The Isotopy Extension Theorem

Once we know that tubular neighborhoods and collar neighborhoods exist, a natural follow – up question is the extent to which they are unique. Even in the simplest cases it is clear that many such neighborhoods exist. For example, if f is a smooth map from $\mathbb{R}^n$ to itself which maps the origin to itself, then one can view f as defining a tubular neighborhood of the origin in $\mathbb{R}^n$. However, it also seems reasonable to say that each such map is equivalent to the standard tubular neighborhood which is given by the identity map on $\mathbb{R}^n$. This clarifies the goal, which is to define a suitable equivalence relationship on tubular neighborhoods such that any two are equivalent. Before doing this, we shall digress to discuss a concept which enters into this equivalence relation.

Definition. Let M and N be two smooth manifolds, and let f and g be smooth embeddings of M into N. We shall say that f and g are ***isotopic*** if there is a smooth map $H: M \times (-\varepsilon, 1+\varepsilon) \to N$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all x. Less formally, the map H, which is called an ***isotopy***, is a smooth homotopy from f to g through smooth embeddings.

The notion of isotopy defines an equivalence relation on the set of all smooth embeddings from one manifold M to another one N; the reflexive and symmetric properties follow immediately, and the transitive property is a consequence of the following exercise.

Exercise. Suppose that f and g are isotopic as above. Prove that there is a smooth isotopy K_t from f to g such that $K(x,t) = f(x)$ if $-\delta < t < \delta$ and $K(x,t) = g(x)$ if $1-\delta < t < 1+\delta$. [***Hint***: Use smooth bump functions to make the original homotopy stationary near $t = 0$ and $t = 1$.]

Notation. One often says that an isotopy is a ***diffeotopy*** if for each t the map H_t is a diffeomorphism. If in addition the "initial diffeomorphism" H_0 is an identity map, the isotopy/diffeotopy is often called an ***ambient diffeotopy*** or an ***ambient isotopy***.

Probably the most elementary examples of isotopies are given in elementary geometry. The intuitive concept of a rigid motion of an object X in some Euclidean space V can be modeled mathematically by a homotopy h_t from X to V such that h_0 is the inclusion of X in V and each map h_t is an isometry. To motivate the main theorem, we shall state a result in Euclidean geometry about rigid motions which one might reasonably expect to be true but is not stated or proved very often.

Isometry Extension Principle. Let A be a subset of $\mathbb{R}^n$, and let $h_t : \mathbb{R}^n \to \mathbb{R}^n$ be a homotopy such that each h_t is an isometry. Then there is a homotopy of isometries $H_t : \mathbb{R}^n \to \mathbb{R}^n$ such that H_0 is the identity and for each a in A and each t we have $H_t(a) = h_t(a)$. If h_t is a smooth homotopy then we can take H_t to be a smooth isotopy.

This result is basically a **1** – parameter version of the ***Isometry Extension Theorem*** which is stated and proved on pages 12 – 13 of the following online document:

http://math.ucr.edu/~res/math205A/metgeom.pdf

In order to avoid going too far off – topic we shall not give the proof here; the argument is fairly elementary but somewhat lengthy.

After giving a simple definition, it will be time to state the first main result.

Definition. Let f be a homeomorphism of a topological space X to itself. Then the **support** of f, written **Supp(*f*)**, is defined to be the ***closure*** of the set of all x in X such that $f(x) \neq x$.

Isotopy Extension Theorem. Let N be a compact smooth manifold, let M be an unbounded smooth manifold, and let h_t be a smooth isotopy of embeddings from N into M for $-\varepsilon < t < 1 + \varepsilon$. Then there exists a smooth ambient isotopy K_t of M with uniformly compact support such that $K_t | \mathrm{N} = h_t$ for $-\varepsilon' < \mathrm{t} < 1 + \varepsilon'$, where $0 < \varepsilon' \leq \varepsilon$.

