F \subset E(1)$ and the section $T_1 = \mu \times S^1 \subset Th$ and gluing their boundary neighborhoods. Parallel copies of F and T_1 will then be identified in M as a symplectically embedded torus, which we will still denote by T_1. The canonical class of the symplectic connected sum satisfies $K_M = K_{E(1)} + K_{Th} + 2T_1 = T_1$. By Taubes [T1], $\pm T_1$ will then be basic classes for M. Szabo then claims that the adjunction inequality will imply that there are no other basic classes. To see this, we use the Mayer-Vietoris sequence to compute a basis for $H_2(M)$:

$$0 \to H_2(\partial) \to H_2(E(1) \setminus N(F)) \oplus H_2(Th \setminus N(T_1)) \to H_2(M) \to \mathbb{Z} \to 0$$

which is

$$0 \to 3\mathbb{Z} \to 11\mathbb{Z} \oplus 3\mathbb{Z} \to H_2(M) \to \mathbb{Z} \to 0$$

There is a basis for $H_2(M) = 12\mathbb{Z}$ given by $S_1, \cdots, S_8, T_1, C, T_2, T_3$. The class C is formed from a punctured section of the fibration for $E(1)$ and a punctured torus fiber of the fibration for Th. The intersection matrix is given by $-E_8 \oplus \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Applying the adjunction inequality for any basic class K to T_i gives $K \cdot T_i = 0$. Applying it to C (using the double torus of square 1 formed by tubing T_1 and C together at their intersection point) gives $K \cdot C = \pm 1$. This means that a basic class must be of the form $\pm T_1 + \sum_{i=1}^{8} a_i S_i$ with a_i even. To rule out the S_i terms, we use a different argument. When we restrict to the M_8 submanifold with boundary $\Sigma(2,3,5)$, the bundle would restrict to the bundle with class $\sum_{i=1}^{8} a_i S_i$ and would be trivial on the boundary. This is a submanifold of the $K3$ surface, which only supports the 0 solution to the Seiberg-Witten equations. Using the fact that $\Sigma(2,3,5)$ is a manifold with positive scalar curvature, we see that when we stretch the neck within $K3$, solutions will tend to the trivial flat connection on the ends. This will allow us to build a solution on the $K3$ surface out of a solution over M_8 and the zero solution on the rest of $K3$ with *spin*^{c} structure corresponding to the class $\sum_{i=1}^{8} a_i S_i$, which is impossible. Thus the only Seiberg-Witten basic classes for M are $\pm T_1$.

For a somewhat more complicated example of using the adjunction inequality to determine the basic classes, see the recent paper of Fintushel and Stern [FS3].

10.2. **Minimal genus estimates in Dolgachev surfaces.** We now focus on the Dolgachev surface $D(p,q)$ which is formed from the rational elliptic surface $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ by performing logarithmic transformations of relatively prime orders p, q on the fiber. The original elliptic fibration $E(1) \to \mathbb{CP}^1$ arises by starting with a pencil $af_0 + bf_1$ of cubic curves (topologically the generic embedded curves are tori by the adjunction formula: $2g - 2 = -3(3) + 3(3) = 0$) in $\mathbb{CP}^2$ which pass through 9 points. Here $[a, b] \in \mathbb{CP}^1$ parametrize the cubics. To remove the intersections we blow up at each of the 9 points. Note that each of the blown up $\overline{\mathbb{CP}^2}$s represents a section of the elliptic fibration as each point of $\overline{\mathbb{CP}^2}$ corresponds to a direction that one of the elements of the original pencil parametrized by $[a, b]$ is passing though one of 9 points. Thus the fibration has 9 sections which represent homologically the classes E_i given by the exceptional fibers. A fiber itself originally came from the class $3H \in H_2(\mathbb{CP}^2)$ and the blowing up will make the tori pass through each of the exceptional fibers transversally in one positive intersection point. This means that the class represented by the fiber is $(3, -1, \cdots, , -1)$ in terms of the basis $H, E_1, \cdots, E_9$. The fibration can be chosen so that it has all regular fibers except for 6 cusp fibers which are topologically spheres with a single singular point where the two generating circles of a torus are pinched to a point.

