Homotopy of paths and line integrals

In multivariable calculus one learns that certain line integrals in the plane of the form

$$\int_\Gamma P\,dx \;+\; Q\,dy$$

depend only on the endpoints of Γ. More precisely, if

$$\frac{\partial P}{\partial y} \;=\; \frac{\partial Q}{\partial x}$$

and P and Q have continuous partial derivatives on a convex open set, then the integral does not depend upon the path. In contrast, if we consider the line integral

$$\int_\Gamma \frac{x\,dy - y\,dx}{x^2 + y^2}$$

over the counterclockwise unit circle $(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$, then direct computation shows that the value obtained is 2π, but if we consider the corresponding line integral over the counterclockwise circle of radius $\frac{1}{3}$ centered at $(\frac{2}{3}, 0)$ with parametrization

$$x(t) \;=\; \frac{2}{3} \;+\; \frac{1}{3}\cos t, \qquad y(t) \;=\; \frac{1}{3}\sin t \qquad (0 \leq t \leq 2\pi)$$

then direct computation shows that the integral's value is zero. Since both of the curves we have described start and end at $(1, 0)$, obviously the line integral does depend upon the path in this case. It is natural to ask the extent to which the line integral does vary with the choice of path; the main result here states that the value depends only on the homotopy class of the path, where it is assumed that the homotopy keeps the endpoints fixed.

MAIN RESULT. *Let U be an open subset of the coordinate plane, let $P(x, y)$ and $Q(x, y)$ be two functions with continuous partials satisfying the previous condition on partial derivatives, and let Γ and Γ' be two piecewise smooth curves in U with the same endpoints such that Γ and Γ' are homotopic by an endpoint preserving homotopy. Then*

$$\int_\Gamma P\,dx + Q\,dy \;=\; \int_{\Gamma'} P\,dx + Q\,dy.$$

In particular, if Γ and Γ' are closed curves, then the line integrals agree if Γ and Γ' determine the same element of $\pi_1(U, \text{endpoint})$.

If we take P and Q to be the specific functions above then they are both defined and continuously differentiable on $\mathbb{R}^2 - \{(0,0)\}$. We claim that *the possible values of the line integral for* *a closed curve starting and ending at* $(1,0)$ *are precisely the integral multiples of* 2π. This is a consequence of the main theorem and the following three observations:

(1) If the curve C is obtained by concatenating A and B, then the line integral of $P\,dx+Q\,dy$ over C is the sum of the corresponding line integrals over A and B.

(2) The fundamental group of $\mathbb{R}^2-\{(0,0)\}$ is infinite cyclic and the counterclockwise unit circle represents a generator.

(3) If C is the constant curve, the line integral of anything over C is zero.

Note that the line integral of the given expression over the smaller circle can be read off from the preceding observations and the Main Theorem without any computations. For the smaller circle lies in the convex region $(0,\infty]\times\mathbb{R}\subset\mathbb{R}^2-\{(0,0)\}$, and hence Γ' represents an element in the image of the homomorphism

$$\pi_1\left(\,(0,\infty]\times\mathbb{R},(1,0)\,\right)\;\longrightarrow\;\pi_1\left(\mathbb{R}^2-\{(0,0)\},(1,0)\,\right).$$

But the fundamental group of a convex set is always the trivial group, and therefore the class of Γ' must be the trivial element.

The following alternate version of the Main Theorem is frequently found in books on multivariable calculus.

ALTERNATE STATEMENT OF MAIN THEOREM. *If P and Q are as above, U is a connected region, and Γ is a piecewise smooth closed curve that is homotopic to a constant in U, then*

$$\int_\Gamma P\,dx+Q\,dy \;=\; 0.$$

This follows immediately from the Main Result, observation (3) above, and the triviality of the fundamental group of U. Conversely, the Main Result follows from the Alternate Statement. To see this, in the setting of the Main Result the curve $\Gamma'+(-\Gamma)$ is a closed piecewise smooth curve that is homotopic to a constant (verify this!), so the Alternate Statement implies that the line integral over this curve is zero. On the other hand, this line integral is also the difference of the line integrals over Γ' and Γ. Combining these observations, we see that the line integrals over Γ' and Γ must be equal.

In fact, it will be more convenient for us to prove the Alternate Statement in the discussion below.

The next result is often also found in multivariable calculus texts.

COROLLARY. *If in the setting of the Main Result and its Alternate Statement we also know that the region U is simply connected then*

(*i*) *for every piecewise smooth closed curve Γ in U we have*

$$\int_\Gamma P\,dx+Q\,dy \;=\; 0$$

(*ii*) *for every pair of piecewise smooth curves Γ, Γ' with the same endpoints we have*

$$\int_\Gamma P\,dx+Q\,dy \;=\; \int_{\Gamma'} P\,dx+Q\,dy.$$

The first part of the corollary follows from the triviality of the fundamental group of U, the Alternate Statement of the Main Result, and the triviality of line integrals over constant curve. The second part follows formally from the first in the same way that the Main Result follows from its Alternate Statement.

Background from multivariable calculus

As noted above, the following result can be found in most multivariable calculus textbooks.