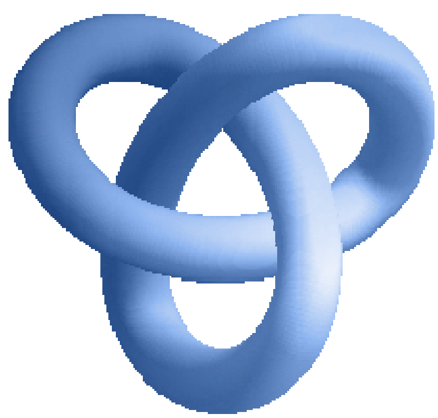
In this result, ***uniformly compact support*** means that the union of the supports of **all** the diffeomorphisms K_t has compact closure.

There is a concise and clearly written proof of this result in Section 8.1 (more precisely, pages 177 – 180) of the following book, which we shall use extensively:

M. W. Hirsch, ***Differential Topology*** (Graduate Texts in Mathematics, Vol. 33). Springer – Verlag, New York ***etc.***, 1976.

This book also proves versions of the Isotopy Extension Theorem which are valid if the boundary of M is nonempty, and it gives several other important applications of this result.

Application to knot theory. The Isotopy Extension Theorem plays an important role in ***knot theory***, which is the study of (say) smoothly embedded simply closed curves in $\mathbb{R}^3$. For all dimensions $n > 1$ except **3**, basic results in topology imply that ***every smoothly embedded simply closed curve in*** $\mathbb{R}^n$ ***is smoothly isotopic to the standard closed curve given by the unit circle in*** $\mathbb{R}^2$; this follows from the Schönflies Theorem if $n = 2$ (which we shall discuss later!), and when $n \geq 4$ it follows from Exercise 10 on page 183 of Hirsch. The basic question of knot theory is this: ***Suppose we are given two simply closed smooth curves in*** $\mathbb{R}^3$. ***Is there an isotopy from one to the other?*** — One particularly simple example along these lines is the trefoil know pictured below; physically it corresponds to tying a simple knot in a rope or string (think about what happens if we glue together the end points of the rope or string; this corresponds to the second picture below), and in physical terms the question corresponds to determining whether one can bend the knot, without cutting it, so that it will lie flat on a plane without crossing itself. This is also equivalent to asking whether one can untie the corresponding knotted rope or string while the end points are held fixed.



http://im-possible.info/images/articles/trefoil-knot/trefoil-knot.gif

If it were possible to find a smooth isotopy h_t from the trefoil knot to the standard closed curve in the plane, then the isotopy extension theorem would yield an ambient isotopy H_t such that $h_t = H_t h_0$, and the map H_1 would define a diffeomorphism from the complement of the trefoil knot to the complement of the standard closed curve. Therefore it would follow that the fundamental groups of the complements of the trefoil knot and the standard closed curve would be isomorphic. However, basic results in knot theory imply that the fundamental group of the complement of the trefoil knot is nonabelian and the fundamental group of the complement of the standard circle is infinite cyclic. The following classic book gives a self – contained but relatively direct account of this result:

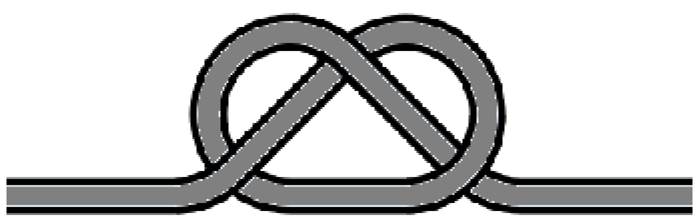
R. H. Crowell and R. H. Fox, ***Introduction to Knot Theory*** (Reprint of the original 1963 edition). Dover Books, New York, 2008.

