COMPUTING FIRST FUNDAMENTAL FORMS

The first example discussed in class was the cylinder with parametrization

$$\mathbf{X}(u,v) \quad = \quad (\cos u, \sin u, v).$$

In this case the partial derivatives of $\mathbf{X}$ are given by $\mathbf{X}_u = (-\sin u, \cos u, 0)$ and $\mathbf{X}_v = (0,0,1)$, so that $E = \mathbf{X}_u \cdot \mathbf{X}_u = 1$, $F = \mathbf{X}_u \cdot \mathbf{X}_v = 0$, and $G = \mathbf{X}_v \cdot \mathbf{X}_v = 1$. Therefore the First Fundamental Form is given by $du\,du \ + \ dv\,dv$. — This has an important conclusion; namely, if β is a curve in the domain of $\mathbf{X}$ and $\gamma(t)$ is the composite $\mathbf{X}(\,\beta(t)\,)$, then the lengths of β and γ are the same. Physically this is not surprising, for the effect of $\mathbf{X}$ is basically to take a flat sheet of paper and roll it up into a tube.

Example 1. Use the First Fundamental Form to find the length of the swirl curve on the cone $z = \sqrt{x^2+y^2}$ with parametrization is given by $\alpha(t) = (t\cos t, t\sin t, t)$, where $0 \leq t \leq 2\pi$.

Strictly speaking, the point $\alpha(0) = \mathbf{0}$ does not lie on the curve, but we can work around this if we replace 0 by some small positive number c and take limits as c goes to 0. The length of the curve will then be the usual integral from 0 to 2π.

SOLUTION. We can parametrize the cone by $\mathbf{X}(u,v) = (v\cos u, v\sin y, v)$, and if we do so then α is the image of the curve $\Delta(t) = (t,t)$ under $\mathbf{X}$. To compute the coefficients of the First Fundamental Form, we must find the partial derivatives $\mathbf{X}_u = (-v\sin u, v\cos u, 0)$ and $\mathbf{X}_v = (\cos u, \sin u, 1)$. We then have $E = \mathbf{X}_u \cdot \mathbf{X}_u = v^2$, $F = \mathbf{X}_u \cdot \mathbf{X}_v = 0$, and $G = \mathbf{X}_v \cdot \mathbf{X}_v = 2$. Therefore the First Fundamental Form is given by $v^2\,du\,du \ + \ 2\,dv\,dv$.

Since $u = v = t$ for the curve Δ, it follows that the length of the curve is given by the integral

$$\int_0^{2\pi} \sqrt{t^2+2}\,dt$$

which one can evaluate using the methods of single variable calculus.

Example 2. Write down the First Fundamental Form for the graph of $z = xy$.

SOLUTION. We can parametrize the cone by $\mathbf{X}(u,v) = (u,v,uv)$, which leads directly to formulas for the partial derivatives of $\mathbf{X}$ and the following formula for the First Fundamental Form:

$$(1+v^2)\,du\,du \ + \ 2uv\,du\,dv \ + \ (1+u^2)\,dv\,dv \ .$$

Example 3. Write down the First Fundamental Form for the torus (or doughnut shaped surface) given by revolving the circle $(x-2)^2 + (y-2)^2 = 1$ about the x-axis.

SOLUTION. We can parametrize the circle by $\alpha(t) = (2+\cos u, 2+\sin u)$, and therefore we can parametrize the torus of revolution by

$$\mathbf{X}(u,v) \quad = \quad \big(\,2+\cos u, \cos v(2+\sin u), \sin v(2+\sin u)\,\big)\ .$$

In this case straightforward computation shows that $E = 1$, $G = 4 + 4\sin u + \sin^2 u$, and $F = \sin v \cos v(2+\sin u)(\sin u - \cos u)$.

Finally here is one that is left to the reader.

Example 4. Write down the First Fundamental Form for the oblique cone whose vertex is the origin and whose intersection with the plane $z = 1$ is the offset circle $(x-2)^2 + (y-2)^2 = 1$.

The first step is to find a parametrization; explain why one can use

$$\mathbf{X}(u,v) \quad = \quad \big(\,v(2+\cos u), v(2+\sin u), v\,\big)\ .$$