PATH INDEPENDENCE THEOREM. *Let U be a rectangular open subset of the coordinate plane of the form $(a_1, b_1) \times (a_2, b_2)$, let P and Q be functions with continuous partials on U such that*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

and let Γ and Γ' be two piecewise smooth curves in U with the same endpoints. Then

$$\int_\Gamma P\,dx + Q\,dy = \int_{\Gamma'} P\,dx + Q\,dy.$$

The underlying idea behind the proof is to construct a function f such that $\nabla f = (P, Q)$. Green's Theorem plays major role in showing that the partials of f have the desired values.

Notational and abuse of language conventions. Given two points $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$ in the coordinate plane, the closed straight line segment joining them is the curve $[\mathbf{p}, \mathbf{q}]$ with parametrization

$$x(t) = tp_1 + (1-t)p_2, \qquad y(t) = tq_1 + (1-t)q_2 \qquad (0 \leq t \leq 1).$$

We would also like to discuss broken line curves, say joining $\mathbf{p}_0$ to $\mathbf{p}_1$ by a straight line segment, then joining $\mathbf{p}_1$ to $\mathbf{p}_2$ by a straight line segment, and so on. The points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, etc. are called the *vertices* of the broken line curve. One technical problem with this involves the choices of linear parametrizations for the pieces. However, since line integrals for such curves do not depend upon such parametrizations and in fact we have

$$\int_C P\,dx + Q\,dy = \Sigma \int_{[\mathbf{p}_{i-1}, \mathbf{p}_i]} P\,dx + Q\,dy$$

we shall not worry about the specific choice of parametrization. Filling in the details will be left as an exercise to a reader who is interested in doing so; this is basically elementary but tedious.

Integrals over broken line inscriptions

First some standard definitions. A *partition* of the interval $[a, b]$ is a sequence of points

$$\Delta : a = t_0 < t_1 < \cdots < t_m = b$$

and the *mesh* of Δ, written $|\Delta|$, is the maximum of the differences $t_i - t_{i-1}$ for $1 \leq i \leq m$. Given a piecewise smooth curve Γ defined on $[a, b]$, the *broken line inscription* $\mathrm{Lin}(\Gamma, \Delta)$ is the broken line curve with vertices

$$\Gamma(a) = \Gamma(t_0),\ \Gamma(t_1),\ \cdots\ \Gamma(t_m) = \Gamma(b).$$

We are now ready to prove one of the key technical steps of the proof of the main result.

LEMMA.. *Let U, P, Q, Γ be as usual, where Γ is defined on $[a,b]$ and P and Q satisfy the condition*

$$\frac{\partial P}{\partial y} \;=\; \frac{\partial Q}{\partial x}.$$

Then there is a positive constant $\delta > 0$ such that for all partitions Δ of $[a,b]$ with $|\Delta| < \delta$ we have

$$\int_\Gamma P\,dx + Q\,dy \;=\; \int_{\mathrm{Lin}(\Gamma,\Delta)} P\,dx + Q\,dy.$$

Proof. If K is the image of Γ then K is a compact subset of the open set U, and therefore there is an $\varepsilon > 0$ so that if $\mathbf{x} \in \mathbb{R}^2$ satisfies $|\mathbf{x}-\mathbf{v}| < \varepsilon$ for some $\mathbf{v} \in K$ then $\mathbf{x} \in U$. It follows that if $\mathbf{v} \in K$ then the inner region for the square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon\sqrt{2}$ lies entirely in U.

By uniform continuity there is a $\delta > 0$ so that if $s, t \in [a,b]$ satisfy $|s-t| < \delta$ then

$$|\Gamma(s) - \Gamma(t)| \;<\; \frac{\varepsilon\sqrt{2}}{2}.$$

Let Δ be a partition of $[a,b]$ whose mesh is less than δ. Then for all i the restriction of Γ to $[t_{i-1}, t_i]$ lies in the open disk of radius $\frac{1}{2}\varepsilon\sqrt{2}$. It follows that both this restriction and the closed straight line segment joining $\Gamma(t_{i-1})$ to $\Gamma(t_i)$ lie in the open square region centered at $\Gamma(t_{i-1})$ with sides parallel to the coordinate axes of length of length $\varepsilon\sqrt{2}$; since the latter lies entirely in U. it follows that P and Q are defined on this square region. Therefore, by the previously quoted result from multivariable calculus we have

$$\int_{\Gamma|[t_{i-1},t_i]} P\,dx + Q\,dy \;=\; \int_{[\Gamma(t_{i-1}),\Gamma(t_i)]} P\,dx + Q\,dy$$

for each i. But the line integral over Γ is the sum of the line integrals over the curves $\Gamma|[t_{i-1}, t_i]$, and the line integral over the broken line inscription is the sum of the line integrals over the line segments $[\Gamma(t_{i-1}), \Gamma(t_i)]$, and therefore it follows that the line integral over Γ is equal to the line integral over the broken line inscription, as required.