There are also many other excellent books on this subject, most of which take it much further; there are too books many to list here, so we shall only give a couple of online references:

http://en.wikipedia.org/wiki/Knot_theory

http://library.thinkquest.org/12295/

A negative example. The Isotopy Extension Theorem does **not** necessarily hold for smooth embeddings of noncompact manifolds, even if one assumes that each embedding in the isotopy is a closed mapping. To construct the standard counterexample, we start with the standard flat embedding of the real line in $\mathbb{R}^3$ as the x – axis and modify it by replacing one piece of a curve with a cut trefoil knot as in the picture below:



We can define a smooth isotopy from this embedding to the standard one by mathematically modeling the geometrical idea of pushing the knot off to infinity (towards the right). One can choose the parametrizations such that the limit of the **1** – parameter smooth embedding family is the previously described flat inclusion and the family extends to a smooth isotopy from the knotted embedding to the flat one. To see that this isotopy does not extend to an ambient isotopy, we need to use some results from knot theory. Specifically, the fundamental group of the complement of the flat line is infinite cyclic (in fact, the complement is diffeomorphic to $S^1 \times \mathbb{R}^2$), but the fundamental group of the complement of the knotted curve turns out to be the fundamental group for the complement of the trefoil knot. As before, if an ambient isotopy existed, then the complements of the two embeddings of the line would be diffeomorphic and hence would have isomorphic fundamental groups. Since their fundamental groups are not isomorphic, no such ambient isotopy exists.

Applications to tubular and collar neighborhoods

We are interested in finding a geometrically meaningful concept of equivalence such that two tubular neighborhoods of a smooth submanifold are related in the prescribed fashion. The organization of this document clearly suggests that isotopy is part of this equivalence concept. However, here is a simple (but extremely important!) example to show that something more is also needed:

Example. Take the embedding of a point in $\mathbb{R}^n$ by inclusion of the origin, and consider the tubular neighborhoods of the origin given by invertible linear transformations of $\mathbb{R}^n$. In particular, consider the identity transformation I and a hyperplane reflection S which fixes all unit vectors except the last one and sends that vector to its negative. We claim that these two tubular neighborhoods are not smoothly isotopic, where we assume that the isotopy is stationary on the origin. Suppose that the tubular neighborhoods are isotopic. Then it will follow that the standard inclusion j of the unit sphere in $\mathbb{R}^n \setminus \{0\}$ and the composite of j followed by S are homotopic as maps into $\mathbb{R}^n \setminus \{0\}$. Now the latter map sends the sphere into itself by reflection through an equatorial hypersurface, and as such its degree is equal to -1. Since the analogous degree for the inclusion map is $+1$, it follows that the tubular neighborhoods defined by the two orthogonal linear transformations cannot be isotopic.

The classical uniqueness theorem for tubular neighborhoods states that the proper notion of equivalence is a combination of isotopy as before with a generalized notion of linear conjugacy corresponding to the issue raised in the preceding paragraph. This result is stated and proved on pages 111 – 113 of Hirsch (***Warning:*** The definition of isotopy for tubular neighborhoods in Hirsch must be read carefully in order to interpret the uniqueness theorem correctly.):

Uniqueness Theorem for Tubular Neighborhoods. Let N be a smooth submanifold of M (both without boundary), and suppose that we are given two tubular neighborhoods $f : E \to M$ and $g : E' \to M$. Then there exists an isotopy $H_t : E \to M$ such that $f = H_0$ and $g = \phi H_1$, where $\phi : E \to E'$ is a ***smooth vector bundle isomorphism***; in other words, ϕ is a diffeomorphism and for each x in N it sends the vector space fiber E_x over x to the fiber E'_x by a linear isomorphism.

We should note that, even in cases where we know ***a priori*** that $E = E'$, the fiberwise linear map Φ need not be isotopic to the identity; the preceding example illustrates this phenomenon.

