Affine transformations and convexity

The purpose of this document is to prove some basic properties of affine transformations involving convex sets. Here are a few online references for background information:

http://math.ucr.edu/∼res/progeom/pgnotes02.pdf

http://math.ucr.edu/∼res/math133/metgeom.pdf

Recall that an *affine transformation* of $\mathbb{R}^n$ is a map of the form $F(\mathbf{x}) = \mathbf{b} + A(\mathbf{x})$, where $\mathbf{b} \in E$ is some fixed vector and A is an invertible linear tranformation of $\mathbb{R}^n$.

Affine transformations satisfy a weak analog of the basic identities which characterize linear transformations.

LEMMA 1. *Let F as above be an affine transformation, let $\mathbf{x}_0, \cdots, \mathbf{x}_k \in \mathbb{R}^n$, and suppose that $t_0, \cdots, t_k \in \mathbb{R}$ satisfy $\sum_j t_j = 1$. Then*

$$F\left(\sum_j t_j \mathbf{x}_j\right) = \sum_j t_j F(\mathbf{x}_j) .$$

Notation. If $t_0, \cdots, t_k \in \mathbb{R}$ satisfy $\sum_j t_j = 1$ and $\mathbf{x}_0, \cdots, \mathbf{x}_k \in \mathbb{R}^n$, then $\sum_j t_j \mathbf{x}_j$ is said to be an *affine combination* of the vectors $\mathbf{x}_0, \cdots, \mathbf{x}_k \in \mathbb{R}^n$.

Proof. Since $\sum_j t_j = 1$ we have

$$F\left(\sum_j t_j \mathbf{x}_j\right) = A\left(\sum_j t_j \mathbf{x}_j\right) + \mathbf{b} = A\left(\sum_j t_j \mathbf{x}_j\right) + \sum_j t_j \mathbf{b} =$$

$$\sum_j t_j A\mathbf{x}_j + \sum_j t_j \mathbf{b} = \sum_j t_j (A\mathbf{x}_j + \mathbf{b}) = \sum_j t_j F(\mathbf{x}_j)$$

which is what we wanted prove.■

We also note the following simple property of affine transformations in $\mathbb{R}^2$:

LEMMA 2. *Let F be an affine transformation of $\mathbb{R}^2$, and let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$ be points such that the lines $\mathbf{xy}$ and $\mathbf{zw}$ are parallel. Then the lines $F(\mathbf{x})F(\mathbf{y})$ and $F(\mathbf{z})F(\mathbf{w})$ are also parallel.*

Proof. Since the two lines are disjoint and F is 1–1, it follows that their images — which are also lines because F is an affine transformation — must also be disjoint.■

CONVEX SETS. Here are the basic definitions we need for convexity:

Definition. If $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, then the *closed segment* $[\mathbf{xy}]$ is the set of all vectors $\mathbf{v}$ such that

$$\mathbf{v} = t\,\mathbf{x} + (1-t)\,\mathbf{y}$$

where $t \in \mathbb{R}$ satisfies $0 < t < 1$.

This corresponds to the intuitive notion of closed line segment in elementary geometry.

Definition. A subset $K \subset \mathbb{R}^n$ is said to be *convex* if $\mathbf{x}$, $\mathbf{y} \in K$ implies that $[\mathbf{xy}]$ is contained in K; in other words, $\mathbf{x}$, $\mathbf{y} \in K$ and $0 \leq t \leq 1$ implies that $t\,\mathbf{x} + (1-t)\,\mathbf{y} \in K$.

The following result suggests that the notions of convexity and affine transformation have some useful interrelationships.

LEMMA 3. *Let $K \subset \mathbb{R}^n$ be convex, let $\mathbf{x}_0, \cdots, \mathbf{x}_m \in K$, and suppose that $t_0, \cdots, t_m \in \mathbb{R}$ satisfy $t_j \geq 0$ and $\sum_j t_j = 1$. Then $\sum_j t_j \mathbf{x}_j \in K$.*

Notation. If $t_0, \cdots, t_m \in \mathbb{R}$ satisfy $t_j \geq 0$ and $\sum_j t_j = 1$ and $\mathbf{x}_0, \cdots, \mathbf{x}_m \in \mathbb{R}^n$, then $\sum_j t_j \mathbf{x}_j$ is said to be a *convex combination* of the vectors $\mathbf{x}_0, \cdots, \mathbf{x}_m \in \mathbb{R}^n$.

