Affine transformations and convexity

The purpose of this document is to prove some basic properties of affine transformations involving convex sets. Here are a few online references for background information:

http://math.ucr.edu/~res/progeom/pgnotes02.pdf

http://math.ucr.edu/~res/math133/metgeom.pdf

Recall that an affine transformation of $\mathbb{R}^n$ is a map of the form $F(x) = b + A(x)$, where $b \in E$ is some fixed vector and $A$ is an invertible linear transformation of $\mathbb{R}^n$.

Affine transformations satisfy a weak analog of the basic identities which characterize linear transformations.

**Lemma 1.** Let $F$ as above be an affine transformation, let $x_0, \cdots, x_k \in \mathbb{R}^n$, and suppose that $t_0, \cdots, t_k \in \mathbb{R}$ satisfy $\sum_j t_j = 1$. Then

$$F \left( \sum_j t_j x_j \right) = \sum_j t_j F(x_j).$$

**Notation.** If $t_0, \cdots, t_k \in \mathbb{R}$ satisfy $\sum_j t_j = 1$ and $x_0, \cdots, x_k \in \mathbb{R}^n$, then $\sum_j t_j x_j$ is said to be an affine combination of the vectors $x_0, \cdots, x_k \in \mathbb{R}^n$.

**Proof.** Since $\sum_j t_j = 1$ we have

$$F \left( \sum_j t_j x_j \right) = A \left( \sum_j t_j x_j \right) + b = A \left( \sum_j t_j x_j \right) + \sum_j t_j b =$$

$$\sum_j t_j A x_j + \sum_j t_j b = \sum_j t_j (A x_j + b) = \sum_j t_j F(x_j)$$

which is what we wanted prove.

We also note the following simple property of affine transformations in $\mathbb{R}^2$:

**Lemma 2.** Let $F$ be an affine transformation of $\mathbb{R}^2$, and let $x, y, z, w$ be points such that the lines $xy$ and $zw$ are parallel. Then the lines $F(x)F(y)$ and $F(z)F(w)$ are also parallel.

**Proof.** Since the two lines are disjoint and $F$ is 1–1, it follows that their images — which are also lines because $F$ is an affine transformation — must also be disjoint.

**Convex Sets.** Here are the basic definitions we need for convexity:

**Definition.** If $x, y \in \mathbb{R}^n$, then the closed segment $[xy]$ is the set of all vectors $v$ such that

$$v = t x + (1-t) y$$

where $t \in \mathbb{R}$ satisfies $0 < t < 1$.

This corresponds to the intuitive notion of closed line segment in elementary geometry.

**Definition.** A subset $K \subset \mathbb{R}^n$ is said to be convex if $x, y \in K$ implies that $[xy]$ is contained in $K$; in other words, $x, y \in K$ and $0 \leq t \leq 1$ implies that $t x + (1-t) y \in K$. 
The following result suggests that the notions of convexity and affine transformation have some useful interrelationships.

**LEMMA 3.** Let \( K \subset \mathbb{R}^n \) be convex, let \( x_0, \ldots, x_m \in K \), and suppose that \( t_0, \ldots, t_m \in \mathbb{R} \) satisfy \( t_j \geq 0 \) and \( \sum_j t_j = 1 \). Then \( \sum_j t_j x_j \in K \).

**Notation.** If \( t_0, \ldots, t_m \in \mathbb{R} \) satisfy \( t_j \geq 0 \) and \( \sum_j t_j = 1 \) and \( x_0, \ldots, x_m \in \mathbb{R}^n \), then \( \sum_j t_j x_j \) is said to be a convex combination of the vectors \( x_0, \ldots, x_m \in \mathbb{R}^n \).

**Proof.** Since a term \( t_j x_j \) makes no contribution to a sum if \( t_j = 0 \), it suffices to consider the case where each \( t_j \) is positive. The proof proceeds by induction on \( m \). If \( m = 1 \) the result is tautological, and if \( m = 2 \) the result follows from the definition of convexity.

