5.A. Some problems from Diophantus' *Arithmetica*

Our purpose here is to look more closely at several problems from Diophantus' *Arithmetica* which are discussed on pages 220–223 of Burton.

Book I, Problem 17. The general form of this problem is to find x, y, z, w which solve the following system, in which A, B, C, D are arbitrary positive rational numbers.

$$\begin{array}{ccccccc} x & + & y & + & z & = & A \\ x & + & y & + & w & = & B \\ x & + & y & + & w & = & C \\ y & + & z & + & w & = & D \end{array}$$

One key step in the solution is to consider the sum S of the four unknowns. If we add all four of the given equations together, we find that the left hand side is equal to $3S$ while the right hand side is equal to $A + B + C + D$, and therefore we have $S = \frac{1}{3}(A + B + C + D)$. If we subtract each of the four given equations from $x + y + z + w = S$, we obtain the following new system:

$$\begin{array}{ccccc} S & - & w & = & A \\ S & - & z & = & B \\ S & - & y & = & C \\ S & - & x & = & D \end{array}$$

Therefore we have $x = S - D$, $y = S - C$, $z = S - B$, and $w = S - A$. To complete the discussion we need to investigate the conditions under which all four of these values are positive. If we reorder the numbers A, B, C, D as A', B', C', D' so that $A' \leq B' \leq C' \leq D'$, then $S - D'$ is the minimum of the values x, y, z, w, so we want this to be positive. In other words, we need

$$0 \quad < \quad \frac{S}{3} \; - \; D' \quad = \quad \frac{A' + B' + C' - 2D'}{3}$$

so that the condition on the four given numbers is $A' + B' + C' \; > \; 2D'$. This holds for the specific values of $20, 22, 24, 27$ in Burton.∎

Book II, Problem 20. The general problem is to find two positive rational numbers x and y so that both $x^2 + y$ and $y^2 + x$ are rational squares. — As is frequently the case in Diophantus' methods, it is useful to restrict our attention to choices of x and y which satisfy some well-chosen constraint. In this case, if we let $y = 2cx + c^2$, then we have $x^2 + y = (x + c)^2$. It follows that $y^2 + x = (2cx + c^2)^2 + x = 4c^2x^2 + (4c^3 + 1)x + c^4$, and the next step is to see if we can find some d such that the right hand side equals $(2cx - d)^2$. Since $(2cx - d) = 4c^2x^2k - 4dcx + d^2$, this means that d must satisfy the equation $(4c^3 + 1)x + c^4 = d^2 - 4cdx + d^2$, or equivalently

$$(4c^3 + 4cd + 1)x \quad = \quad d^2 - c^4 \; .$$

Again working backwards, if we are given positive rational numbers c and d such that $d > c^2$ and $4c^3 + 4cd + 1 > 0$, then we obtain values of x and $y = 2cx + c^2$ which satisfy the desired conditions. Now the first inequality implies the second, so for each choice of c and d such that $d > c^2$ we obtain suitable values of x and y. Note that there are infinitely many different choices of x and y which satisfy the conditions in the problem.

The discussion in Burton concentrates on the special case where $d = 2$ and $c = 1$.∎

Book II, Problem 13. The problem is to find a positive rational number x such that both $x-c$ and $x-d$ are squares of positive rational numbers, where c and d are two fixed positive rational numbers such that $c < d$.

We want to find a, b, x such that $x - c = a^2$ and $x - d = b^2$. Since $0 < c < d$, it follows that $a^2 > b^2$. The desired values a, b must satisfy $a^2 - b^2 = d - c$, where the right hand side is a fixed quantity. Let p and q be arbitrary positive rational numbers such that $p < q$ and $pq = d - c$. We then need to find a and b such that $pq = d - c = a^2 - b^2 = (a + b)(a - b)$, and in particular we shall see if it is possible to choose a and b such that $p = a - b$ and $q = a + b$. The latter equations imply that $a = \frac{1}{2}(p + q)$ and $b = \frac{1}{2}(q - p)$. Finally, if we set $x = c + a^2$ for this choice of a, then it also follows that $x - d = b^2$ (since $a^2 - b^2 = d - c$). Note that for a given choice of c and d there are always infinitely many values of x which have the desired properties.∎

Book III, Problem 21. The problem is to write a positive rational numbers C as a sum $x + y$ of two positive rational numbers such that there is some third positive rational number z for which both $x + z^2$ and $y + z^2$ are (rational) squares.

In this problem one starts with a change of variables $z = u + 1$, so that $z^2 = u^2 + 2u + 1$. If p and q are arbitrary rational numbers greater than 1, then we have

$$(z + p)^2 = z^2 + (2p - 1)u + (p^2 - 1)\ , \qquad (z + q)^2 = z^2 + (2q - 1)u + (q^2 - 1)$$

and consequently if we choose x and y so that $x + y = C$ and

$$x \quad = \quad (2p - 1)u + (p^2 - 1)\ , \qquad y \quad = \quad (2q - 1)u + (q^2 - 1)$$

then we are done if we can solve for u (this will yield z). But if we add the last two equations together we obtain $C = (2p + 2q - 2)u + (p^2 + q^2 - 2)$, which yields the solution

$$u \quad = \quad \frac{C + 2 - p^2 - q^2}{2p + 2q - 2}\ .$$

There are plenty of choices for p and q such that this expression is positive; all we need to do is choose p and q such that $p^2 + q^2 < C + 2$. Since C is positive, there are infinitely many ways of choosing $p, q > 1$ so that this condition holds.∎