Proof. Since a term $t_j \mathbf{x}_j$ makes no contribution to a sum if $t_j = 0$, it suffices to consider the case where each t_j is positive. The proof proceeds by induction on m. If $m = 1$ the result is tautological, and if $m = 2$ the result follows from the definition of convexity.

Assume now that the result is true for $m \geq 2$, and suppose we are given scalars $t_0, \cdots, t_{m+1} \in \mathbb{R}$ satisfying $t_j > 0$ and $\sum_j t_j = 1$ together with vectors $\mathbf{x}_0, \cdots, \mathbf{x}_{m+1} \in K$. Set σ equal to $\sum_{i \leq m} t_i$, and for $0 \leq s \leq m$ set s_j equal to t_j/σ. Then it follows that $s_j > 0$ and $\sum_j s_j = 1$, so by induction we know that $\mathbf{y} = \sum_j s_j \mathbf{x}_j$ is in K. By construction we have $0 < \sigma < 1$ and $\sigma + t_{m+1} = 1$, and therefore it follows that

$$\sum_j t_j \mathbf{x}_j = \left(\sum_{j \leq m} t_j \mathbf{x}_j \right) + t_{m+1} \mathbf{x}_{m+1} =$$

$$\sigma \mathbf{y} + t_{m+1} \mathbf{x}_{m+1} \in K$$

which is what we wanted to prove.∎

COROLLARY 4. *If F is an affine transformation of $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is convex, then the image $F[A]$ is also convex.*

Proof. Suppose that $\mathbf{x}$, $\mathbf{y} \in A$ and $0 \leq t \leq 1$. Then Lemma 1 implies that

$$F(t\mathbf{x} + (1-t)\mathbf{y}) = t F(\mathbf{x}) + (1-t) F(\mathbf{y})$$

and hence the segment $[F(\mathbf{x})F(\mathbf{y})]$ is contained in $F[A]$.

Since every pair of points in $F[A]$ can be expressed as $F(\mathbf{x})$ and $F(\mathbf{y})$ for some $\mathbf{x}$, $\mathbf{y} \in A$, the preceding sentence implies that $F[A]$ must be convex.∎

Extreme points. This is a fundamental concept involving convex sets.

Definition. A point $\mathbf{p}$ in a convex set K is said to be an *extrme point* if it cannot be written in the form $\mathbf{p} = t\mathbf{x} + (1-t)\mathbf{y}$ where $\mathbf{x}$ and $\mathbf{y}$ are distinct points of K and $0 < t < 1$; informally speaking, this means $\mathbf{p}$ is not between two other points of K.

EXAMPLE 0. Let $a < b \in \mathbb{R}$, and let $X \subset \mathbb{R}$ be the closed interval $[a, b]$. We claim that a and b are the extreme points of X. — First of all, if $a < x < b$ and

$$t = \frac{x-a}{b-a}$$

then $0 < t < 1$ and $x = (1-t)a + tb$, so the two endpoints are the only possible extreme points. To see that each is an extreme point, suppose we are given a point x which is **NOT** an extreme point. Choose distinct points u and v in $[a, b]$ and t in the open interval $(0, 1)$ such that $x = (1-t)u + tv$; without loss of generality we may as well assume $u < v$ (note that $t \in (0, 1)$ implies $1 - t \in (0, 1)$ and $1 - (1-t) = t$). The inequalities in the preceding sentence imply that $u < x < v$, and since

a and b are minimal and maximal points of the interval $X = [a, b]$ it follows that $x \neq a, b$, which means that a and b are extreme points of X.

EXAMPLE 1. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are noncollinear points and X is the solid triangular region consisting of all convex combinations of these vectors, then the extreme points of X are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. — First of all, this set is convex because Lemma 3 implies that a convex combination of convex combinations is again a convex combination. To prove the assertion about extreme points, note that if $t\,\mathbf{a} + u\,\mathbf{b} + v\,\mathbf{c}$ is a convex combination in which at least two coefficients are positive, then an argument like the inductive step of Lemma 3 implies that this convex combination is between two others, and therefore the only possible extreme points are the original vectors. Furthermore, if $\mathbf{p} = t\,\mathbf{x} + (1-t)\,\mathbf{y}$ where $\mathbf{x}$ and $\mathbf{y}$ are convex combinations and $0 < t < 1$, then one can check directly that at least two barycentric coordinates of $\mathbf{p}$ must be positive (this is a bit messy but totally elementary). Therefore a point that is not an extreme point cannot be one of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and hence these must be the extreme points of X.