Assume now that the result is true for \( m \geq 2 \), and suppose we are given scalars \( t_0, \ldots, t_{m+1} \in \mathbb{R} \) satisfying \( t_j > 0 \) and \( \sum_j t_j = 1 \) together with vectors \( x_0, \ldots, x_{m+1} \in K \). Set \( \sigma \) equal to \( \sum_{i \leq m} t_i \), and for \( 0 \leq s \leq m \) set \( s_j = t_j / \sigma \). Then it follows that \( s_j > 0 \) and \( \sum_j s_j = 1 \), so by induction we know that \( y = \sum_j s_j x_j \) is in \( K \). By construction we have \( 0 < \sigma < 1 \) and \( \sigma + t_{m+1} = 1 \), and therefore it follows that

\[
\sum_j t_j x_j = \left( \sum_{j \leq m} t_j x_j \right) + t_{m+1} x_{m+1} = \sigma y + t_{m+1} x_{m+1} \in K
\]

which is what we wanted to prove.

**COROLLARY 4.** If \( F \) is an affine transformation of \( \mathbb{R}^n \) and \( A \subset \mathbb{R}^n \) is convex, then the image \( F[A] \) is also convex.

**Proof.** Suppose that \( x, y \in A \) and \( 0 \leq t \leq 1 \). Then Lemma 1 implies that

\[
F(t x + (1 - t) y) = t F(x) + (1 - t) F(y)
\]

and hence the segment \( [F(x)F(y)] \) is contained in \( F[A] \).

Since every pair of points in \( F[A] \) can be expressed as \( F(x) \) and \( F(y) \) for some \( x, y \in A \), the preceding sentence implies that \( F[A] \) must be convex.

**Extreme points.** This is a fundamental concept involving convex sets.

**Definition.** A point \( p \) in a convex set \( K \) is said to be an extreme point if it cannot be written in the form \( p = t x + (1 - t) y \) where \( x \) and \( y \) are distinct points of \( K \) and \( 0 < t < 1 \); informally speaking, this means \( p \) is not between two other points of \( K \).

**EXAMPLE 0.** Let \( a < b \in \mathbb{R} \), and let \( X \subset \mathbb{R} \) be the closed interval \( [a, b] \). We claim that \( a \) and \( b \) are the extreme points of \( X \). — First of all, if \( a < x < b \) and

\[
t = \frac{x - a}{b - a}
\]

then \( 0 < t < 1 \) and \( x = (1 - t)a + tb \), so the two endpoints are the only possible extreme points. To see that each is an extreme point, suppose we are given a point \( x \) which is NOT an extreme point. Choose distinct points \( u \) and \( v \) in \( [a, b] \) and \( t \) in the open interval \((0, 1)\) such that \( x = (1 - t)u + tv \); without loss of generality we may as well assume \( u < v \) (note that \( t \in (0, 1) \) implies \( 1 - t \in (0, 1) \) and \( 1 - (1 - t) = t \)). The inequalities in the preceding sentence imply that \( u < x < v \), and since
a and b are minimal and maximal points of the interval \(X = [a, b]\) it follows that \(x \neq a, b\), which means that \(a\) and \(b\) are extreme points of \(X\).

**EXAMPLE 1.** If \(a, b, c\) are noncollinear points and \(X\) is the solid triangular region consisting of all convex combinations of these vectors, then the extreme points of \(X\) are \(a, b,\) and \(c\). First of all, this set is convex because Lemma 3 implies that a convex combination of convex combinations is again a convex combination. To prove the assertion about extreme points, note that if \(t a + u b + v c\) is a convex combination in which at least two coefficients are positive, then an argument like the inductive step of Lemma 3 implies that this convex combination is between two others, and therefore the only possible extreme points are the original vectors. Furthermore, if \(p = t x + (1 - t) y\) where \(x\) and \(y\) are convex combinations and \(0 < t < 1\), then one can check directly that at least two barycentric coordinates of \(p\) must be positive (this is a bit messy but totally elementary). Therefore a point that is not an extreme point cannot be one of \(a, b, c\) and hence these must be the extreme points of \(X\).

**EXAMPLE 2.** Let \(X\) be the solid rectangular region in \(\mathbb{R}^2\) given by \([0, p] \times [0, q]\) where \(0 \leq q \leq p\). In this case we claim that \(X\) is convex and the extreme points are the vertices \((0, 0), (p, 0), (0, q)\) and \((p, q)\). This will be a consequence of Example 0 and the following result:

**PROPOSITION 5.** Let \(K_1\) and \(K_2\) be convex subsets of \(\mathbb{R}^n\) and \(\mathbb{R}^m\) respectively. Then \(K_1 \times K_2 \subset \mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}\) is convex. Furthermore, a point \((p_1, p_2)\) is an extreme point of \(K_1 \times K_2\) if and only if \(p_1\) is an extreme point of \(K_1\) and \(p_2\) is an extreme point of \(K_2\).