Proof of the Alternate Statement of the Main Result

We may as well assume that Γ is defined on the unit interval $[0,1]$ since we can always arrange this by a linear change of variables. Let $H : [0,1] \times [0,1] \to U$ be a continuous map such that $H(s,0) \;=\; \Gamma(s)$ for all s and H is constant on both $[0,1] \times \{1\}$ and $\{0,1\} \times [0,1]$.

If L is the image of H then L is a compact subset of the open set U, and as in the proof of the lemma there is an $\varepsilon' > 0$ so that if $\mathbf{x} \in \mathbb{R}^2$ satisfies $|\mathbf{x}-\mathbf{v}| < \varepsilon'$ for some $\mathbf{v} \in L$ then $\mathbf{x} \in U$. It follows that if $\mathbf{v} \in L$ then the inner region for the square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$ lies entirely in U.

By uniform continuity there is a $\delta' > 0$ so that if $\mathbf{s}, \mathbf{t} \in [0,1] \times [0,1]$ satisfy $|\mathbf{s}-\mathbf{t}| < \delta'$ then

$$|H(\mathbf{s}) - H(\mathbf{t})| \;<\; \frac{\varepsilon'\sqrt{2}}{2}\;.$$

Without loss of generality we may assume that δ' is no greater than the δ in the previous lemma. Let Δ be a partition of $[a,b]$ whose mesh is less than $\frac{1}{2}\delta'\sqrt{2}$, and choose a positive integer N such that

$$\frac{1}{N} < \frac{\delta'\sqrt{2}}{2}.$$

Then for all i such that $1 \le i \le m$ and all j such that $1 \le j \le N$ the restriction of H to $[t_{i-1}, t_i] \times [\frac{j-1}{N}, \frac{j}{N}]$ lies in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$.

A special case. To motivate the remainder of the argument, we shall first specialize to the case where H extends to a map on an open set containing the square $[0,1] \times [0,1]$ and has continuous partials on this open set. For each i such that $0 \le i \le m$ and each j such that $1 \le j \le N$ let $A(i,j)$ be the broken line curve in the square with vertices

$$(0, \tfrac{j-1}{N}),\ \dots\ (t_i, \tfrac{j-1}{N}),\ (t_i, \tfrac{j}{N}),\ \dots\ (1, \tfrac{j}{N})\ .$$

In other words, this curve is formed by starting with a horizontal line segment from $(0, \frac{j-1}{N})$ to $(t_i, \frac{j-1}{N})$, then concatenating with a vertical line segment from $(t_i, \frac{j-1}{N})$ to $(t_i, \frac{j}{N})$, and finally concatenating with a horizontal line segment from $(t_i, \frac{j}{N})$ to $(1, \frac{j}{N})$. If $W(i,j)$ denotes the composite $H \circ A(i,j)$, then it follows that $W(i,j)$ is a piecewise smooth closed curve in U. Furthermore, $W(m,1)$ is just the concatenation of Γ with a constant curve and $W(0,N)$ is just a constant curve, so the proof of the main result reduces to showing that the line integrals of the expression $P\,dx + Q\,dy$ over the curves $W(m,1)$ and $W(0,N)$ are equal. We claim this will be established if we can show the following hold for all i and j:

(1) The corresponding line integrals over the curves $W(0,j-1)$ and $W(m,j)$ are equal.

(2) The corresponding line integrals over the curves $W(i-1,j)$ and $W(i,j)$ are equal.

To prove the claim, first note that (2) implies that the value of the line integral over $W(i,j)$ is a constant z_j that depends only on j, and then note that (1) implies $z_{j-1} = z_j$ for all j. Thus the two assertions combine to show that the line integrals over all the curves $W(i,j)$ have the same value.

We begin by verifying (1). Since H is constant on $\{0,1\} \times [0,1]$, it follows that $W(m,j)$ is formed by concatenating $H|[0,1] \times \{\frac{j}{m}\}$ and a constant curve (in that order), while $W(0,j-1)$ is formed by concatenating a constant curve and $H|[0,1] \times \{\frac{j}{m}\}$ (again in the given order). Thus the line integrals over both $W(0,j-1)$ and $W(m,j)$ are equal to the line integral over $H|[0,1] \times \{\frac{j}{m}\}$, proving (1).

Turning to (2), since the broken line curves $A(i,j)$ and $A(i-1,j)$ differ only by one vertex, it follows that the difference

$$\int_{W(i,j)} P\,dx + Q\,dy \ - \int_{W(i-1,j)} P\,dx + Q\,dy$$

is equal to

$$\int_{V(i,j)} P\,dx + Q\,dy \ - \int_{V'(i,j)} P\,dx + Q\,dy$$

where $V(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j}{N}\right)$$

and $V'(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad \left(t_{i-1}, \tfrac{j}{N}\right), \quad \left(t_i, \tfrac{j}{N}\right).$$

Our hypotheses imply that both of these curves lie in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$ and thus also in the open square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$; by construction the latter retion lies entirely in U. Therefore by the previously quoted result from multivariable calculus we have

$$\int_{V(i,j)} P\,dx + Q\,dy \;=\; \int_{V'(i,j)} P\,dx + Q\,dy$$

for each i and j, so that the difference of the line integrals vanishes. Since this difference is also the difference between the line integrals over $W(i,j)$ and $W(i-1,j)$, it follows that the line integrals over the latter two curves must be equal.