Gompf [G] shows that the nucleus $N(1)$ contains all of the vital information of the surface up to diffeomorphism. If we call the fiber class $f = (3, -1, \cdots, -1)$ and use the section $s = (0, 1, 0, \cdots, 0)$, then there is a decomposition of the quadratic form as $\{f, s\} \oplus \{f, s\}^{\perp}$. The quadratic form of the first part is $Q(N(1)) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, and the second is $-E_8$.

When we perform the two log transforms on $E(1)$ to form $E(1; p, q)$ this can be done within the nucleus to give a decomposition $E(1; p, q) = M_8 \cup N(1; p, q)$ and a principal idea from [G] is that the diffeomorphism type of the result is completely determined by $N(1; p, q)$. The intersection form is unchanged by the operation of log transform, and the existence of the section is used to show that the result is still simply connected if p, q are relatively prime. Freedman's classification of topological 4-manifolds [F] implies that there is a homeomorphism of $N(1; p, q)$ to $N(1)$ which extends to a homeomorphism from $E(1; p, q)$ to $E(1)$. In $N(1; p, q)$ there are two multiple fibers F_p, F_q and homologically the original fiber $f = pF_p = qF_q$. There is a new elliptic fibration $E(1; p, q) \to \mathbb{CP}^1$, which is a Kähler surface with canonical class $-f + (p-1)F_p + (q-1)F_q$ [FM]. Each of the classes f, F_p, F_q is a multiple of a primitive class g which is a linear combination $aF_p + bF_q$ with $aq + bp = 1$. In terms of g, the canonical class is $K_{p,q} = (-pq + (p-1)q + (q-1)p)g = ((p-1)(q-1) - 1)g = n(p, q)g$.

The homeomorphism from $N(1; p, q) \to N(1)$ must send g to a class $cf + ds$ with square 0. The only possibilities are $\pm f, \pm(f + 2s)$. However, since s is represented by a sphere with square -1 there is a diffeomorphism of $N(1)$ given by reflection through s which sends f to $f + 2s$ – on homology $R(\xi) = \xi + 2(\xi \cdot s)s$. Hence we may assume that g is mapped to $\pm f$. By a further diffeomorphism of $E(1)$, we can identify $E(1; p, q)$ up to homeomorphism with $E(1)$ so that the class g gets identified with $f = (3, -1, \cdots, -1)$. Hence the canonical class $K_{p,q}$ gets identified with the class $n(p, q)f$. We use this homeomorphism to identify elements of the second homology of $E(1; p, q)$ with that of $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$.

By Donaldson or Seiberg-Witten invariant calculations, $E(1, p, q), p, q \geq 2$ is known not to possess any embedded spheres of nonnegative square. The fibers f, F_p, F_q are all represented by embedded tori of square 0. We want to look at classes of positive square and examine what the adjunction inequality implies about them. For Σ a class of positive square, we identify Σ with a homology class in $H_2(\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2})$ as $\Sigma = \Sigma_0 + af + bs$. The canonical class is then identified with $n(p, q)f$. Then the adjunction inequality says that as long as $\Sigma \cdot \omega > 0$,

$$2g - 2 \geq K \cdot \Sigma + \Sigma \cdot \Sigma = n(p, q)b + \Sigma \cdot \Sigma$$

Now any class of positive square has to have a non-zero component of s since otherwise the square would be less than or equal to zero. The condition $\Sigma \cdot \omega > 0$ guarantees that $K \cdot \Sigma \geq 0$, so b is positive. The condition of positive square then requires a to be positive as well. Thus we see that the minimal genus of any class $\Sigma = \Sigma_0 + af + bs, a > 0, b > 0$ of positive square must be greater than or equal to

$$g(\Sigma) = 1 + \frac{1}{2}(n(p, q)b + \Sigma \cdot \Sigma).$$

Note that $\Sigma \cdot \Sigma = (2a-b)b - |\Sigma_0 \cdot \Sigma_0|$ and so we must have $2a > b$ to have a class of positive square. Note also that the component Σ_0 only enters into the estimate through $\Sigma \cdot \Sigma$.