Sketch of proof. The first step is to modify f by an isotopy to a tubular neighborhood f_1 so that $f_1[E]$ is contained in $g[E']$. To do this, first let W be the open set $f^{-1}\big[g[E']\big]$, so that W is a neighborhood of the zero section. Put a Riemannian metric on E; then there is a smooth positive valued smooth function $\delta(x)$ on M such that for each x in M the set of all vectors in E with length less than $\delta(x)$ is contained in W. We can now construct a shrinking isotopy K_t such that K_0 is the identity and K_1 maps each vector space E_x onto the set of all vectors in E_x of length less than $\delta(x)$; we can construct this isotopy so that each K_t will be the identity on the set of vectors in E_x of length less than $\delta(x)/2$. The map $f_1 = f K_1$ is then a tubular neighborhood which is isotopic to f and maps E into the image of E'. Since isotopy is a transitive relation, it will suffice to prove the uniqueness theorem for tubular neighborhoods satisfying this additional condition, so henceforth we shall assume it holds for f. In such instances it will be enough to prove the result when $N = E'$ and g is the identity map (if we know this special case, then we can compose with g to retrieve the general case).

To summarize the preceding, we need only consider the special case of a tubular neighborhood $f : E \rightarrow E'$ which sends the zero section to itself by the identity. Define a partial smooth isotopy H_t for $0 \leq t < 1$ by the formula

$$H_t(\mathrm{v}) = (1-t)^{-1} f\big((1-t)\,\mathrm{v}\big).$$

We want to extend this to an interval of the form $-\varepsilon < t < 1 + \varepsilon$, and the main point is to determine what happens when $t = 1$.

This is basically a local problem, so consider the special case when N is an open subset of $\mathbb{R}^n$, the tubular neighborhood E is a product bundle $\mathrm{N} \times \mathbb{R}^{n-m}$ and N corresponds to $N \times \{0\}$ in $\mathrm{E}' = N \times \mathbb{R}^{n-m}$; actually, we need to consider a slightly shrunken version N_0 in N because the mappings H_t are not necessarily fiber preserving. In this local setting, one can use the definition of derivative to conclude that H_1 should be given by the map sending (x, v) to

$$\big(x,\ \Pi_2\,[Df(x)]\,(0, \mathrm{v})\big)$$

where Π_2 is projection onto the last $m - n$ coordinates of $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{n-m}$. By construction, $Df(x)$ is an isomorphism for all x, and since f is the identity map on $N \times \{0\}$ it follows that $Df(x)$ maps $\mathbb{R}^n \times \{0\}$ to itself to the identity; these observations combine to show that the composite $\Pi_2 Df(\mathbf{x})$ is a linear isomorphism on $\{0\} \times \mathbb{R}^{n-m}$. Thus we can construct H_1 locally; by continuity these definitions must agree on overlapping charts, and therefore it follows that one obtains a well – defined diffeomorphism from $E \to E'$ which is the identity on the zero section and maps fibers to fibers by linear isomorphisms.

The case of collar neighborhoods

A similar argument yields the following result:

Theorem. Let M be a manifold with boundary, and let $f, g : \partial M \times \mathbb{R}_+ \to M$ be collar neighborhoods of the boundary. Then f and g are isotopic by an isotopy which fixes the boundary (note that the half open interval $[0, 1)$ is diffeomorphic to $\mathbb{R}_+$).

Proof. The same methods as above show that f is isotopic to a collar neighborhood of the form $g(x, h(x)t)$ where h is a positive valued smooth function on the boundary. This is true because (i) the argument proving uniqueness of tubular neighborhoods implies that f is isotopic to g composed with a vector bundle automorphism of $\partial M \times \mathbb{R}_+$, (ii) every linear isomorphism of $\mathbb{R}$ sending $\mathbb{R}_+$ to itself is multiplication by a positive constant, (iii) by the preceding observations, the linear automorphism must have the form described above. There is an obvious straight line homotopy from h to the constant function $k(x) = 1$, and this defines a smooth isotopy of vector bundle automorphisms from $g(x, h(x)t)$ to $g(x, t)$; by the transitivity of isotopy, we see that f and g must be isotopic.

