Affine transformations and convexity

The purpose of this document is to prove some basic properties of affine transformations involving convex sets. Here are a few online references for background information:

http://math.ucr.edu/~res/progeom/pgnotes02.pdf
http://math.ucr.edu/~res/math133/metgeom.pdf

Recall that an affine transformation of $\mathbb{R}^n$ is a map of the form $F(x) = b + A(x)$, where $b \in E$ is some fixed vector and $A$ is an invertible linear transformation of $\mathbb{R}^n$.

Affine transformations satisfy a weak analog of the basic identities which characterize linear transformations.

**Lemma 1.** Let $F$ as above be an affine transformation, let $x_0, \cdots, x_k \in \mathbb{R}^n$, and suppose that $t_0, \cdots, t_k \in \mathbb{R}$ satisfy $\sum_j t_j = 1$. Then

$$F\left(\sum_j t_j x_j\right) = \sum_j t_j F(x_j).$$

**Notation.** If $t_0, \cdots, t_k \in \mathbb{R}$ satisfy $\sum_j t_j = 1$ and $x_0, \cdots, x_k \in \mathbb{R}^n$, then $\sum_j t_j x_j$ is said to be an affine combination of the vectors $x_0, \cdots, x_k \in \mathbb{R}^n$.

**Proof.** Since $\sum_j t_j = 1$ we have

$$F\left(\sum_j t_j x_j\right) = A\left(\sum_j t_j x_j\right) + b = A\left(\sum_j t_j x_j\right) + \sum_j t_j b =$$

$$\sum_j t_j A x_j + \sum_j t_j b = \sum_j t_j (A x_j + b) = \sum_j t_j F(x_j)$$

which is what we wanted prove.

We also note the following simple property of affine transformations in $\mathbb{R}^2$:

**Lemma 2.** Let $F$ be an affine transformation of $\mathbb{R}^2$, and let $x, y, z, w$ be points such that the lines $xy$ and $zw$ are parallel. Then the lines $F(x)F(y)$ and $F(z)F(w)$ are also parallel.

**Proof.** Since the two lines are disjoint and $F$ is 1–1, it follows that their images — which are also lines because $F$ is an affine transformation — must also be disjoint.

**Convex Sets.** Here are the basic definitions we need for convexity:

**Definition.** If $x, y \in \mathbb{R}^n$, then the closed segment $[xy]$ is the set of all vectors $v$ such that

$$v = tx + (1-t)y$$

where $t \in \mathbb{R}$ satisfies $0 < t < 1$.

This corresponds to the intuitive notion of closed line segment in elementary geometry.

**Definition.** A subset $K \subset \mathbb{R}^n$ is said to be convex if $x, y \in K$ implies that $[xy]$ is contained in $K$; in other words, $x, y \in K$ and $0 \leq t \leq 1$ implies that $tx + (1-t)y \in K$. 

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The following result suggests that the notions of convexity and affine transformation have some useful interrelationships.

**LEMMA 3.** Let $K \subseteq \mathbb{R}^n$ be convex, let $x_0, \ldots, x_m \in K$, and suppose that $t_0, \ldots, t_m \in \mathbb{R}$ satisfy $t_j \geq 0$ and $\sum_j t_j = 1$. Then $\sum_j t_j x_j \in K$.

**Notation.** If $t_0, \ldots, t_m \in \mathbb{R}$ satisfy $t_j \geq 0$ and $\sum_j t_j = 1$ and $x_0, \ldots, x_m \in \mathbb{R}^n$, then $\sum_j t_j x_j$ is said to be a convex combination of the vectors $x_0, \ldots, x_m \in \mathbb{R}^n$.

**Proof.** Since every pair of points in $A$ can be expressed as a convex combination $\sum t_j x_j$ of the extreme points, the segment $[x_j, x_k]$ is contained in $K$. By construction we have $0 < t < 1$, and therefore it follows that

$$\sum_j t_j x_j = \left( \sum_{j \leq m} t_j x_j \right) + t_{m+1} x_{m+1} = \sigma y + t_{m+1} x_{m+1} \in K$$

which is what we wanted to prove.

**COROLLARY 4.** If $F$ is an affine transformation of $\mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$ is convex, then the image $F[A]$ is also convex.

**Proof.** Suppose that $x, y \in A$ and $0 \leq t \leq 1$. Then Lemma 1 implies that

$$F(t x + (1-t)y) = t F(x) + (1-t) F(y)$$

and hence the segment $[F(x)F(y)]$ is contained in $F[A]$.

Since every pair of points in $F[A]$ can be expressed as $F(x)$ and $F(y)$ for some $x, y \in A$, the preceding sentence implies that $F[A]$ must be convex.