If we just look at classes of square 1, then this says that the minimal genus is

$$g \geq 1 + \frac{1}{2}(n(p,q)b + 1)$$

One interesting case to compare to is the class which was originally represented by a sphere of square 1 in $E(1)$. We first have to rewrite this class in terms of our new basis to determine a, b. Doing this gives $a = b = 3$. Thus the estimate is $g \geq 1 + \frac{1}{2}(3n(p,q)+1)$. Note that any other class of square 1 will have to have $b \geq 1$ and so the minimal genus will always have to be greater than equal to $1 + \frac{1}{2}(n(p,q)+1)$.

11. Final comments

In [L] there were some conjectures which have since been solved which we want to comment upon. Conjecture 3 says that a necessary condition for a divisible class to be smoothly representable by a sphere is that the underlying primitive class of which it is a multiple is smoothly representable by a sphere. It is related to the Thom-Kronheimer-Mrowka conjecture in the case that the square is nonnegative. For if Σ has minimal genus $g > 0$ and the minimal genus for $m\Sigma$ is given by the conjecture, then $m\Sigma$ would have minimal genus equal to $m^2n + m(2g-2-n) \geq m^2n - mn = mn(m-1)$ which is greater than 0 for $m \geq 2$. Thus whenever the Thom- Kronheimer-Mrowka conjecture holds this conjecture will hold as well. One situation where this applies is when we are dealing with classes of nonnegative square where the minimal genus is governed by the generalized adjunction inequality and we have equality for each primitive class. An example would be the $K3$ surface. The key idea needed is that the formula

$$b_\kappa(\xi) = 2g(\xi) - 2 - \xi \cdot \xi - \kappa \cdot \xi$$

satisfies $b_\kappa(m\xi) = mb_\kappa(\xi) = 0$ whenever $b_\kappa(\xi) = 0$ and the adjunction inequality says $b_\tau(\xi) \geq 0$ whenever κ is a basic class.

As for Conjecture 4, the last statement about bounds on the squares of characteristic classes which can be smoothly represented by spheres in almost definite 4-manifolds is strongly true in that our application of Furuta's theorem showed that in fact we must have the square equal to the signature. For general manifolds, the Rochlin genus inequality will constrain the square in terms of the signature, the genus of a representative, and the second Betti number.

Although we have surveyed here a large number of advances on the minimal genus problem, there remains much work to be done. The adjunction inequality (in a variety of forms) is the most powerful tool in getting bounds on the minimal genus, and we have given a few examples where it has been applied quite successfully. We have shown how it allows the complete determination of the minimal genus for classes of nonnegative square in the $K3$ surface, rational surfaces X_m with $m \leq 6$, and symplectically represented classes in symplectic manifolds. However, there still should be much more information available for other classes of 4-manifolds. For example, the adjunction inequality completely determines the minimal genus for the $K3$ surface, but little has been done for homotopy $K3$s, of which there are many families that have been studied thoroughly in other ways. Morgan and Szabo [MSz] have recently shown that homology $K3$s must have $SW(0) \neq 0$, and so must satisfy

$$g(\Sigma) \geq 1 + \frac{\Sigma \cdot \Sigma}{2}$$

for classes of positive square, as does $K3$. However, most known families of homotopy $K3$ surfaces have other basic classes and satisfy stronger bounds than this. In fact, it is quite plausible that the $K3$ surface is distinguished among all homotopy $K3$s for having the minimal genus given exactly by the above formula.