The general case. If H is an arbitrary continuous function the preceding proof breaks down because we do not know if the continuous curves $W(i,j)$ are well enough behaved to define line integrals. We shall circumvent this by using broken line approximations to these curves and appealing to the previous lemma to relate the value of the line integrals over these approximations to the value on the original curve. Since the proof is formally analogous to that for the special case we shall concentrate on the changes that are required.

Let $X(i,j)$ denote the broken line curve with vertices

$$H(0, \tfrac{j-1}{N}),\ \ldots\ H(t_i, \tfrac{j-1}{N}),\ H(t_i, \tfrac{j}{N}),\ \ldots\ H(1, \tfrac{j}{N}).$$

By our choice of Δ these broken lines all lie in U, and the constituent segments all lie in suitably small open disks inside U.

We claim that it will suffice to prove that the line integrals over the curves $X(0, j-1)$ and $X(m,j)$ are equal for all j and for each j the corresponding line integrals over the curves $X(i-1,j)$ and $X(i,j)$ are equal. As before it will follow that the line integrals over all the broken line curves $X(i,j)$ have the same value. But $X(m,N)$ is a constant curve, so this value is zero. On the other hand, by construction the curve $X(m,1)$ is formed by concatenating $\mathrm{Lin}(\Gamma,\Delta)$ and a constant curve, so this value is also the value of the line integral over $\mathrm{Lin}(\Gamma,\Delta)$. But now the Lemma implies that the values of the corresponding line integrals over Γ and $\mathrm{Lin}(\Gamma,\Delta)$ are equal, and therefore the value of the line integral over the original curve Γ must also be equal to zero.

The first set of equalities follow from the same sort argument used previously for $W(0, j-1)$ and $W(m,j)$ with the restriction of Γ replaced by the broken line curve with vertices

$$H(0, \tfrac{j}{N}),\ \ldots\ H(1, \tfrac{j}{N}).$$

To verify the second set of equalities, note that the difference between the values of the line integrals over $X(i,j)$ and $X(i-1,j)$ is given by

$$\int_{C(i,j)} P\,dx + Q\,dy \;-\; \int_{C'(i,j)} P\,dx + Q\,dy$$

where $C(i,j)$ is the broken line curve with vertices

$$H\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad H\left(t_i, \tfrac{j-1}{N}\right), \quad H\left(t_i, \tfrac{j}{N}\right)$$

and $C'(i,j)$ is the broken line curve with vertices

$$H\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad H\left(t_{i-1}, \tfrac{j}{N}\right), \quad H\left(t_i, \tfrac{j}{N}\right).$$

By the previously quoted result from multivariable calculus we have

$$\int_{C(i,j)} P\,dx + Q\,dy \;=\; \int_{C'(i,j)} P\,dx + Q\,dy$$

for each i and j, and therefore the difference between the values of the line integrals must be zero. Therefore the difference between the values of the line integrals over $X(i,j)$ and $X(i-1,j)$ must also be zero, as required. This completes the proof.

The three dimensional case

The preceding results have natural extensions to higher dimensions. We shall indicate the changes that are needed to obtain the three dimensional case.

First of all, we need to deal with line integrals of the form $\int_\Gamma P\,dx + Q\,dy + R\,dz$, and the condition we need on the functions P, Q, R is that the curl of the associated vector field $\mathbf{F} = (P, Q, R)$ is zero. Under these circumstances there is an analog of the path independence property from two dimensional multivariable calculus for rectangular open regions of the form

$$(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$$

the main difference being that one needs to apply Stokes' Theorem instead of Green's Theorem to show that a certain function f satisfies $\nabla f = \mathbf{F}$. Throughout the epsilon-delta manipluations in the proof one also needs to make some minor changes in factors of the form $\frac{1}{2}\sqrt{2}$ or $\sqrt{2}$; the underlying geometric fact is that a 3-dimensional disk of radius r about a point now contains the inside of a cube of side $\frac{2}{3}\sqrt{3}$ centered at the given point, and the latter in turn contains an open disk of radius $\frac{1}{3}\sqrt{3}$ about the point.

If L is the image of H then L is a compact subset of the open set U, and as in the proof of the lemma there is an $\varepsilon' > 0$ so that if $\mathbf{x} \in \mathbb{R}^2$ satisfies $|\mathbf{x} - \mathbf{v}| < \varepsilon'$ for some $\mathbf{v} \in L$ then $\mathbf{x} \in U$. It follows that if $\mathbf{v} \in L$ then the inner region for the square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$ lies entirely in U.