**Extreme points.** This is a fundamental concept involving convex sets.

**Definition.** A point $p$ in a convex set $K$ is said to be an extreme point if it cannot be written in the form $p = t x + (1-t)y$ where $x$ and $y$ are distinct points of $K$ and $0 < t < 1$; informally speaking, this means $p$ is not between two other points of $K$.

**EXAMPLE 0.** Let $a < b \in \mathbb{R}$, and let $X \subseteq \mathbb{R}$ be the closed interval $[a, b]$. We claim that $a$ and $b$ are the extreme points of $X$. — First of all, if $a < x < b$ and

$$t = \frac{x-a}{b-a}$$

then $0 < t < 1$ and $x = (1-t)a + tb$, so the two endpoints are the only possible extreme points. To see that each is an extreme point, suppose we are given a point $x$ which is NOT an extreme point. Choose distinct points $u$ and $v$ in $[a, b]$ and $t$ in the open interval $(0, 1)$ such that $x = (1-t)u + tv$; without loss of generality we may as well assume $u < v$ (note that $t \in (0, 1)$ implies $1-t \in (0, 1)$ and $1 - (1-t) = t$). The inequalities in the preceding sentence imply that $u < x < v$, and since
a and b are minimal and maximal points of the interval X = [a, b] it follows that x ≠ a, b, which means that a and b are extreme points of X.

EXAMPLE 1. If a, b, c are noncollinear points and X is the solid triangular region consisting of all convex combinations of these vectors, then the extreme points of X are a, b, and c. — First of all, this set is convex because Lemma 3 implies that a convex combination of convex combinations is again a convex combination. To prove the assertion about extreme points, note that if t a + u b + v c is a convex combination in which at least two coefficients are positive, then an argument like the inductive step of Lemma 3 implies that this convex combination is between two others, and therefore the only possible extreme points are the original vectors. Furthermore, if p = t x + (1 - t) y where x and y are convex combinations and 0 < t < 1, then one can check directly that at least two barycentric coordinates of p must be positive (this is a bit messy but totally elementary). Therefore a point that is not an extreme point cannot be one of a, b, c and hence these must be the extreme points of X.

EXAMPLE 2. Let X be the solid rectangular region in ℝ² given by [0, p] × [0, q] where 0 ≤ q ≤ p. In this case we claim that X is convex and the extreme points are the vertices (0,0), (p,0), (0,q) and (p,q). — This will be a consequence of Example 0 and the following result:

PROPOSITION 5. Let K₁ and K₂ be convex subsets of ℝⁿ and ℝᵐ respectively. Then K₁ × K₂ ⊂ ℝⁿ × ℝᵐ ∼= ℝⁿ⁺ᵐ is convex. Furthermore, a point (p₁,p₂) is an extreme point of K₁ × K₂ if and only if p₁ is an extreme point of K₁ and p₂ is an extreme point of K₂.

Proof. The first step is to prove that K₁ × K₂ is convex. Suppose that t ∈ (0,1) and that (x₁,x₂) and (y₁,y₂) belong to K₁ × K₂. Then

\[(1 - t) \cdot (x₁,x₂) + t \cdot (y₁,y₂) = \((1 - t) \cdot x₁ + t \cdot y₁),(1 - t) \cdot x₂ + t \cdot y₂\)

and by convexity the first and second coordinates belong to K₁ and K₂ respectively.

The statement about extreme points will follow if we can prove the contrapositive: A point p in K₁ × K₂ is not an extreme point if and only if at least one of its coordinates is not an extreme point of the corresponding factor. — Write p = (p₁,p₂). If p is not an extreme point then we have

\[p = (p₁,p₂) = (1 - t) \cdot (x₁,x₂) + t \cdot (y₁,y₂)\]

where 0 < t < 1 and (x₁,x₂) and (y₁,y₂) are distinct points of K₁ × K₂. By the definition of an ordered pair, it follows that either the first or second coordinates of (x₁,x₂) and (y₁,y₂) are distinct; if we choose i = 1 or 2 such that the i^{th} coordinates are distinct, then it follows that pᵢ cannot be an extreme point of Kᵢ. Conversely, suppose that one coordinate pᵢ of p is not an extreme point of the corresponding convex set Kᵢ. Without loss of generality, we may as well assume that i = 1 (if i = 2, reverse the roles of 1 and 2 in the argument). Choose x₁ ≠ y₁ ∈ K₁ and t ∈ (0,1) such that p₁ = (1 - t)x₁ + t y₁. Then we also have

\[p = (p₁,p₂) = (1 - t) \cdot (x₁,p₂) + t \cdot (y₁,p₂)\]

and therefore p is not an extreme point of K₁ × K₂.■

The final result reflects the importance of extreme points.

THEOREM 6. Let A ⊂ ℝⁿ be a convex set, and suppose that F is an affine transformation of ℝⁿ. Then F maps the extreme points of A onto the extreme points of F[A].
Proof. We shall prove the following contrapositive statement: If \( p \in A \), then \( p \) is not an extreme point of \( A \) if and only if \( F(p) \) is not an extreme point of \( F[A] \). Note that every point \( q \in F[A] \) is \( F(p) \) for some \( p \in A \).