There also remains the question of finding how strictly the bounds given by the adjunction inequality hold. Li and Li have recently given an example where the adjunction inequality doesn't determine the minimal genus. It uses the product of two surfaces of genus > 1 and a homology class using sums of products of one dimensional classes from the two surfaces. One particularly promising area of study is how to combine results on the action of the diffeomorphism group with adjunction inequality results.

Another area which needs further exploration is restrictions on the minimal genus for classes of negative square. There are some results due to Fintushel and Stern that were discussed in Section 6 which deal ultimately with embedded spheres of negative self intersection in terms of Seiberg-Witten invariants. They can be used to give some restrictions on embedded spheres, but say nothing about surfaces of other genus. The results using versions of Furuta's theorem also apply to classes of negative square, but they apply mainly to characteristic classes. One interesting question is what the bounds are on the square of a class represented by a surface of genus g. A simple case is the question of how negative the square can be in $K3$ and still have a representative by a sphere. Mikhalkin has found classes of square -86 in $K3$ which are represented by an embedded sphere. It is conjectured that there should be a lower bound on the square of any sphere in $K3$. More generally, there should be a lower bound on the square possible for a given genus in $K3$. Since the diffeomorphism group for $K3$ acts transitively on classes of given square and type, the problem can be rexpressed by getting a bound on the minimal genus expressed in terms of the self intersection number. Note that for characteristic classes we are able to get such bounds coming from Rochlin's genus theorem and applications of variations of Furuta's theorem.

Another area where there is much potential for further progress is the development of constructive techniques. Here the work of Li and Li [Li] has provided some useful techniques. Also, Mikhalkin has successfully used constructive techniques of real algebraic geometry in attacking the minimal genus problem. Hopefully, constructive techniques can help us to understand how tight are the bounds given by the generalized adjunction inequality for non-negative classes. They should also be useful in studying how various forms of stabilization and other geometric constructions such as fiber sum affect the minimal genus.

There is still little known about the case of definite 4-manifolds other than $\mathbb{CP}^2$. The simplest case which is still unresolved in the class $(3,2) \in H_2(2\mathbb{CP}^2)$. This class is represented by an embedded torus, but it is unknown whether it is represented by an embedded sphere. Seiberg-Witten invariants don't apply here, as they all vanish, and techniques such as using Furuta's theorem applied to the orbifold don't help since the orbifold is still definite. Moreover, once one stabilizes twice by taking connected sum with $2\overline{\mathbb{CP}^2}$, results of Wall can be used to show that the resulting class (in any of 3 different forms of stabilization) will be represented by an embedded sphere. What happens with one stabilization is connected to the question above about classes of negative square in X_2. One approach which can be used for divisible classes is to look at branched covers to which one can apply one of the other techniques. For example, the minimal genus of the class $(6,2)$ in $H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ can be shown to be 10 using Bryan's Theorem 46 [B2]. This particular example can also be handled by noting that if there were a genus 9 representative, the double branched cover would be a homology $K3$ surface, and the covering genus 9 surface would violate the adjunction inequality which holds using the result of [MSz] that $SW(0) \neq 0$ for a homology $K3$.

Finally, we close with some speculation concerning the role the genus bounds in determining a 4-manifold up to diffeomorphism. We restrict our attention to simply connected 4-manifolds. First, there is some evidence that knowing the minimal genus for all homology classes of non-negative square determines completely all of the nontrivial Seiberg-Witten classes. A more adventurous conjecture is that it would determine the manifold up to diffeomorphism. This would say that a homeomorphism $h : X \to Y$ so that $h(\xi)$ and ξ have the same minimal genus smooth representative for all 2-dimensional homology classes ξ would be homotopic to a diffeomorphism, at least up to connected sum with a homotopy 4-sphere. More likely, however, one would have to enlarge the knowledge of the minimal genus to configuations of pairs of classes where we have knowledge not only about the minimal genus representatives of a given class but also about the minimal number of intersections between representatives of different classes. Enlarging the information to configurations of classes itself is a promising area for applications of the techniques which we have presented in this paper.

