ADDENDUM ON THE CAUCHY – SCHWARZ INEQUALITY

NOTE. *The following material should be inserted between pages* 4 *and* 5 *of the file* `geometrynotes1.pdf`.

We shall now show that equality holds in the Cauchy – Schwarz Inequality

$$|\langle \mathbf{x},\, \mathbf{y}\rangle| \quad = \quad |\mathbf{x}| \cdot |\mathbf{y}|$$

if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Since equality obviously holds if either $\mathbf{x}$ or $\mathbf{y}$ is zero, it will suffice to consider cases where both vectors are nonzero.

If the vectors are linearly dependent, then each is a nonzero multiple of the other; in particular, we have $\mathbf{y} \ = \ c\,\mathbf{x}$ for a suitable nonzero scalar $\mathbf{y}$. This means that

$$|\langle \mathbf{x},\, \mathbf{y}\rangle| \quad = \quad |\langle \mathbf{x},\, c\,\mathbf{x}\rangle| \quad = \quad |c| \,\cdot |\langle \mathbf{x},\, \mathbf{x}\rangle| \quad = \quad |c| \cdot |\mathbf{x}|^2 \quad =$$

$$|\mathbf{x}| \cdot (\, |c|\, |\mathbf{x}|\,) \quad = \quad |\mathbf{x}| \cdot (\, |c\,\mathbf{x}|\,) \quad = \quad |\mathbf{x}| \cdot |\mathbf{y}|$$

which is what we wanted to verify.

Conversely, suppose that we have

$$|\langle \mathbf{x},\, \mathbf{y}\rangle| \quad = \quad |\mathbf{x}| \cdot |\mathbf{y}| \ .$$

Then direct calculation shows that

$$0 \quad = \quad \left| \mathbf{x} \ - \ \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\langle \mathbf{y}, \mathbf{y}\rangle}\, \mathbf{y} \right|^2$$

and this equation implies that $\mathbf{x}$ is a multiple of $\mathbf{y}$ (since both vectors are nonzero, $\mathbf{x}$ must in fact be a *nonzero* multiple of $\mathbf{y}$).■