# 2.A. Calculus and the area of Hippocrates' lune

The methods of integral calculus provide techniques for computing the areas of arbitrary lunes, so we shall indicate how this applies to the special example computed by Hippocrates of Chios. We shall refer to the figure in the main unit.

We shall take the smaller semicircular region to be the one bounded by the circle $x^2 + y^2 = 2$ and the $x$-axis. Having made that choice, we must take the second circle to be the one whose radius is $\sqrt{2}$ times the given circle and contains the end points of the semicircle. Since the radius of the original circle is $\sqrt{2}$ and the end points are $(\pm\sqrt{2}, 0)$, we have enough information to write down the equation of the circle, and in fact it is given as follows:

$$x^2 \ + \ (y + \sqrt{2})^2 \quad = \quad 4$$

The smaller circle is the upper curve for this region and the larger circle is the lower curve. The equations for the pieces of these curves in the upper half plane are given by $y = \sqrt{2 - x^2}$ and $y = \sqrt{4 - x^2} - \sqrt{2}$. The region of interest to us lies between the vertical lines $x = \pm\sqrt{2}$. Therefore the area of the lune is given by the following integral:

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left( \sqrt{2 - x^2} + \sqrt{2} - \sqrt{4 - x^2} \right) \, dx$$

It might be worthwhile to compute this definite integral and see how easy or difficult it is to retrieve Hippocrates' result using calculus. Of course, one could also apply this to other lunes that Hippocrates studied as well.

*Another approach*

Computing the definite integral for the lune's area involves some messy calculations, so we shall give another approach using more sophisticated technique; namely, Green's Theorem, which relates the line integral over a closed curve $\Gamma$ to a double integral over the closed region $D$ which $\Gamma$ bounds. For the sake of simplicity we shall assume that the four vertices of the square are $(1, 1)$, $(-1, 1)$ $(-1, -1)$ and $(1, -1)$ so that the center of the square is the origin.

According to Green's Theorem, the area of the region bounded by $D$ is equal to the line integral

$$\frac{1}{2} \int_\Gamma x \, dy \ - \ y \, dx$$

where $\Gamma$ is parametrized in a counterclockwise sense. We shall let $D$ be the region at the top of the drawing in `history02.pdf` which is labeled $\mathsf{C}$. We can parametrize the lower arc in a counterclockwise sense by the formula

$$\alpha(t) \quad = \quad \sqrt{2}\,(\cos t, \sin t\,) \ , \quad \pi/4 \ \leq \ t \ \leq \ 3\pi/4 \ .$$

Similarly, we can parametrize the the upper arc in the counterclockwise sense by the formula

$$\beta(t) \quad = \quad \big(\cos u, \sin u + 1\big) \ . \quad 0 \ \leq \ t \ \leq \ \pi \ .$$

By Green's Theorem the area bounded by the region is equal to

$$\frac{1}{2} \left( \int_\beta x \, dy \ - \ y \, dx \ - \ \int_\alpha x \, dy \ - \ y \, dx \right) \ .$$

The integrals in this equation are very straightforward to compute, and the result is equal to 1. On the other hand, the closed region $\mathsf{A}$ underneath $\mathsf{C}$ is a solid region bounded by a triangle whose vertices are $(0, 0)$, $(-1, 1)$ and $(1, 1)$; its horizontal base has length 2 and vertical height 1, and therefore the area of the closed region $\mathsf{A}$ is also equal to 1. Thus we have shown that the regions $\mathsf{A}$ and $\mathsf{C}$ have the same areas.