Suppose that \( p \) is not an extreme point of \( A \). Then \( p = tx + (1 - t)y \) where \( x \) and \( y \) are distinct points of \( A \) and \( 0 < t < 1 \). By Lemma 1 we then have

\[
F(p) = tF(x) + (1 - t)F(y)
\]

and since \( F \) is 1–1 it follows that \( F(p) \) is not an extreme point of \( F[A] \). To prove the converse, combine this argument with the fact that \( F^{-1} \) is also affine.

**COROLLARY 7.** If \( 0 \leq p, q \) and \( 0 \leq r, s \) and \( F \) is an affine equivalence mapping \([0, p] \times [0, q] \) onto \([0, r] \times [0, s] \), then \( F \) sends the vertices of the first solid rectangular region to the vertices of the second.

This follows immediately from the theorem and Example 2.

**Convex hulls.** Given a subset \( X \) in \( \mathbb{R}^n \), the convex hull is defined so that it will be the unique smallest convex subset containing \( X \).

**Definition.** If \( X \subset \mathbb{R}^n \), then the convex hull of \( X \), written \( \text{Conv}(X) \), is the set of all convex combinations \( \sum_j t_j x_j \) where \( x_0, \cdots, x_m \in X \) and \( t_0, \cdots, t_m \in \mathbb{R} \) satisfy \( t_j \geq 0 \) and \( \sum_j t_j = 1 \).

Here are some elementary properties of convex hulls; they combine to prove that the convex hull is in fact the unique smallest convex subset of \( \mathbb{R}^n \) containing \( X \).

**LEMMA 8.** The convex hull has the following properties:

(i) If \( X \subset \mathbb{R}^n \), then \( \text{Conv}(X) \) is a convex subset of \( \mathbb{R}^n \).

(ii) If \( X \) is convex, then \( X = \text{Conv}(X) \).

(iii) If \( X \subset Y \subset \mathbb{R}^n \), then \( \text{Conv}(X) \subset \text{Conv}(Y) \).

**Proof.** The third statement follows immediately from the definition, and the second follows immediately from Lemma 3.

To prove the first statement, let \( y_i \) (where \( 1 \leq i \leq n \)) be points of \( \text{Conv}(X) \), and let \( s_i \geq 0 \) satisfy \( \sum_i s_i = 1 \). We can then find finitely many \( x_j \in X \) such that for each \( i \) we have

\[
y_i = \sum_j t_{i,j} x_j
\]

where each \( t_{i,j} \) is nonnegative and \( \sum_j t_{i,j} = 1 \), and hence we also have the following:

\[
\sum_i s_i y_i = \sum_i s_i \left( \sum_j t_{i,j} x_j \right) = \sum_j \left( \sum_i s_i t_{i,j} \right) x_j
\]

We claim that the sum of the coefficients in the right hand expression is equal to 1; this will prove that the vector in question belongs to \( \text{Conv}(X) \), which is what we want to prove. This may be verified as follows:

\[
\sum_j \left( \sum_i s_i t_{i,j} \right) = \sum_i s_i \left( \sum_j t_{i,j} \right) = \sum_i s_i \cdot 1 = 1
\]
As noted above, this shows that Conv \((X)\) is closed under taking convex combinations and hence is convex.

Finally, the following result is often very useful for studying the effects of affine transformations on geometrical figures, especially when combined with Theorem 6.

**THEOREM 9.** If \(X \subseteq \mathbb{R}^n\) and \(F\) is an affine transformation of \(\mathbb{R}^n\), then \(F\) maps Conv \((X)\) onto Conv \((F[X])\).

**Proof.** We shall first show that \(F\) maps Conv \((X)\) into Conv \((F[X])\). To see this, note that \(v \in \text{Conv} (X)\) implies that \(v = \sum_j t_j x_j\) where \(x_0, \cdots, x_m \in X\) and \(t_0, \cdots, t_m \in \mathbb{R}\) satisfy \(t_j \geq 0\) and \(\sum_j t_j = 1\), and since \(F\) is an affine transformation we have

\[
F \left( \sum_j t_j x_j \right) = \sum_j t_j F(x_j) \in \text{Conv} (F[X]).
\]

To see that every point in Conv \((F[X])\) comes from a point in Conv \((X)\), note that a point \(y\) in Conv \((F[X])\) has the form \(\sum_j t_j F(x_j)\) for suitable \(t_j\) and \(x_j\), and by Lemma 1 this expression is equal to \(F \left( \sum_j t_j x_j \right)\); since the expression inside the parentheses lies in Conv \((X)\), it follows that \(y \in F[\text{Conv} (X)]\) as required.