By uniform continuity there is a $\delta' > 0$ so that if $\mathbf{s}, \mathbf{t} \in [0,1] \times [0,1]$ satisfy $|\mathbf{s} - \mathbf{t}| < \delta'$ then

$$|H(\mathbf{s}) - H(\mathbf{t})| \;<\; \frac{\varepsilon'\sqrt{2}}{2}.$$

Without loss of generality we may assume that δ' is no greater than the δ in the lemma on page 17 of the reader. Let Δ be a partition of $[a,b]$ whose mesh is less than $\frac{1}{2}\delta'\sqrt{2}$, and choose a positive integer N such that

$$\frac{1}{N} \;<\; \frac{\delta'\sqrt{2}}{2}.$$

Then for all i such that $1 \leq i \leq m$ and all j such that $1 \leq j \leq N$ the restriction of H to $[t_{i-1}, t_i] \times [\frac{j-1}{N}, \frac{j}{N}]$ lies in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$.

A special case. To motivate the remainder of the argument, we shall first specialize to the case where H extends to a map on an open set containing the square $[0,1] \times [0,1]$ and has continuous partials on this open set. For each i such that $0 \leq i \leq m$ and each j such that $1 \leq j \leq N$ let $A(i,j)$ be the broken line curve in the square with vertices

$$(0, \tfrac{j-1}{N}), \ \ldots\ (t_i, \tfrac{j-1}{N}),\ (t_i, \tfrac{j}{N}), \ldots\ (1, \tfrac{j}{N})\ .$$

In other words, this curve is formed by starting with a horizontal line segment from $(0, \frac{j-1}{N})$ to $(t_i, \frac{j-1}{N})$, then concatenating with a vertical line segment from $(t_i, \frac{j-1}{N})$ to $(t_i, \frac{j}{N})$, and finally concatenating with a horizontal line segment from $(t_i, \frac{j}{N})$ to $(1, \frac{j}{N})$. If $W(i,j)$ denotes the composite $H \circ A(i,j)$, then it follows that $W(i,j)$ is a piecewise smooth closed curve in U. Furthermore, $W(m,1)$ is just the concatenation of Γ with a constant curve and $W(0,N)$ is just a constant curve, so the proof of the main result reduces to showing that the line integrals of the expression $P\,dx + Q\,dy$ over the curves $W(m,1)$ and $W(0,N)$ are equal. We claim this will be established if we can show the following hold for all i and j:

(1) The corresponding line integrals over the curves $W(0, j-1)$ and $W(m,j)$ are equal.

(2) The corresponding line integrals over the curves $W(i-1, j)$ and $W(i,j)$ are equal.

To prove the claim, first note that (2) implies that the value of the line integral over $W(i,j)$ is a constant z_j that depends only on j, and then note that (1) implies $z_{j-1} = z_j$ for all j. Thus the two assertions combine to show that the line integrals over all the curves $W(i,j)$ have the same value.

We begin by verifying (1). Since H is constant on $\{0,1\} \times [0,1]$, it follows that $W(m,j)$ is formed by concatenating $H|[0,1] \times \{\frac{j}{m}\}$ and a constant curve (in that order), while $W(0, j-1)$ is formed by concatenating a constant curve and $H|[0,1] \times \{\frac{j}{m}\}$ (again in the given order). Thus the line integrals over both $W(0, j-1)$ and $W(m,j)$ are equal to the line integral over $H|[0,1] \times \{\frac{j}{m}\}$, proving (1).

Turning to (2), since the broken line curves $A(i,j)$ and $A(i-1,j)$ differ only by one vertex, it follows that the difference

$$\int_{W(i,j)} P\,dx + Q\,dy \ - \int_{W(i-1,j)} P\,dx + Q\,dy$$

is equal to

$$\int_{V(i,j)} P\,dx + Q\,dy \ - \int_{V'(i,j)} P\,dx + Q\,dy$$

where $V(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j}{N}\right)$$

and $V'(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \ \left(t_{i-1}, \tfrac{j}{N}\right), \ \left(t_i, \tfrac{j}{N}\right).$$

Our hypotheses imply that both of these curves lie in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$ and thus also in the open square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$; by construction the latter retion lies entirely in U. Therefore by the previously quoted result from multivariable calculus we have

$$\int_{V(i,j)} P\,dx + Q\,dy \;=\; \int_{V'(i,j)} P\,dx + Q\,dy$$

for each i and j, so that the difference of the line integrals vanishes. Since this difference is also the difference between the line integrals over $W(i,j)$ and $W(i-1,j)$, it follows that the line integrals over the latter two curves must be equal.

The general case. If H is an arbitrary continuous function the preceding proof breaks down because we do not know if the continuous curves $W(i,j)$ are well enough behaved to define line integrals. We shall circumvent this by using broken line approximations to these curves and appealing to the previous lemma to relate the value of the line integrals over these approximations to the value on the original curve. Since the proof is formally analogous to that for the special case we shall concentrate on the changes that are required.

Let $X(i,j)$ denote the broken line curve with vertices

$$H(0, \tfrac{j-1}{N}), \;\ldots\; H(t_i, \tfrac{j-1}{N}),\; H(t_i, \tfrac{j}{N}),\; \ldots\; H(1, \tfrac{j}{N}).$$

By our choice of Δ these broken lines all lie in U, and the constituent segments all lie in suitably small open disks inside U.