**Proof.** The first step is to prove that \(K_1 \times K_2\) is convex. Suppose that \(t \in (0, 1)\) and that \((x_1, x_2)\) and \((y_1, y_2)\) belong to \(K_1 \times K_2\). Then

\[
(1 - t) \cdot (x_1, x_2) + t \cdot (y_1, y_2) = ((1 - t) \cdot x_1 + t \cdot y_1, (1 - t) \cdot x_2 + t \cdot y_2)
\]

and by convexity the first and second coordinates belong to \(K_1\) and \(K_2\) respectively.

The statement about extreme points will follow if we can prove the contrapositive: A point \(p\) in \(K_1 \times K_2\) is not an extreme point if and only if at least one of its coordinates is not an extreme point of the corresponding factor. Write \(p = (p_1, p_2)\). If \(p\) is not an extreme point then we have

\[
p = (p_1, p_2) = (1 - t) \cdot (x_1, x_2) + t \cdot (y_1, y_2)
\]

where \(0 < t < 1\) and \((x_1, x_2)\) and \((y_1, y_2)\) are distinct points of \(K_1 \times K_2\). By the definition of an ordered pair, it follows that either the first or second coordinates of \((x_1, x_2)\) and \((y_1, y_2)\) are distinct; if we choose \(i = 1\) or \(2\) such that the \(i\)th coordinates are distinct, then it follows that \(p_i\) cannot be an extreme point of \(K_i\). Conversely, suppose that one coordinate \(p_i\) of \(p\) is not an extreme point of the corresponding convex set \(K_i\). Without loss of generality, we may as well assume that \(i = 1\) (if \(i = 2\), reverse the roles of 1 and 2 in the argument we shall give to obtain the same conclusion in that case). Choose \(x_1 \neq y_1 \in K_1\) and \(t \in (0, 1)\) such that \(p_1 = (1 - t) x_1 + t y_1\). Then we also have

\[
p = (p_1, p_2) = (1 - t) \cdot (x_1, p_2) + t \cdot (y_1, p_2)
\]

and therefore \(p\) is not an extreme point of \(K_1 \times K_2\.\)

The final result reflects the importance of extreme points.

**THEOREM 6.** Let \(A \subset \mathbb{R}^n\) be a convex set, and suppose that \(F\) is an affine transformation of \(\mathbb{R}^n\). Then \(F\) maps the extreme points of \(A\) onto the extreme points of \(F[A]\).
**Proof.** We shall prove the following contrapositive statement: If \( p \in A \), then \( p \) is not an extreme point of \( A \) if and only if \( F(p) \) is not an extreme point of \( F[A] \). Note that every point \( q \in F[A] \) is \( F(p) \) for some \( p \in A \).

Suppose that \( p \) is not an extreme point of \( A \). Then \( p = tx + (1-t)y \) where \( x \) and \( y \) are distinct points of \( A \) and \( 0 < t < 1 \). By Lemma 1 we then have

\[
F(p) = tF(x) + (1-t)F(y)
\]

and since \( F \) is 1-1 it follows that \( F(p) \) is not an extreme point of \( F[A] \). To prove the converse, combine this argument with the fact that \( F^{-1} \) is also affine.

**COROLLARY 7.** If \( 0 \leq p, q \) and \( 0 \leq r, s \) and \( F \) is an affine equivalence mapping \([0, p] \times [0, q] \) onto \([0, r] \times [0, s] \), then \( F \) sends the vertices of the first solid rectangular region to the vertices of the second.

This follows immediately from the theorem and Example 2.

**Convex hulls.** Given a subset \( X \) in \( \mathbb{R}^n \), the convex hull is defined so that it will be the unique smallest convex subset containing \( X \).

**Definition.** If \( X \subset \mathbb{R}^n \), then the convex hull of \( X \), written \( \text{Conv} (X) \), is the set of all convex combinations \( \sum_j t_j x_j \) where \( x_0, \ldots, x_m \in X \) and \( t_0, \ldots, t_m \in \mathbb{R} \) satisfy \( t_j \geq 0 \) and \( \sum_j t_j = 1 \).