EXAMPLE 2. Let X be the solid rectangular region in $\mathbb{R}^2$ given by $[0, p] \times [0, q]$ where $0 \leq q \leq p$. In this case we claim that X is convex and the extreme points are the vertices $(0, 0)$, $(p, 0)$, $(0, q)$ and (p, q). — This will be a consequence of Example 0 and the following result:

PROPOSITION 5. *Let K_1 and K_2 be convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then $K_1 \times K_2 \subset \mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ is convex. Furthermore, a point $(\mathbf{p}_1, \mathbf{p}_2)$ is an extreme point of $K_1 \times K_2$ if and only if $\mathbf{p}_1$ is an extreme point of K_1 and $\mathbf{p}_2$ is an extreme point of K_2*

Proof. The first step is to prove that $K_1 \times K_2$ is convex. Suppose that $t \in (0, 1)$ and that $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ belong to $K_1 \times K_2$. Then

$$(1-t) \cdot (\mathbf{x}_1, \mathbf{x}_2) \;+\; t \cdot (\mathbf{y}_1, \mathbf{y}_2) \quad = \quad \big((1-t) \cdot \mathbf{x}_1 + t \cdot \mathbf{y}_1, (1-t) \cdot \mathbf{x}_2 + t \cdot \mathbf{y}_2\,\big)$$

and by convexity the first and second coordinates belong to K_1 and K_2 respectively.

The statement about extreme points will follow if we can prove the contrapositive: *A point $\mathbf{p}$ in $K_1 \times K_2$ is not an extreme point if and only if at least one of its coordinates is not an extreme point of the corresponding factor.* — Write $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$. If $\mathbf{p}$ is not an extreme point then we have

$$\mathbf{p} \quad = \quad (\mathbf{p}_1, \mathbf{p}_2) \quad = \quad (1-t) \cdot (\mathbf{x}_1, \mathbf{x}_2) \;+\; t \cdot (\mathbf{y}_1, \mathbf{y}_2)$$

where $0 < t < 1$ and $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ are distinct points of $K_1 \times K_2$. By the definition of an ordered pair, it follows that either the first or second coordinates of $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ are distinct; if we choose $i = 1$ or 2 such that the i^{th} coordinates are distinct, then it follows that $\mathbf{p}_i$ cannot be an extreme point of K_i. Conversely, suppose that one coordinate $\mathbf{p}_i$ of $\mathbf{p}$ is not an extreme point of the corresponding convex set K_i. Without loss of generality, we may as well assume that $i = 1$ (if $i = 2$, reverse the roles of 1 and 2 in the argument we shall give to obtain the same conclusion in that case). Choose $\mathbf{x}_1 \neq \mathbf{y}_1 \in K_1$ and $t \in (0, 1)$ such that $\mathbf{p}_1 = (1-t)\,\mathbf{x}_1 + t\,\mathbf{y}_1$. Then we also have

$$\mathbf{p} \quad = \quad (\mathbf{p}_1, \mathbf{p}_2) \quad = \quad (1-t) \cdot (\mathbf{x}_1, \mathbf{p}_2) \;+\; t \cdot (\mathbf{y}_1, \mathbf{p}_2)$$

and therefore $\mathbf{p}$ is not an extreme point of $K_1 \times K_2$.■

The final result reflects the importance of extreme points.

THEOREM 6. *Let $A \subset \mathbb{R}^n$ be a convex set, and suppose that F is an affine transformation of $\mathbb{R}^n$. Then F maps the extreme points of A onto the extreme points of $F[A]$.*

Proof. We shall prove the following contrapositive statement: *If $\mathbf{p} \in A$, then $\mathbf{p}$ is not an extreme point of A if and only if $F(\mathbf{p})$ is not an extreme point of $F[A]$.* — Note that every point $\mathbf{q} \in F[A]$ is $F(\mathbf{p})$ for some $\mathbf{p} \in A$.

Suppose that $\mathbf{p}$ is not an extreme point of A. Then $\mathbf{p} = t\,\mathbf{x} + (1-t)\,\mathbf{y}$ where $\mathbf{x}$ and $\mathbf{y}$ are distinct points of A and $0 < t < 1$. By Lemma 1 we then have

$$F(\mathbf{p}) \;=\; t\,F(\mathbf{x}) + (1-t)\,F(\mathbf{y})$$

and since F is 1–1 it follows that $F(\mathbf{p})$ is not an extreme point of $F[A]$. To prove the converse, combine this argument with the fact that F^{-1} is also affine.■

COROLLARY 7. *If $0 \leq p, q$ and $0 \leq r, s$ and F is an affine equivalence mapping $[0,p] \times [0,q]$ onto $[0,r] \times [0,s]$, then F sends the vertices of the first solid rectangular region to the vertices of the second.*

This follows immediately from the theorem and Example 2.■

Convex hulls. Given a subset X in $\mathbb{R}^n$, the convex hull is defined so that it will be the unique smallest convex subset containing X.