We claim that it will suffice to prove that the line integrals over the curves $X(0, j-1)$ and $X(m,j)$ are equal for all j and for each j the corresponding line integrals over the curves $X(i-1,j)$ and $X(i,j)$ are equal. As before it will follow that the line integrals over all the broken line curves $X(i,j)$ have the same value. But $X(m,N)$ is a constant curve, so this value is zero. On the other hand, by construction the curve $X(m,1)$ is formed by concatenating $\mathrm{Lin}(\Gamma,\Delta)$ and a constant curve, so this value is also the value of the line integral over $\mathrm{Lin}(\Gamma,\Delta)$. But now the Lemma implies that the values of the corresponding line integrals over Γ and $\mathrm{Lin}(\Gamma,\Delta)$ are equal, and therefore the value of the line integral over the original curve Γ must also be equal to zero.

The first set of equalities follow from the same sort argument used previously for $W(0, j-1)$ and $W(m,j)$ with the restriction of Γ replaced by the broken line curve with vertices

$$H(0, \tfrac{j}{N}), \;\ldots\; H(1, \tfrac{j}{N}).$$

To verify the second set of equalities, note that the difference between the values of the line integrals over $X(i,j)$ and $X(i-1,j)$ is given by

$$\int_{C(i,j)} P\,dx + Q\,dy \;-\; \int_{C'(i,j)} P\,dx + Q\,dy$$

where $C(i,j)$ is the broken line curve with vertices

$$H\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad H\left(t_i, \tfrac{j-1}{N}\right), \quad H\left(t_i, \tfrac{j}{N}\right)$$

and $C'(i,j)$ is the broken line curve with vertices

$$H\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad H\left(t_{i-1}, \tfrac{j}{N}\right), \quad H\left(t_i, \tfrac{j}{N}\right).$$

By the previously quoted result from multivariable calculus we have

$$\int_{C(i,j)} P\,dx + Q\,dy \;=\; \int_{C'(i,j)} P\,dx + Q\,dy$$

for each i and j, and therefore the difference between the values of the line integrals must be zero. Therefore the difference between the values of the line integrals over $X(i,j)$ and $X(i-1,j)$ must also be zero, as required. This completes the proof.

The three dimensional case

The preceding results have natural extensions to higher dimensions. We shall indicate the changes that are needed to obtain the three dimensional case.

First of all, we need to deal with line integrals of the form $\int_\Gamma P\,dx + Q\,dy + R\,dz$, and the condition we need on the functions P, Q, R is that the curl of the associated vector field $\mathbf{F} = (P, Q, R)$ is zero. Under these circumstances there is an analog of the path independence property from two dimensional multivariable calculus for rectangular open regions of the form

$$(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$$

the main difference being that one needs to apply Stokes' Theorem instead of Green's Theorem to show that a certain function f satisfies $\nabla f = \mathbf{F}$. Throughout the epsilon-delta manipluations in the proof one also needs to make some minor changes in factors of the form $\frac{1}{2}\sqrt{2}$ or $\sqrt{2}$; the underlying geometric fact is that a 3-dimensional disk of radius r about a point now contains the inside of a cube of side $\frac{2}{3}\sqrt{3}$ centered at the given point, and the latter in turn contains an open disk of radius $\frac{1}{3}\sqrt{3}$ about the point.

ADDENDUM TO Homotopy of paths and line integrals

The purpose of this note is to describe how one can obtain a mild strengthening of the main result on homotopy invariance of line integrals over closed curves.

MODIFIED MAIN RESULT. *Let U be an open subset of the coordinate plane, let $P(x, y)$ and $Q(x, y)$ be two functions with continuous partials satisfying the previous condition on partial derivatives, and let Γ and Γ' be two piecewise smooth closed curves in U with the same endpoints such that Γ and $Gamma'$ are homotopic as closed curves. Then*

$$\int_\Gamma P\,dx + Q\,dy \;=\; \int_{\Gamma'} P\,dx + Q\,dy.$$

This turns out to be a corollary of the main result.

Proof of the Modified Main Result

Let $H : [0, 1] \times [0, 1] \to U$ be a continuous map such that $H(s, 0) \;=\; \Gamma(s)\; H(s, 1) \;=\; \Gamma'(s)$ for all s and $H(0, t) = H(1, t)$ for all $t \in [0, 1]$.

Let $\Lambda(t) = H(0, t)$. Then the restriction of H to the boundary of $[0, 1] \times [0, 1]$ in the counterclockwise sense (with basepoint at the origin) is basepoint preservingly homotopic to

$$B := \;\Gamma + \Lambda + (-\Gamma') + (-\Lambda).$$

Choose a partition of $[0, 1] \times [0, 1]$ into squares of side $\frac{1}{N}$ for a large integer N as in the proof of the main result, let Δ denote the induced partition on the boundary, and let $\mathrm{Lin}(\Delta, B)$ be the broken line inscription for B corresponding to Δ. Since B extends to the solid square, the argument establishing the main result shows that

$$\int_{\mathrm{Lin}(B,\Delta)} P\,dx + Q\,dy \;=\; 0.$$

By homotopy invariance, the definition of B, and the additivity of line integrals with respect to concatenations we have

$$0 \;=\; \int_{\mathrm{Lin}(\Gamma,\Delta)} P\,dx+Q\,dy + \int_{\mathrm{Lin}(\Lambda,\Delta)} P\,dx+Q\,dy - \int_{\mathrm{Lin}(\Gamma',\Delta)} P\,dx+Q\,dy - \int_{\mathrm{Lin}(\Lambda,\Delta)} P\,dx+Q\,dy.$$