Here are some elementary properties of convex hulls; they combine to prove that the convex hull is in fact the unique smallest convex subset of \( \mathbb{R}^n \) containing \( X \).

**LEMMA 8.** The convex hull has the following properties:

(i) If \( X \subset \mathbb{R}^n \), then \( \text{Conv} (X) \) is a convex subset of \( \mathbb{R}^n \).

(ii) If \( X \) is convex, then \( X = \text{Conv} (X) \).

(iii) If \( X \subset Y \subset \mathbb{R}^n \), then \( \text{Conv} (X) \subset \text{Conv} (Y) \).

**Proof.** The third statement follows immediately from the definition, and the second follows immediately from Lemma 3.

To prove the first statement, let \( y_i \) (where \( 1 \leq i \leq n \)) be points of \( \text{Conv} (X) \), and let \( s_i \geq 0 \) satisfy \( \sum_i s_i = 1 \). We can then find finitely many \( x_j \in X \) such that for each \( i \) we have

\[
y_i = \sum_j t_{i,j} x_j
\]

where each \( t_{i,j} \) is nonnegative and \( \sum_j t_{i,j} = 1 \), and hence we also have the following:

\[
\sum_i s_i y_i = \sum_i s_i \left( \sum_j t_{i,j} x_j \right) = \sum_j \left( \sum_i s_i t_{i,j} \right) x_j
\]

We claim that the sum of the coefficients in the right hand expression is equal to 1; this will prove that the vector in question belongs to \( \text{Conv} (X) \), which is what we want to prove. This may be verified as follows:

\[
\sum_j \left( \sum_i s_i t_{i,j} \right) = \sum_i s_i \left( \sum_j t_{i,j} \right) = \sum_i s_i \cdot 1 = 1
\]
As noted above, this shows that \text{Conv}(X) is closed under taking convex combinations and hence is convex. ■

Finally, the following result is often very useful for studying the effects of affine transformations on geometrical figures, especially when combined with Theorem 6.

**Theorem 9.** If \( X \subset \mathbb{R}^n \) and \( F \) is an affine transformation of \( \mathbb{R}^n \), then \( F \) maps \( \text{Conv}(X) \) onto \( \text{Conv}(F[X]) \).

**Proof.** We shall first show that \( F \) maps \( \text{Conv}(X) \) into \( \text{Conv}(F[X]) \). To see this, note that \( v \in \text{Conv}(X) \) implies that \( v = \sum_j t_j x_j \) where \( x_0, \ldots, x_m \in X \) and \( t_0, \ldots, t_m \in \mathbb{R} \) satisfy \( t_j \geq 0 \) and \( \sum_j t_j = 1 \), and since \( F \) is an affine transformation we have

\[
F \left( \sum_j t_j x_j \right) = \sum_j t_j F(x_j) \in \text{Conv}(F[X]).
\]

To see that every point in \( \text{Conv}(F[X]) \) comes from a point in \( \text{Conv}(X) \), note that a point \( y \) in \( \text{Conv}(F[X]) \) has the form \( \sum_j t_j F(x_j) \) for suitable \( t_j \) and \( x_j \), and by Lemma 1 this expression is equal to \( F \left( \sum_j t_j x_j \right) \); since the expression inside the parentheses lies in \( \text{Conv}(X) \), it follows that \( y \in F[\text{Conv}(X)] \) as required. ■
Affine transformations and convexity – II

We shall now use the preceding material to show that affine transformations also preserve several other fundamental types of convex sets. The first result deals with the two half-spaces determined by a hyperplane in \( H \) in \( \mathbb{R}^n \). If \( n = 2 \) or \( 3 \), these are just the two “sides” of a line or a plane respectively; for the sake of completeness, we shall formulate things more generally.

**Lemma 10.** Let \( H \subset \mathbb{R}^n \) be a hyperplane. Then there is a unit vector \( \mathbf{n} \in \mathbb{R}^n \) such that \( \mathbf{n} \) is perpendicular to every vector of the form \( \mathbf{x} - \mathbf{y} \), where \( \mathbf{y} \) and \( \mathbf{y} \) are in \( H \). This vector is unique up to multiplication by \( \pm 1 \), and \( H \) is the set of all vectors \( \mathbf{x} \) satisfying the equation \( \mathbf{n} \cdot \mathbf{x} = k \) for some real number \( k \).

