Orthogonal transformations and cross products

One of the equivalent definitions of orthogonal linear transformations on $\mathbb{R}^n$ and unitary linear transformations on $\mathbb{C}^n$ is that they preserve the real or complex inner product. In some situations it is useful to know how the standard vector cross product on $\mathbb{R}^3$ behaves with respect to orthogonal transformations. The answer is given by the following result:

THEOREM. *Let A be a 3×3 orthogonal matrix with real entries. Then for all $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^3$ we have*

$$A\mathbf{x} \ \times \ A\mathbf{y} \quad = \quad (\det A)\big(\mathbf{x} \ \times \ \mathbf{y}\big) \ .$$

In particular, if $\det A = 1$ *then A preserves the cross product.*

We need the following basic observation:

LEMMA. *Let $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^3$. Then $\mathbf{x} = \mathbf{y}$ if and only if for all $\mathbf{z} \in \mathbb{R}^3$ we have $\langle \mathbf{x},\, \mathbf{z}\rangle = \langle \mathbf{y},\, \mathbf{z}\rangle$.*

This follows from the basic identity $\mathbf{w} \ = \ \sum \langle \mathbf{w},\, \mathbf{e}_i\rangle\, \mathbf{e}_i$, where $\{\, \mathbf{e}_i \,\}$ denotes the standard basis of unit vectors.■

Proof of the theorem. We shall use the lemma and show that

$$\langle A\mathbf{x} \ \times \ A\mathbf{y},\, \mathbf{z}\rangle \quad = \quad (\det A) \cdot \langle \mathbf{x} \times \mathbf{y},\, \mathbf{z}\rangle$$

for all $\mathbf{z}$.

Let $\mathbf{z} \in \mathbb{R}^3$ be arbitrary. Since A is onto we may write $\mathbf{z} = A\mathbf{z}'$ for some $\mathbf{z}'$, so that

$$\langle A\mathbf{x} \ \times \ A\mathbf{y},\, \mathbf{z}\rangle \quad = \quad \langle A\mathbf{x} \ \times \ A\mathbf{y},\, A\mathbf{z}'\rangle \quad = \quad \big[A\mathbf{x}, A\mathbf{y}, A\mathbf{z}'\big]$$

where the right hand side is the determinant of the matrix whose rows or columns are given by the three vectors inside the bracket (taken in order). Now the matrix in question is just the product of A with the matrix whose rows or columns in order are $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}'$, and therefore the product rule for determinants yields the identity

$$\big[A\mathbf{x}, A\mathbf{y}, A\mathbf{z}'\big] \quad = \quad \det(A) \cdot \big[\mathbf{x}, \mathbf{y}, \mathbf{z}'\big] \quad = \quad \det(A) \cdot \langle \mathbf{x} \times \mathbf{y},\, \mathbf{z}'\rangle \ .$$

Since orthogonal matrices preserve dot products, the latter is equal to

$$\det(A) \cdot \langle\, A(\mathbf{x} \times \mathbf{y}),\, A\mathbf{z}'\rangle \quad = \quad \det(A) \cdot \langle\, A(\mathbf{x} \times \mathbf{y}),\, \mathbf{z}\rangle \ .$$

Combining these, we have that

$$\langle A\mathbf{x} \ \times \ A\mathbf{y},\, \mathbf{z}\rangle \quad = \quad (\det A) \cdot \langle \mathbf{x} \ \times \ \mathbf{y},\, \mathbf{z}\rangle$$

for all $\mathbf{z}$, and as noted above the latter suffices to prove the cross product identity in the theorem.■