Definition. If $X \subset \mathbb{R}^n$, then the *convex hull* of X, written $\mathrm{Conv}\,(X)$, is the set of all convex combinations $\sum_j t_j\,\mathbf{x}_j$ where $\mathbf{x}_0, \cdots, \mathbf{x}_m \in X$ and $t_0, \cdots, t_m \in \mathbb{R}$ satisfy $t_j \geq 0$ and $\sum_j t_j = 1$.

Here are some elementary properties of convex hulls; they combine to prove that the convex hull is in fact the unique smallest convex subset of $\mathbb{R}^n$ containing X.

LEMMA 8. *The convex hull has the following properties:*

(i) If $X \subset \mathbb{R}^n$, then $\mathrm{Conv}\,(X)$ is a convex subset of $\mathbb{R}^n$.

(ii) If X is convex, then $X = \mathrm{Conv}\,(X)$.

(iii) If $X \subset Y \subset \mathbb{R}^n$, then $\mathrm{Conv}\,(X) \subset \mathrm{Conv}\,(Y)$.

Proof. The third statement follows immediately from the definition, and the second follows immediately from Lemma 3.

To prove the first statement, let $\mathbf{y}_i$ (where $1 \leq i \leq n$) be points of $\mathrm{Conv}\,(X)$, and let $s_i \geq 0$ satisfy $\sum_i s_i = 1$. We can then find finitely many $\mathbf{x_j} \in X$ such that for each i we have

$$\mathbf{y}_i \;=\; \sum_j t_{i,j}\,\mathbf{x}_j$$

where each $t_{i,j}$ is nonnegative and $\sum_j t_{i,j} = 1$, and hence we also have the following:

$$\sum_i s_i\,\mathbf{y}_i \;=\; \sum_i s_i \left(\textstyle\sum_j t_{i,j}\,\mathbf{x}_j\right) \;=\; \sum_j \left(\textstyle\sum_i s_i\,t_{i,j}\right)\mathbf{x}_j$$

We claim that the sum of the coefficients in the right hand expression is equal to 1; this will prove that the vector in question belongs to $\mathrm{Conv}\,(X)$, which is what we want to prove. This may be verified as follows:

$$\sum_j \left(\textstyle\sum_i s_i\,t_{i,j}\right) \;=\; \sum_i s_i \left(\textstyle\sum_j t_{i,j}\right) \;=\; \sum_i s_i \cdot 1 \;=\; 1$$

As noted above, this shows that $\mathrm{Conv}\,(X)$ is closed under taking convex combinations and hence is convex.■

Finally, the following result is often very useful for studying the effects of affine transformations on geometrical figures, especially when combined with Theorem 6.

THEOREM 9. *If $X \subset \mathbb{R}^n$ and F is an affine transformation of $\mathbb{R}^n$, then F maps* $\mathrm{Conv}\,(X)$ *onto* $\mathrm{Conv}\,(F[X])$.

Proof. We shall first show that F maps $\mathrm{Conv}\,(X)$ into $\mathrm{Conv}\,(F[X])$. To see this, note that $\mathbf{v} \in \mathrm{Conv}\,(X)$ implies that $\mathbf{v} = \sum_j t_j\,\mathbf{x}_j$ where $\mathbf{x}_0, \cdots, \mathbf{x}_m \in X$ and $t_0, \cdots, t_m \in \mathbb{R}$ satisfy $t_j \geq 0$ and $\sum_j t_j = 1$, and since F is an affine transformation we have

$$F\left(\sum_j t_j\,\mathbf{x}_j\right) \quad = \quad \sum_j t_j\,F\,(\mathbf{x}_j) \quad \in \quad \mathrm{Conv}\,(F[X]) \ .$$

To see that every point in $\mathrm{Conv}\,(F[X])$ comes from a point in $\mathrm{Conv}\,(X)$, note that a point $\mathbf{y}$ in $\mathrm{Conv}\,(F[X])$ has the form $\sum_j t_j\,F(\mathbf{x}_j)$ for suitable t_j and $\mathbf{x}_j$, and by Lemma 1 this expression is equal to $F\left(\sum_j t_j\,\mathbf{x}_j\right)$; since the expression inside the parentheses lies in $\mathrm{Conv}\,(X)$, it follows that $\mathbf{y} \in F[\mathrm{Conv}\,(X)]$ as required.■