Free actions of finite groups on Hausdorff spaces

The notion of a group action on a topological space is defined on page 54 of Munkres. For our purposes it will suffice to take a group and to view it as topological groups with respect to the discrete topology. If G is such a group and X is a topological space, the group action itself is given by a continuous mapping $\Phi : G \times X \to X$, with $\Phi(g, x)$ usually abbreviated to $g \cdot x$ or gx, such that $1 \cdot x = x$ for all x and $(gh) \cdot x = g \cdot (h \cdot x)$ for all g, h and x. One can then define an equivalence relation on X by stipulating that $y \sim x$ if and only if $y = g \cdot x$ for some $g \in G$, and the quotient space with respect to this relation is called the *orbit space* of the group action and written X/G. The equivalence classes (often called the **orbits** of the group action) are the sets $G\,x$ of all points of the form $g \cdot x$ where $g \in G$. Note that if X is Hausdorff and G is finite, then $G\,x$ is a closed compact (in fact, finite!) subset for each $x \in X$

PROPOSITION. *Suppose that G is a finite group acting on a Hausdorff space X. Then the orbit space X/G is Hausdorff.*

Proof. One fast way of proving this result is to use the following fact about Hausdorff spaces: *If E and F are disjoint compact subsets of a Hausdorff space X, then there are disjoint open subsets U and V such that $E \subset U$ and $F \subset V$.* — The proof of this is a fairly straightforward exercise. Note that this result applies if E and F are both finite.

Let $x, y \in X$ be such that $G\,x \neq G\,y$, and apply the result in the preceding paragraph to $\{x\}$ and $G\,y$ to obtain disjoint open neighborhoods U_0 and V_0 of these compact subsets. Let $V = \cap_g \; g \cdot V_0$, where g runs through all the elements of G. The V is a G-invariant open neighborhood of $G\,y$ and $U_0 \cap V = \emptyset$. Let $U = \cup_g \; g\,U_0$. Then U and V are disjoint G-invariant open neighborhoods of $G\,x$ and $G\,y$ respectively.

Let $\pi : X \to X/G$ denote the quotient projection. We claim that $\pi[U]$ and $\pi[V]$ are disjoint open neighborhoods of $\pi(x)$ and $\pi(y)$ respectively. The two sets in question are open in the quotient topology because

$$\pi^{-1}\left[\,\pi[U]\,\right] \;=\; U \;, \quad \pi^{-1}\left[\,\pi[V]\,\right] \;=\; V$$

and U and V are open in X. Likewise, $\pi[U]$ and $\pi[V]$ are disjoint, for if $\pi(z)$ lies in their intersection then $G\,z \in U \cap V$ and we know that the latter is empty.■

If we are given a group action as above and A is a subset of X, then for a given $g \in G$ it is customary to let $g \cdot A$ (the translate of A by g) be the set $\Phi[\,\{g\} \times A\,]$; this is the set of all points expressible as $g \cdot a$ for the fixed g and some $a \in A$.

Definition. We shall say that a group action Φ as above is a **free action** (or G acts freely) if for every $x \in X$ the only solution to the equation $g \cdot x = x$ is the trivial solutions for which $g = 1$. — If $X = S^2$ as above and G is the order two subgroup $\{\pm 1\}$ of the real numbers (with respect to multiplication), then scalar multiplication defines a free action of G on S^2, and the quotient space is just $\mathbf{RP}^2$. Of course, there are also similar examples for which 2 is replaced by an arbitrary positive integer n, and in this case the quotient space $S^n/\{\pm 1\}$ is called *real projective n-space*. Some links to the motivation for this definition are given in the following course directory document:

`projective-spaces-links.pdf`

Further information about the relationship is contained in Exercises V.1.2 and V.1.3 on page 12 of the following document:

`http://math.ucr.edu/~res/math205A/gentopexercises2008.pdf`

Solutions are given on page 6 of the document `math205Asolutions4.pdf` in the same directory.

The next result implies that the orbit space projections $S^n \to \mathbf{RP}^n$ are covering space projections.

THEOREM. *Let G be a finite group which acts freely on the Hausdorff topological space X, and let $\pi : X \to X/G$ denote the orbit space projection. Then π is a covering space projection.*

Proof. Let $x \in X$ be arbitrary, and let $g \neq 1$ in G. Then there are open neighborhoods $U_0(g)$ of x and $V_0(g)$ of $g \cdot x$ that are disjoint. If we let $W(g) = U(g) \cap g^{-1} \cdot V(g)$ is another open set containing x, while $g \cdot W(g)$ is an open set containing $g \cdot x$, and we have $W(g) \cap g \cdot W(g) = \emptyset$. Let

$$W \quad = \quad \bigcap_{h \neq 1} W(h)$$

so that W is an open set containing x.

We claim that if $g_1 \neq g_2$, then $g_1 \cdot W \cap g_2 \cdot W \quad = \quad \emptyset$. If we know this, then it will follow immediately that $\pi[W]$ is an open set in X/G whose inverse image is the open subset of X given by $\cup_g \, g \cdot W$. This and the definition of the quotient topology imply that $\pi[W]$ is an evenly covered open neighborhood of x, and therefore it will follow that π is a covering space projection.

Thus it remains to prove the statement in the first sentence of the preceding paragraph. Note first that it will suffice to prove this in the special case where $g_1 = 1$; assuming we know this, in the general case we then have

$$g_1 \cdot W \cap g_2 \cdot W \quad = \quad g_1 \left(W \cap (g_1^{-1} g_2) \cdot W \right)$$

and the coefficient of g_1 on the right hand side is empty by the special case when $g_1 = 1$ and the fact that $g_1 \neq g_2$ implies $1 \neq g_1^{-1} \cdot g_2$. — But if $g \neq 1$ then we have $W \cap g \cdot W \subset W(g) \cap g \cdot W(g)$, and we know that the latter is empty by construction. Therefore $W \cap g \cdot W = \emptyset$, and as noted before this completes the proof.∎