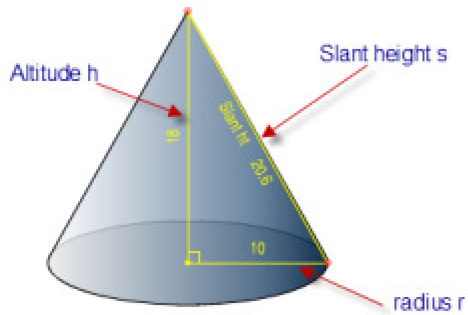
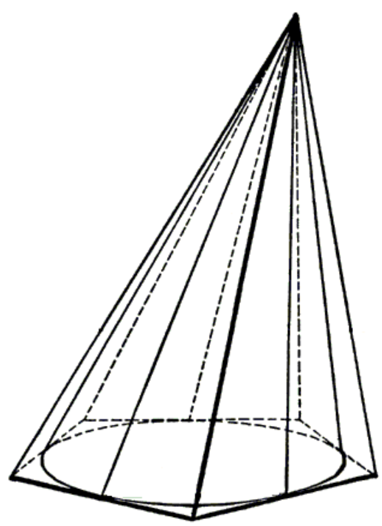
The lateral surface area of a cone

The purpose here is to explain the standard formula for the lateral (upper) area of a standard right circular cone. The important dimensions of the cone are the height **h,** the radius **r** and the slant height **s,** all of which are indicated in the figure below**:**



(***Source:*** **http://www.mathopenref.com/images/coneslantheight/8-5-2011%205-04-57%20PM.png**)

We shall explain why *the lateral area of such a cone is equal to the product* **πrs.** This derivation is different from the one appearing in the course text**,** and it is based upon the standard definition of surface area in multivariable calculus texts**.** According to that definition**,** one can find the surface area using pyramids circumscribed about the cone. As in the drawing below**,** the base of such a pyramid is a polygon whose sides are tangent to the cone's base circle**,** and whose summit vertex is the summit vertex of the cone**.**

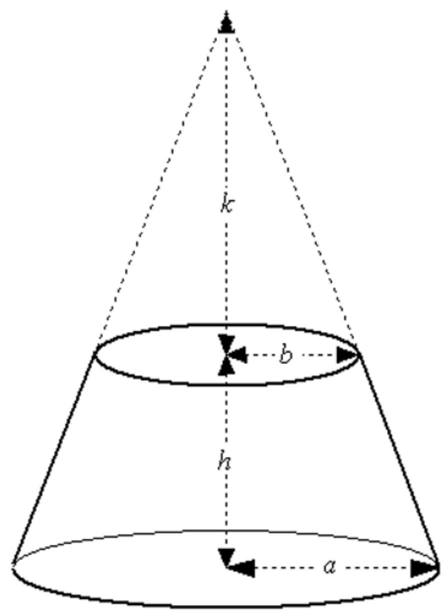


(***Source:*** **http://etc.usf.edu/clipart/42200/42235/conepoly_42235_lg.gif**)

Note that the altitudes of the lateral triangles agree with the slant heights of the cone**.** If we assume that the base polygon of the pyramid is a regular **n –** gon and its perimeter is equal to $\mathbf{P_n}$**,** then it follows that the lateral area of the pyramid**,** which is the sum of the areas of the lateral triangular faces**,** is equal to $\mathbf{½P_n s}$**,** where **s** is the slant height**.** The multivariable calculus definition of surface area implies that the lateral surface area of the cone is the limit of the lateral surface areas of the pyramids with regular polygonal bases. As **n** approaches infinity, the limit of the perimeters $\mathbf{P_n}$ is equal to the circumference of the circle, which is $\mathbf{2\pi r}$. Therefore the limit of the lateral areas as **n** approaches infinity is equal to $\mathbf{\frac{1}{2}(2\pi r)s} = \mathbf{\pi rs}$, which is what we wanted to prove.

The lateral area of a frustrum of a cone

We are going to need a consequence of the area for the lateral area of the solid formed from a cone by cutting off the top at some fixed altitude. Such an object is called a ***frustrum of a cone***. In the drawing below we start with a right circular cone of altitude **h** and chop off a right circular cone of altitude **k** from the top. The radii of the upper and lower boundary disks are denoted by **b** and **a** respectively.



http://mordochai.tripod.com/graphics/kiyyors7.gif

Obviously the lateral area of the frustrum is just the difference between the areas of the larger and smaller cones. The derivation in the text shows that this difference is equal to