**Proof.** Write \( H = \mathbf{v} + V \) where \( V \) is an \((n-1)\)-dimensional vector subspace of \( \mathbb{R}^n \). Then the orthogonal complement \( V^\perp \) is 1-dimensional and hence spanned by some unit vector \( \mathbf{n} \). If \( \mathbf{x} \) and \( \mathbf{y} \) are in \( H \), write these vectors as \( \mathbf{x} = \mathbf{v} + \mathbf{x}_0 \) and \( \mathbf{y} = \mathbf{v} + \mathbf{y}_0 \) where \( \mathbf{x}_0, \mathbf{y}_0 \in V \). Then \( \mathbf{x} - \mathbf{y} = \mathbf{x}_0 - \mathbf{y}_0 \), and since the right hand side lies in \( V \) it follows that the difference vector is perpendicular to \( \mathbf{n} \).

Since \( V \) is uniquely determined by \( H \), so is \( V^\perp \), and since the latter has exactly two unit vectors (which are the negatives of each other), the uniqueness statement follows. Finally, we know that \( V \) is defined by the equation \( \mathbf{n} \cdot \mathbf{z} = 0 \) and that \( \mathbf{v} \in V \). If \( k = \mathbf{n} \cdot \mathbf{v} \), then it follows that \( \mathbf{n} \cdot \mathbf{x} = k \) if and only if \( \mathbf{n} \cdot (\mathbf{x} - \mathbf{v}) = 0 \), which in turn is true if and only if \( \mathbf{x} - \mathbf{v} \in V \), and the latter is true if and only if \( \mathbf{x} \in \mathbf{v} + V = H \).

**Definition.** Let \( H \) be a hyperplane, let \( \mathbf{n} \) be one of the two unit vectors as in Lemma 10, and let \( k \) be such that \( H \) is defined by the equation \( \mathbf{n} \cdot \mathbf{x} = k \). The two half-spaces determined by \( H \) are the sets defined by the strict inequalities \( \mathbf{n} \cdot \mathbf{x} < k \) and \( \mathbf{n} \cdot \mathbf{x} > k \). We also say that \( H \) separates \( \mathbb{R}^n \) into these half-spaces.

We claim that the half-spaces in the definition do not depend upon the choices of \( \mathbf{n} \) or \( k \). First of all, if we fix \( \mathbf{n} \), there is a unique \( k \) such that \( H \) is defined by \( \mathbf{n} \cdot \mathbf{x} = k \), for if \( k \neq k' \) then the sets defined by \( \mathbf{n} \cdot \mathbf{x} = k \) and \( \mathbf{n} \cdot \mathbf{x} = k' \) are disjoint. Next, if we replace \( \mathbf{n} \) by its negative, then \( H \) will be defined by the equation \( -\mathbf{n} \cdot \mathbf{x} = -k \), and the two half-planes in this case are defined by the inequalities \( -\mathbf{n} \cdot \mathbf{x} < -k \) and \( -\mathbf{n} \cdot \mathbf{x} > -k \). Since these are equivalent to \( \mathbf{n} \cdot \mathbf{x} > k \) and \( \mathbf{n} \cdot \mathbf{x} < k \) respectively, we obtain the same subsets if we use \( -\mathbf{n} \) instead of \( \mathbf{n} \).

Note further that if \( \mathbf{c} \cdot \mathbf{x} = M \) is any linear equation defining \( H \), then the two half-spaces are defined by the inequalities \( \mathbf{c} \cdot \mathbf{x} < M \) and \( \mathbf{c} \cdot \mathbf{x} > M \). This is true because \( \mathbf{C} = \mathbf{L} \mathbf{n} \) where \( \mathbf{L} > 0 \) and \( \mathbf{u} \) is a unit vector, so that the two inequalities given in the preceding sentence are equivalent to \( \mathbf{n} \cdot \mathbf{x} < M/L \) and \( \mathbf{n} \cdot \mathbf{x} > M/L \).