This simplifies to

$$0 \;=\; \int_{\mathrm{Lin}(\Gamma,\Delta)} P,\,dx + Q\,dy \;-\; \int_{\mathrm{Lin}(\Gamma',\Delta)} P\,dx + Q\,dy$$

which by previous results on close broken line approximations implies that

$$0 \;=\; \int_{\Gamma} P\,dx + Q\,dy \;-\; \int_{\Gamma'} P\,dx + Q\,dy$$

which is the desired conclusion.

Application to the Fundamental Theorem of Algebra

The first step is a familar limit formula.

Lemma. *If $p(z)$ is a nonconstant monic polynomial in the complex plane then* $\lim_{z\to\infty} |p(z)| = \infty$.

Sketch of proof. Use the identity

$$p(z) \;=\; z^n \cdot \left(1 + \frac{c_{n-1}}{z} + \frac{c_{n-2}}{z^2} + \cdots + \frac{c_1}{z^{n-1}} + \frac{c_0}{z^n}\right)$$

and the fact that the limit of the term inside the parentheses is zero.

Corollary. *In the setting above there is an $r > 0$ such that $R \geq r$ implies that $p(z)$ is never zero on a circle C_R of radius R about the origin.*

Now let $\Gamma(p,R)$ be the closed curve given by $R\cdot p(\exp(2\pi i t))$ for $0 \leq t \leq 1$, so that $\Gamma(p,R)$ just describes the behavior of the polynomial p on the circle of radius R about the origin. Consider the so-called *winding number integral*

$$\int_{\Gamma(p,R)} \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

The proof of the Fundamental Theorem of Algebra has two remaining steps.

(1) *If $p(z) \neq 0$ for all z satisfying $|z| \leq R$, then the winding number integral is zero.*

(2) *If p has degree n then the winding number integral is equal to n.*

Proof of first statement. By construction $\Gamma(p, R)$ lies in the punctured plane $\mathbb{C} - \{0\}$. Since p has no zero points it follows that p also defines a map into the punctured plane, and the restriction of p to the solid disk of radius R defines a basepoint preserving homotopy from $\Gamma(p, R)$ to the constant curve. Therefore by homotopy invariance we know that the corresponding line integrals over $\Gamma(p, R)$ and the constant curve are equal. Since the latter integral is zero it follows that the original winding number integral is also zero.

Proof of second statement. First of all, if $p(z) = z^n$ then it follows that the winding number is n by direct calculation. It will suffice to show that for R sufficiently large the closed curves $\Gamma(p, R)$ and z^n are homotopic, for then we can use the modified form of the main result to show that the line integrals associated to the two polynomials are equal.

To prove this we use the identity

$$p(z) \;=\; z^n \cdot \left(1 + \frac{c_{n-1}}{z} + \frac{c_{n-2}}{z^2} + \cdots + \frac{c_1}{z^{n-1}} + \frac{c_0}{z^n}\right)$$

to conclude that

$$\lim_{z \to \infty} \frac{p(z)}{z^n} \;=\; 1.$$

In particular, there is an $S > 0$ such that $R > S$ implies that

$$\left| \frac{p(z)}{z^n} - 1 \right| \;<\; \frac{1}{2}.$$

This in turn implies that if $|z| = R$ then the line segment joining 1 to

$$\frac{p(z)}{z^n}$$

lies entirely in the punctured plane. If $h(z, t)$ is this straight line homotopy on the circle $|z| = R$ then $z^n h(z, t)$ defines a homotopy between $\Gamma(p, R)$ and the closed curve defined by the restriction of z^n to the circle of radius R. As noted before, this completes the proof of the second statement and of the Fundamental Theorem of Algebra.

Exercises

1. Let U be an open subset of $\mathbb{R}^n$. Prove that U is connected if and only if every pair of points in U can be joined by a broken line curve that lies entirely in U.

2. Let U be a simply connected open set in $\mathbb{R}^2$, and let P and Q be two functions with continuous partials on U such that

$$\frac{\partial P}{\partial y} \;=\; \frac{\partial Q}{\partial x}.$$

Prove that there is a continuous function f on U such that $\nabla f = (P, Q)$. [*Hint:* Pick some point $\mathbf{u} \in U$, and for each $\mathbf{v} \in U$ define $f(\mathbf{v})$ to be the line integral of $P\,dx + Q\,dy$ over some broken line curve joining $\mathbf{u}$ to $\mathbf{v}$. Why do such curves exist, and why is the value of the line integral independent of the choice of broken line curve?]