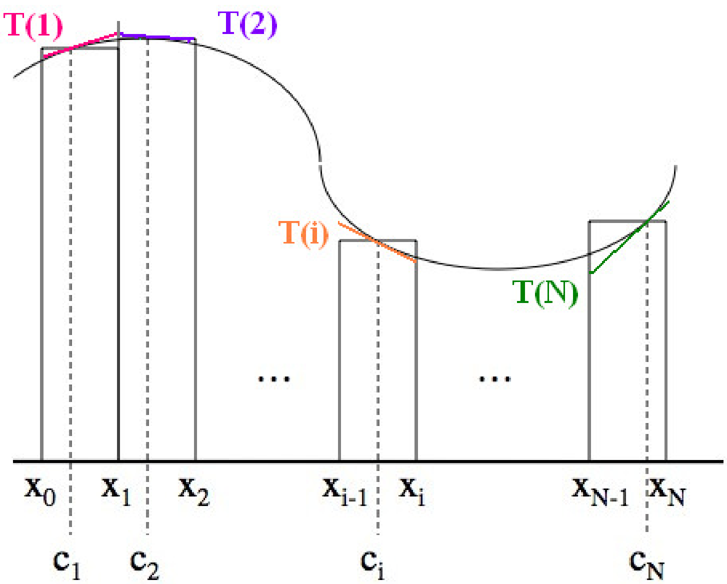
$$2\pi\left(\frac{a+b}{2}\right)\cdot\sqrt{(a-b)^2+h^2} \quad = \quad \pi(a+b)\sqrt{(a-b)^2+h^2}$$

and hence this gives the lateral area of the frustrum in the drawing.

We must now apply this to analyze the surface area of a surface of revolution generated by a graph curve **y** = **f(x)**, where **x** ranges between **a** and **b**; we need to assume that **f** is always positive valued. As usual, when trying to derive an integral formula for some quantity, one key step is to slice the surface into thin pieces and find reasonable first order approximations to the surface area of these pieces. The precise method of doing this is slightly complicated, and we shall illustrate it using a variant of the drawing in

http://www.askamathematician.com/2011/04/q-why-is-the-integralantiderivative-the-area-under-a-function/

which was also used earlier for Section 7.2.



Specifically, over the interval from $\mathbf{x_{i-1}}$ to $\mathbf{x_i}$ we approximate the curve by the tangent line **T(i)** to the graph for some value $\mathbf{C_i}$ between $\mathbf{x_{i-1}}$ and $\mathbf{x_i}$. Let **S(i)** denote the closed segment on **T(i)** which lies over the given interval, and let **L(i)** be the length of this segment. Over each interval we shall approximate the original surface by the surface of revolution given by the curve **S(i).** This is not quite the same as the approximation described and illustrated on page 225 of the text, but our approximations and the texts' look very similar. We have taken a slightly different approach because it is consistent with the general approach to surface area in multivariable calculus courses.

We must now approximate the surface area of the piece of the surface of revolution over the given interval, and as one might expect we do this using the surface area for the previously defined frustrum of a cone. The upper and lower radii of the latter are given by $\mathbf{f(x_{i-1})}$ and $\mathbf{f(x_i)}$; in any given case we do not know which of these radii is the larger, but this does not really matter in the lateral area formula. By the preceding discussions the area of the frustrum is equal to

$$\pi(y_i + y_{i-1})L(i) \quad = \quad \pi(y_i + y_{i-1})\sqrt{(y_i - y_{i-1})^2 + (x_i - x_{i-1})^2}$$

where $\mathbf{(x_{i-1}, y_{i-1})}$ and $\mathbf{(x_i, y_i)}$ are the endpoints of the segment **S(i).** Since the slope of the tangent line is equal to $\mathbf{f'(C_i)}$, it follows that

$$y_i - y_{i-1} \quad = \quad f'(C_i)\cdot(x_i - x_{i-1})$$

where $\mathbf{(x_{i-1}, y_{i-1})}$ and $\mathbf{(x_i, y_i)}$ are the endpoints of the segment **S(i).** We can use this to rewrite the lateral surface area of the frustrum as

$$\pi\left(2y_{i-1} + f'(C_i)\Delta x_i\right)\cdot\Delta x_i\sqrt{1 + f'(C_i)^2}$$

where $\Delta x_i = x_i - x_{i-1}$, and if we add these terms, then a first order approximation to the entire area is given by the (Riemann) sum

$$\sum_i 2\pi y_{i-1} \cdot \sqrt{1 + f'(C_i)^2} \cdot \Delta x_i$$

where "first order" means that we ignore terms involving higher powers of Δx_i. Finally, as usual, the limit of these terms as the Δx_i go to zero is equal to the surface area and also to the definite integral

$$2\pi \int_a^b f(x) \cdot \sqrt{1 + f'(x)^2}\, dx$$

which is precisely the formula in the text. Also, as noted in the text there is a similar formula for the area of a surface of revolution obtained by rotating the graph curve about the **y** – axis:

$$2\pi \int_a^b x \cdot \sqrt{1 + f'(x)^2}\, dx$$

Why second – order terms can be disregarded. For the sake of completeness, there is a file in the course directory which explains why the second order terms of Riemann – like sums do not affect the limiting values:

http://math.ucr.edu/~res/math009B-2012/second-order.pdf

The mathematical level of this discussion is that of an introductory undergraduate real variables course like Mathematics 151A; in this course it is enough simply to know that "second order terms involving Δx (in other words, squares of the latter) don't matter when we take limits."