Closed tubular neighborhoods

Up to this point we have only discussed open tubular neighborhoods; since closed neighborhoods are often extremely useful in point set topology, it is natural to look for closed analogs of open tubular neighborhoods. These can be constructed fairly easily using Riemannian metrics. Since tubular neighborhoods are given by total spaces of smooth vector bundles, it suffices to consider the case where M is the zero section in E, where $\pi: E \to M$ is a smooth vector bundle over M. The following elementary observation has some far – reaching consequences:

Lemma. Suppose that U is open in $\mathbb{R}^k$ for some n and g is a smooth Riemannian metric on the product bundle $U \times \mathbb{R}^k$ for some k. Then there is an inner product preserving vector bundle isomorphism from $U \times \mathbb{R}^k$ with the trivial Riemannian metric and $U \times \mathbb{R}^k$ with the Riemannian metric g.

Sketch of proof. The main idea is to show that $U \times \mathbb{R}^k$ has a family of smooth cross sections σ_k such that are orthonormal everywhere. Such a family can be constructed by starting with the unit cross sections $\sigma_k(u) = (u, \mathbf{e}_k)$ and using the Gram – Schmidt process to obtain an orthonormal family.

Corollary. Let $\pi: E \to M$ be a smooth k – dimensional vector bundle over M, let g be a smooth Riemannian metric on this bundle, let $D(\pi)$ denote the set of all vectors of length ≤ 1, and let $S(\pi)$ denote the set of all vectors of length 1. Then each point x in M has an open neighborhood U and a fiber – preserving diffeomorphism from $\pi^{-1}[U] \to U \times \mathbb{R}^k$ such that $D(\pi)$ and $S(\pi)$ correspond to $U \times D^k$ and $U \times S^{k-1}$ respectively. In particular, it follows that $D(\pi)$ is a manifold with boundary, and its boundary is $S(\pi)$. Likewise, the complementary subspace $E \setminus \mathrm{INT}\, D(\pi)$ is a manifold with boundary, and its boundary is $S(\pi)$.

Notation. The subspaces/submanifolds $D(\pi)$ and $S(\pi)$ are called the ***associated unit disk and sphere bundles*** respectively.

Definition. A closed tubular neighborhood of a submanifold N in M (both without boundary) is a smooth embedding f of an associated disk bundle $D(\alpha)$ into M such that the restriction of f to the zero section is the inclusion mapping. The existence of smooth Riemannian metrics and the usual tubular neighborhood theorem imply that closed tubular neighborhoods always exist.

Obviously, we would like to have a more refined uniqueness theorem for closed tubular neighborhoods. However, we begin with some elementary consequences of the definition.

Exercises. **1.** Suppose that N and M as above are connected, and let f be a closed tubular neighborhood with domain $D(\alpha)$. Prove that f extends to an open tubular neighborhood whose domain is the total space $E(\alpha)$.

2. In the setting of the preceding exercise, prove that $M \setminus S(\alpha)$ has precisely two (connected/path) components. What are they?

Orthogonalizing vector bundle isomorphisms. At this point we need to address a fundamental problem: Suppose we are given two vector bundles A, B over the same space X, and assume that we are given a Riemannian metric on each one. If $\mathbf{T}: A \to B$ is a vector bundle isomorphism (sending fibers to fibers by linear isomorphisms), can $\mathbf{T}$ be deformed to an isomorphism which preserves the Riemannian metrics? One can ask this question in both the topological and smooth categories.

The first step is to establish the following local result:

Lemma. Suppose that W is an open disk of radius 3 in $\mathbb{R}^n$ for some n and that $F: V \times \mathbb{R}^k \to V \times \mathbb{R}^k$ is a (continuous or smooth) vector bundle isomorphism which is an orthogonal isomorphism on a closed subset A. Then F is (continuously or smoothly) homotopic, through vector bundle isomorphisms, to a vector bundle isomorphism G which is an orthogonal isomorphism on $A \cup D^n$ and agrees with F over the set B of all u in W such that $|u| \geq 2$. Furthermore, one can choose the homotopy so that it is stationary over $A \cup B$.