3. Let A be a finite set or points in $\mathbb{R}^3$ such that no two points have the same first coordinate. Prove that $\mathbb{R}^3 - A$ is simply connected. [*Hint:* Let $\mathbf{v}_1, ..., \mathbf{v}_k$ be the finite set of points, ordered so that their first coordinates satisfy $x_1 < \ ... \ < x_k$, and let U_j be the open set

$$(x_{j-1}, x_{j+1}) \times \mathbb{R} \times \mathbb{R}$$

for $1 \leq j \leq k$, with the convention that $x_0 = -\infty$ and $x_{k+1} = +\infty$. Then each U_j is simply connected. What can one say about the unions $U_1 \cup ... \cup U_j$?]

> NOTES. The same conclusion holds even if the assumption on the first coordinates is dropped. One way to see this is to use the following result: *If A is a finite subset of $\mathbb{R}^n$, then there is a homeomorphism h from $\mathbb{R}^n$ to itself such that $h(A)$ is a set of points whose first coordinates are all distinct.* Given this result, the conclusion follows because $\mathbb{R}^3 - A$ is homeomorphic to $\mathbb{R}^3 - h(A)$. The homeomorphism h may be constructed as follows: First, show that there is a homeomorphism of the solid disk to itself such that the boundary is sent to itself by the identity and the origin is sent to an arbitrary nonzero point that is close to the origin. Next, find small pairwise disjoint solid disks D_j about the points of A, and let h_j be a self homeomorphism of D_j of the type described. Extend h_j to h by defining it to be the identity off the union of the disks. For suitable choices of the nonzero points, the homeomorphism h will have the desired properties. Finally, the homeomorphism of the solid unit disk may be constructed by writing a point in polar form as $t\mathbf{v}$ where $0 \leq t \leq 1$ and $|\mathbf{v}| = 1$ and sending it to $t\mathbf{v} + (1-t)\mathbf{p}$, where $\mathbf{p}$ is the nonzero point in the open disk. It is left as an exercise for the reader to verify that such a map defines a homeomorphism that is the identity on the boundary.
>
> One can combine the remarks in the previous paragraph with the three dimensional analog of Exercise 2 to prove an analog of the latter for open sets in $\mathbb{R}^3$ of the form $\mathbb{R}^3 - A$ for some finite set A.

4. Let $A = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a finite set or points in $\mathbb{R}^2$ such that no two points have the same first coordinate. Prove that the fundamental group of $\mathbb{R}^2 - A$ is generated by the classes of the curves $C_1, \ ... \ ,C_k$ where C_j is a small counterclockwise circle about the point $\mathbf{v}_j \in A$. [*Hint:* Imitate the proof of Exercise 3.]

5. Let A be as in the preceding exercise without the assumption on first coordinates. Prove that there are functions P_j, Q_j on $\mathbb{R}^2 - A$ such that the partial of P_j with respect to y equals partial of Q_j with respect to x for all j and the integral of $P_j\,dx + Q_j\,dy$ over C_i is equal to 2π if $i = j$ and 0 if $i \neq j$.

6. Let A and B be finite subsets of $\mathbb{R}^2$ with pairwise distinct first coordinates, and assume that B contains fewer points than A. Prove that $\mathbb{R}^2 - A$ and $\mathbb{R}^2 - B$ are not homeomorphic. [*Hint:* Let k and ℓ be the numbers of points in A and B respectively. Using Exercise 4, prove that the image of $\pi_1(\mathbb{R}^2 - B)$ in a real vector space always has dimension at most ℓ. On the other hand, using Exercise 5 show that the line integrals over the closed curves C_1 etc. in $\mathbb{R}^2 - A$ define a homomorphism into $\mathbb{R}^k$ whose image spans the latter.]

> NOTES. As with Exercise 3, the same conclusion holds even if the assumption on the first coordinates is dropped.

7. Let U be a connected open subset of $\mathbb{R}^2$, let P and Q be functions with continuous partials defined on U that satisfy

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

and let Γ be a continuous curve in U defined on $[a, b]$. Prove that there is a positive constant $\delta > 0$ such that for all partitions Δ of $[a, b]$ with $|\Delta| < \delta$ the value of

$$\int_{\mathrm{Lin}(\Gamma,\Delta)} P\,dx + Q\,dy$$

is independent of Δ. [*Hint:* Look at the proof of the lemma. A partition Δ' is said to be a refinement of Δ if every point of Δ is also in Δ'. Why does it suffice to prove that the integrals for Δ and Δ' are the same? Why does every pair of partitions have a common refinement? Why does it suffice to consider the special case where Δ' is obtained by adjoining a single point to Δ? How can this be established? Similar sorts of arguments can be found in the discussion of arc length in many real variables texts.]

8. Let p be an analytic function defined on a neighborhood of the disk of radius $\leq R$ such that p has no zeros on the boundary circle. Prove that p has a zero in the disk if the winding number integral is nonzero.

9. (A version of Rouché's Theorem.) In the situation above, suppose that $|q| < |p|$ for $|z| \leq R$ and that the winding number integral for p is nonzero. Prove that the winding number integral for $p + q$ is also nonzero.