**Theorem 11.** If \( H \subset \mathbb{R}^n \) is a hyperplane and \( F \) is an affine transformation of \( \mathbb{R}^n \), then \( F \) maps the each half-space \( W \) in \( \mathbb{R}^n - H \) to a half-space \( V \) in \( \mathbb{R}^n - F[H] \). Furthermore, if \( \mathbf{z} \in \mathbb{R}^n \) is not in \( H \), then \( F \) sends the half-space for \( H \) containing \( \mathbf{z} \) to the half-space for \( F[H] \) containing \( F(\mathbf{z}) \).

**Proof.** It suffices to prove the second statement. Choose a nonzero vector \( \mathbf{c} \) and a scalar \( k \) such that \( H \) is defined by the equation \( \mathbf{c} \cdot \mathbf{x} = k \). We shall need a formula for the affine transformation \( F^{-1} \). If \( F(\mathbf{x}) \) is given by \( A\mathbf{x} + \mathbf{b} \) where \( A \) is an invertible matrix and \( \mathbf{b} \) is some vector, then we may solve the equation \( \mathbf{y} = F(\mathbf{x}) \) to obtain the following:

\[
\mathbf{x} = F^{-1}(\mathbf{y}) = A^{-1}\mathbf{y} - A^{-1}\mathbf{b}
\]
If we rewrite the equation defining $H$ in the matrix form $\mathcal{T}_c \mathbf{x} = k$, then the formula for the inverse function yields the equation

$$\mathcal{T}_c A^{-1} \mathbf{y} = k + \mathcal{T}_c A^{-1} \mathbf{b}$$

which can be rewritten in the form

$$\mathcal{T}_d \mathbf{y} = k + \mathcal{T}_c A^{-1} \mathbf{b} = m$$

where $\mathbf{d} = \mathcal{T}_c A^{-1} \mathbf{c}$; this is a defining equation for $F[H]$. By our hypotheses and the formulas given above, we know that $\mathcal{T}_c \mathbf{x} < k$ and $\mathcal{T}_c \mathbf{x} > k$ are equivalent to $\mathcal{T}_d \mathbf{y} < m$ and $\mathcal{T}_d \mathbf{y} > m$ respectively, and therefore $F$ sends the two half-spaces determined by $H$ into the two half-spaces determined by $F[H]$.\[\blacksquare\]

The preceding theorem shows that affine transformations preserve half-spaces. Here are some further examples of sets in $\mathbb{R}^2$ which are preserved by affine transformations.

**Theorem 12.** Let $F$ be an affine transformation of $\mathbb{R}^2$, and let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be noncollinear points. Then the following hold:

(i) $F$ sends the interior of $\triangle \mathbf{a} \mathbf{b} \mathbf{c}$ to the interior of $\triangle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$.

(ii) $F$ sends the interior of $\Delta \mathbf{a} \mathbf{b} \mathbf{c}$ to the interior of $\Delta F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$.

**Proof.** (i) By Theorem 11 we know that $F$ sends the half-plane $S(\mathbf{c})$ for the line $\mathbf{a} \mathbf{b}$ containing $\mathbf{c}$ to the half-plane $S(F(\mathbf{c}))$ for the line $F(\mathbf{a})F(\mathbf{b})$ containing $F(\mathbf{c})$. Similarly, by Theorem 11 we know that $F$ sends the half-plane $S(\mathbf{a})$ for the line $\mathbf{b} \mathbf{c}$ containing $\mathbf{a}$ to the half-plane $S(F(\mathbf{a}))$ for the line $F(\mathbf{b})F(\mathbf{c})$ containing $F(\mathbf{a})$. Hence $F$ sends the intersection of $S(\mathbf{c})$ and $S(\mathbf{a})$, which is the interior of $\triangle \mathbf{a} \mathbf{b} \mathbf{c}$, to the intersection of $S(F(\mathbf{c}))$ and $S(F(\mathbf{a}))$, which is the the interior of $\triangle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$.

(ii) Since $F$ preserves intersections, as in the first part of the proof we know that $F$ maps the intersection of the interiors of $\triangle \mathbf{a} \mathbf{b} \mathbf{c}$ and $\triangle \mathbf{b} \mathbf{c} \mathbf{a}$ — which is the interior of $\Delta \mathbf{a} \mathbf{b} \mathbf{c}$ — to the intersection of the interiors of $\triangle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$ and $\triangle F(\mathbf{b})F(\mathbf{c})F(\mathbf{a})$ — which is the interior of $\Delta F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$.\[\blacksquare\]