Sketch of proof. This begins with yet another elaboration of the Gram – Schmidt process. Specifically, we shall use the latter to show that the orthogonal group O_k is a strong deformation retract of the group $GL(k, \mathbb{R})$ of all invertible matrices. If P is an invertible matrix, then its columns form a basis for $\mathbb{R}^k$. Let Q be the orthonormal matrix obtained by applying the Gram – Schmidt process. The formulas for obtaining the orthonormal basis imply that $Q = PC$, where C is a $k \times k$ lower triangular matrix with positive entries down the diagonal. We can now write C as a product $T_1 E_1$, where T_1 is lower triangular with ones down the diagonal and E_1 is a diagonal matrix with positive entries. It follows that we have $P = QDU$, where D is a diagonal matrix with positive entries and U is lower triangular with ones down the diagonal. Furthermore, by the Gram – Schmidt process and Cramer's Rule the entries of Q, D and U are all rational functions of the entries of P; furthermore, if P is orthogonal then D and U are identity matrices. Suppose that the positive diagonal entries of D are d_i; then there is a canonical curve joining D to the identity through diagonal matrices with positive entries, and the entries of $D(t)$ are given by $t + (1-t)d_i$. Likewise, if we write $U = I + N$, then N is strictly triangular and hence nilpotent, and we have a canonical curve joining U to the identity through similar matrices which is given by $U(t) = I + tN$. It follows immediately that the mapping sending P to Q defnes a smooth retraction onto the subgroup of orthogonal matrices and the homotopy $H_t(P) = QD(t)U(t)$ defines a smooth homotopy from the identity on $GL(k, \mathbb{R})$ to the composite $GL(k, \mathbb{R}) \to O_k \to GL(k, \mathbb{R})$, where the first map is the retraction from $GL(k, \mathbb{R})$ to O_k and the second is inclusion.

Express F in the form $F(u, \mathrm{v}) = (u, C(u)\mathrm{v})$ where C is a (continuous or smooth) map from W to $GL(k, \mathbb{R})$. Let η be a smooth function on $\mathbb{R}$ with values in the interval $[0, 1]$ such that $\eta = 1$ for $t \leq 1$ and $\eta = 0$ for $t \geq 2$, and define a new vector bundle isomorphism by

$$G(u, \mathrm{v}) = (\mathbf{u}, H_{1-\eta[t]}[C(u)]\,\mathrm{v}).$$

There is a homotopy from F to G given by $H_{s(1-\eta(t))}$, and it has the following properties:

1. The homotopy is stationary over $A \cup B$.
2. The initial map H_0 correspond to F.
3. The final map H_1 corresponds to G, and G is an orthogonal bundle isomorphism over D^n.

This is the homotopy that we want.

We now also have the following approximation result.

Theorem. Let M be a smooth manifold, let E and E' be smooth vector bundles over M with smooth Riemannian metrics, and suppose that $T: E \to E'$ is a vector bundle isomorphism. Then T is smoothly isotopic to an orthogonal vector bundle isomorphism.

Sketch of proof. This is a standard globalization argument. Take a locally finite open covering by smooth charts defined on open disks of radius **3**, such that the subdisks of radius **1** also define an open covering, and put them in order. Assume that one has constructed a smooth isotopy to a map which is orthogonal over the union A_m of the first m closed disks (we can clearly find such a map when $q = 0$). We can then use the lemma to define an isotopy to a map which agrees with the given one on A_m and is orthogonal over A_{m+1}. By local finiteness we obtain a well – defined limit map which is orthogonal everywhere and is isotopic to the original one.

This has an immediate consequence for tubular neighborhoods.