References

[Ac] Acosta, D., personal communication.

[A] Akbulut, S., *Lectures on Seiberg-Witten Invariants*, Proceedings of Gökova Geometry-Topology Conference 1995, 95-118.

[At-Bo] M. Atiyah, R. Bott, *A Lefschetz fixed point formula for elliptic complexes: II. Applications*, Ann. of Math. (2) 88 1968 451–491.

[B1] Bryan, J., *Seiberg-Witten à la Furuta and genus bounds for classes with divisibility*, Turkish J. Math. (to appear)

[B2] Bryan, J., *Extending Furuta's technique in the presence of symmetry and an application to new genus bounds*, MSRI talk, January, 1997.

[D1] Donaldson, S., *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Diff. Geom. **26** (1987), 397–428.

[D2] Donaldson, S., *The Seiberg-Witten equations and 4-manifold topology*, Bull. Amer. Math. Soc. **33** (1996), 45–70.

[Fi] Fintushel, R., *private communication*

[FS1] Fintushel, R., and Stern, R., *Immersed Spheres in 4-Manifolds and the Immersed Thom Conjecture*, Turkish J. Math. 19 (1995), 145–157.

[FS2] Fintushel, R., and Stern, R., Rational blowdowns of smooth 4-manifolds, Jour. Diff. Geom. (to appear).

[FS3] Fintushel, R., and Stern, R., Knots, links, and 4-manifolds, preprint.

[F] Freedman, M., The topology of four-dimensional manifolds. J. Differential Geom. 17 (1982), no. 3, 357–453

[FM] Friedman, R., and Morgan, J., *Algebraic Surfaces and Seiberg-Witten invariants*, preprint.

[Fu] Furuta, M., *Monopole Equation and the 11/8 Conjecture*, preprint.

[G] Gompf, R., *Nuclei of elliptic surfaces*, Topology 30 (1991), no. 3, 479–511.

[Ga] Gao, H., Representing homology classes of almost definite 4-manifolds, Topology and its Appl. 52 (1993), 109–120.

[GG] Gan, D.Y., and Guo, J. H., Embeddings and immersions of a 2-sphere in 4-manifolds, Proc. A. M. S. 118 (1993), 1323-1330.

[K1] Kikuchi, K., *Representing positive homology classes of* $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ *and* $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$, Proc. A.M.S. 117 (1993), 861–869.

[K2] Kikuchi, K., *Positive 2-spheres in 4-manifolds of signature* $(1, n)$, Pacific J. Math. 160 (1993), 245–258.

[K] Kotschick, D., *Orientations and geometrisations of compact complex surfaces*, Bull. Lond. Math. Soc. (to appear)

[KoM] Kotschick, D., and Matic, G., *Embedded surfaces in four-manifolds, branched covers, and* $SO(3)$*-invariants*, Math. Proc. Cambridge Philos. Soc. 117 (1995), 275–286.

[KM1] Kronheimer, P.B. and Mrowka, T.S., *The Genus of Embedded Surfaces in the Projective Plane*, Math. Res. Letters **1** (1994), 797-808.

[KM2] Kronheimer, P.B., and Mrowka, T.S., *Gauge theory for embedded surfaces I,II*, Topology **32** (1993), 773–826; Topology **34** (1995), 37–97.

[KM3] Kronheimer, P.B., and Mrowka, T.S., *Embedded surfaces and the structure of Donaldson polynomial invariants*, Jour. Diff. Geom. **41** (1995), 573–734.

[LM] Lawson, H.B., and Michelson, M.-L., *Spin Geometry*, Princeton University Press, Princeton, N.J., 1989.

[L] Lawson, T., *Smooth Embeddings of 2-Spheres in 4-Manifolds*, Expos. Math. **10** (1992), 289–309.

