LUSTERNIK – SCHNIRELMANN CATEGORY AND CUP PRODUCTS

Every compact topological n-manifold is a union of fintely many open subsets U_i such each U_i is homeomorphic to $\mathbb{R}^n$. Since each such open subset is noncompact, it is clear that one needs at least two such open subsets, and of course S^n is an example where the minimum number is exactly two. More generally, one can ask the following question:

> *Suppose that X is a compact Hausdorff space which has at least one open covering consisting entirely of contractible sets. What is the* MINIMUM *number of such sets that are needed to form an open covering of X?*

The following homotopy-theoretic concept is closely related to this question:

Definition. Let X be a second countable, locally compact, Hausdorff space. Then X is said to have *Lusternik-Schnirelmann* or **LS** *category* $\leq m$ if X is a union of m subsets U_i such that the inclusions $U_i \subset M$ are nullhomotopic.

Note. Frequently one finds slightly different spellings of the names "Lusternik" and "Schnirelmann" based upon different conventions for transliterating the Cyrillic letters into Latin counterparts.

Definition. We shall say that X has Lusternik-Schnirelmann or **LS** category equal to k if it has **LS** category $\leq k$ but does not have **LS** category $\leq k-1$. Similarly, we shall say that X has **LS** category $\geq k$ it X does not have **LS** category $\leq k-1$.

If X is a compact topological n-manifold which has a covering by k open subsets, each homeomorphic to $\mathbb{R}^n$, then it follows immediately that X has **LS** category $\leq k$.

The **LS** *category of T^n*

Here is the main result:

THEOREM 1. *The n-torus T^n has* **LS** *category equal to $n+1$.*

The proof that T^n has **LS** category $\geq n+1$ will be a consequence of the following general observation.

THEOREM 2. *Suppose that X is an arcwise connected, second countable, locally compact, Hausdorff space with* **LS** *category $\leq m$, and let $u_1 \in H^{d(1)}(X,\mathbb{F}), \cdots, u_m \in H^{d(m)}(X;\mathbb{F})$ with $d(i) > 0$ for all i. Then $u_1 \cdots u_m = 0$.*

If the conclusion of the theorem holds for an arcwise connected space X, we shall say that X *has* cuplength $\leq m$ because every product of m positive-dimensional cohomology classes in X is equal to zero.

Proof. Let $W_1, \cdot, W_m$ be a covering of X such that each inclusion $W_i \to X$ is nullhomotopic. Since each cohomology restriction map $H^{m(i)}(X;\mathbb{F}) \to H^{m(i)}(W_i;\mathbb{F})$ is trivial, the classes u_i lift to classes v_i in the relative cohomology groups $H^{m(i)}(X, W_i;\mathbb{F})$. It follows that $u_1 \cdots u_m$ is the image of $v_1 \cdots v_m$ is the group

$$H^*(X, \cup_i U_i;\mathbb{F}) \quad = \quad H^*(X,X;\mathbb{F}) \quad = \quad 0$$

and hence this product equals zero.■

Since there are n classes in $H^1(T^n;\mathbb{F})$ whose cup product is nonzero, Theorem 2 implies that T^n has **LS** category $\geq n+1$.

It follows immediately that every open covering of T^n by sets homeomorphic to $\mathbb{R}^n$ consists of at least $n+1$ sets.

Conversely, one can construct an explicit open covering of T^n with $n+1$ open sets as follows: Let $p:\mathbb{R}^n \to T^n$ be the usual universal covering projection sending $(t_1, \cdots, t_n)$ to $(\exp 2\pi\, i\, t_1, \cdots, \exp 2\pi\, i\, t_n)$, and let $a_0, \cdots, a_n$ be distinct points in the half-open interval $[0,1)$, so that the points $z_k = \exp 2\pi\, i\, a_k \in S^1$ are distinct. Now let $W_k \subset \mathbb{R}^n$ be the set of all points such that $a_k < t_k < a_k + 1$ for all k, and take $V_k \subset T^n$ to be the image of W_k under p. By construction each set V_k is contractible. A point of T^n will lie in $T^n - V_k$ if and only if at least one of its coordinates is equal to z_k. The intersection of the sets $T^n - V_k$ will consist of all points $(b_1, \cdots, b_n)$ such that for each k, there is some j for which $b_j = z_k$. Since there are $n+1$ values of z_k and only n coordinates b_j, this is impossible. Therefore $\cap_k\ (T^n - V_k) = \emptyset$, so that $T^n = \cup_k\ V_k$.

It follows immediately that T^n has Lusternik-Schnirelmann category equal to $n+1$. — We shall conclude this document with a general statement about **LS** category for a general n-manifold.

Estimating the **LS** *category of n-manifolds*

The principal result is easy to state.

THEOREM 3. *If M is a (second countable) arcwise connected topological n-manifold, then M has* **LS** *category $\leq n+1$.*

Theorem 1 follows immediately from Theorem 3 and the statement after the proof of Theorem 2. We have seen that T^n has **LS** category $n+1$, but in general the inequality in the theorem is strict; for example, the n-sphere has **LS** category equal to 2,

The proof of Theorem 3 depends upon the following result about open coverings of n-manifolds:

PROPOSITION 4. *Let M^n be a (second countable) arcwise connected topological n-manifold, and let $\mathcal{U}$ be an open covering of M. Then there is a finite open covering of M by open subsets $V_0, \cdots, V_n$ such that each V_i is a union of open sets $W_{i,j}$ such that (a) if $j \neq k$ then $W_{i,j} \cap W_{i,k} = \emptyset$, (b) the family $\mathcal{W} = \{W_{i,j}\}$ is locally finite, (c) each open set $W_{i,j}$ is contained in some element of $\mathcal{U}$.*

A proof of this result is given on the first half of page 19 in the following online notes:

`http://maths.ed.ac.uk/~aar/papers/difftop.pdf`

The concept of Lebesgue covering dimension plays a crucial role in this argument.

Proof that Proposition 4 implies Theorem 3. Let $\mathcal{U}$ be an open covering of M consisting of open sets which are each homeomorphic to $\mathbb{R}^n$, and take $V_0, \cdots, V_n$ and $\mathcal{W} = \{W_{i,j}\}$ as in the statement of Proposition 4. Then each inclusion $W_{i,j} \to X$ is nullhomotopic because $W_{i,j} \subset U_{\alpha(i,j)}$ for some $\alpha(i,j)$ and the sets $U_{\alpha(i,j)}$ are all contractible. Since each V_i is a union of the pairwise disjoint open subsets $W_{i,j}$ (with i held fixed), these nullhomotopies piece together to form a nullhomotopy for the inclusion map $V_i \subset M$. Hence for every open set V_i the inclusion into M is nullhomotopic, and therefore it follows from the definition that M has **LS** category $\leq n+1$.■

References for further information

The *Wikipedia* article

`http://en.wikipedia.org/wiki/Lustenik%E2%80%Schnirelmann_category`

is a good starting point for learning more about the concept of Lusternik-Schnirelmann category, and it gives several good references for further information on the topic. The book by Cornea, Lupton, Oprea and Tanré (cited in that article) contains a very thorough treatment of this subject.