[LW1] Lee, R., and Wilczynski, D., *Locally flat 2-spheres in 4-manifolds*, Comment. Math. Helv. 65 (1990), 388-412; Correction, Ibid. **67** (1992), 334–335.

[LW2] Lee, R., and Wilczynski, D., *Representing homology classes by locally flat 2-spheres*, K-Theory **7** (1993), 33-367.

[LW3] Lee, R., and Wilczynski, D., *Representing homology classes by locally flat surfaces of minimum genus*, Amer. J. Math. (to appear).

[LW4] Lee, R., and Wilczynski, D., *Genus inequalities and four-dimensional surgery*, preprint.

[Li] Li, B.H., *Embeddings of surfaces in* 4*-manifolds (II)*, Chinese Science Bulletin **36** (1991), 2030–2033.

[LL1] Li, B.H., and Li,T.-J., *Minimal genus embeddings in* $S^2 \times S^2$ *and* $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ *with* $n \leq 8$, preprint.

[LL2] Li, B.H., and Li,T.-J., *Minimal genus smooth embeddings in* S^2*-bundles over surfaces*, preprint.

[LLu1] Li, T.-J., and Liu, A., *Symplectic Structure on Ruled Surfaces and a Generalized Adjunction Formula*, Math. Res. Lett. **2** (1995), 453–471.

[LLu2] , Li, T.-J., and Liu, A., *General Wall Crossing Formula*, Math. Res. Lett. 2 (1995), 797–810.

[MS] McDuff, D., and Salamon, D., *A survey of symplectic 4-manifolds with* $b^+ = 1$, Proceedings of Gökova Geometry-Topology Conference 1995, 47–60.

[M1] Mikhalkin, G., *Surfaces in the neighborhoods of other surfaces in smooth 4-manifolds*, Turkish J. Math. 19 (1995), no. 2, 201–206..

[M2] Mikhalkin, G., *J-holomorphic curves in almost complex surfaces do not always minimize the genus*, Proc. A. M. S. (to appear).

[M] Morgan, J., *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*, Princeton Mathematical Notes **44**, Princeton University Press, 1996.

[MSz] Morgan, J., and Szabo, Z., Homotopy $K3$ surfaces and mod 2 Seiberg-Witten invariants, preprint.

[MST] Morgan, J., Szabo, Z., and Taubes, C., A Product Formula for the Seiberg-Witten Invariants and the Generalized Thom Conjecture, preprint.

[R] Ruberman, D., *The minimal genus of an embedded surface of non-negative square in a rational surface*, Proceedings of Gökova Geometry-Topology Conference 1995, 129–133.

[Rub] Ruberman, D., *Smooth 2-spheres in homology K3 surfaces*, Topology Appl. 59 (1994), no. 1, 87–99

[S] Salamon, D., *Spin Geometry and Seiberg-Witten invariants*, preprint.

[Sz] Szabo, Z., *Exotic 4-manifolds with* $b_2^+ = 1$, preprint.

[T1] Taubes, C., *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Letters **1** (1994), 809–822.

[T2] Taubes, C., *More constraints on symplectic forms from the Seiberg-Witten equations*, Math. Res. Letters **2** (1995), 9-14.

[T3] Taubes, C., *The Seiberg-Witten and the Gromov invariants*, Math. Research Letters **2** (1995), 221-238.

[Th] Thurston, W., *Some simple examples of symplectic manifolds*, Proc. A.M.S. 55 (1976), 467–468.

[W] Witten, E., *Monopoles and 4-manifolds*, Math. Res. Letters **1** (1994), 760–796.

[Y] Yasuhara,A., *Connecting lemmas and representing homology classes of simply connected 4-manifolds*, Tokyo J. Math. 19 (1996), 245–261.

Department of Mathematics
Tulane University
New Orleans, LA 70118
E-mail address: `tcl@math.tulane.edu`