Uniqueness of closed tubular neighborhoods. Let N be a smooth submanifold of M (both without boundary), and suppose that we are given two closed tubular neighborhoods $f : D(\alpha) \to M$ and $g : D(\alpha') \to M$. Then there exists an isotopy $H_t : D(\alpha) \to M$ such that $f = H_0$ and $g = \phi H_1$, where $\phi : D(\alpha) \to D(\alpha')$ is a ***smooth orthogonal vector bundle isomorphism***; in other words, ϕ is a diffeomorphism and for each x in N it sends the disk fiber $D(\alpha)_x$ over x to the fiber $D(\alpha')_x$ by an orthogonal isomorphism. Furthermore, there is an ambient isotopy L_t of M such that $L_t H_0 = H_t$.

Corollary. In the preceding notation, the closed complements $M \setminus \mathrm{INT}\, D(\alpha)$ and $M \setminus \mathrm{INT}\, D(\alpha')$ are diffeomorphic.

Example. The complements of **open** tubular neighborhoods are not necessarily diffeomorphic; one can construct simple examples where the submanifold is a point and $M = \mathbb{R}^n$, in which case one tubular neighborhood is the identity (so the complement is empty), and the other maps $\mathbb{R}^n$ diffeomorphically to a disk of bounded diameter such that $\mathbf{0}$ is sent to itself so the complement is nonempty).

Modifications for closed collar neighborhoods

Here is the corresponding uniqueness result for closed collar neighborhoods. The proof is similar to its counterpart for closed tubular neighborhoods.

Theorem. Let M be a manifold with boundary, and let $f, g : \partial M \times [0, 1] \to M$ be closed collar neighborhoods of the boundary. Then f and g are ambient isotopic by an ambient isotopy which fixes the boundary.

An important special case

Perhaps surprisingly, the one of the most trivial special cases — where the submanifold is a point — is particularly important in studying the structure of manifolds.

Cerf – Palais Disk Theorem (J. Cerf, R. S. Palais). Let M be a connected smooth n – manifold without boundary, and let $f, g : D^n \to M$ be smooth embeddings (it follows that these extend to smooth embeddings on slightly larger open disks). Then either f and g are ambient isotopic or else f and gS are ambient isotopic, where S is the reflection diffeomorphism of D^n defined by the diagonal matrix whose diagonal entries are $(1, \dots, 1, -1)$.

We have already seen an example in which the embeddings f and fS are not isotopic, and therefore the second option is necessary.

The proof of this theorem requires the following basic result:

__Lemma.__ For all $n > 0$ the orthogonal group O_n has two components, and two matrices lie in the same component if and only if their determinants are equal.

<u>Sketch of proof for the lemma.</u> Since we know that O_n is a deformation retract of $GL(n, \mathbb{R})$, it will suffice to prove that two invertible matrices lie in the same component of the latter group if and only if their determinants have the same sign, or equivalently that a matrix lies in the component of the identity if and only if its determinant is positive.

We know that every invertible matrix is a product of elementary matrices obtained from the identity by one of a few basic operations:

1. Interchange two rows.
2. Multiply one row by a positive constant.
3. Multiply one row by -1.
4. Add a nonzero multiple of one row to another.

If we are given a matrix of the second type, then clearly there is a continuous curve in the space of invertible matrices joining the given matrix to the identity. Likewise, if we are given a matrix of the fourth type we may express it in the form $I + kE_{u,v}$, where k is a nonzero constant and $E_{u,v}$ is the matrix which has a 1 in the (u, v) position and zeros elsewhere. An explicit curve joining this matrix to the identity through invertible matrices is given by $I + (1-t)kE_{u,v}$. If we insert these curves into the factorization of the matrix into elementary matrix, we see that the original example lies in the same arc component as a product of matrices of the first and third types. The determinant of each such matrix is equal to -1, and since we started with a matrix with positive determinant, it follows that there must be an even number of such matrices in the factorization.

At this point it will be helpful to make two observations. First of all, the $\mathbf{2 \times 2}$ matrix $-\boldsymbol{I}$ can be joined to the identity by the following continuous curve, which always lies in the group of orthogonal matrices:

$$\begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}$$

Likewise, the orthogonal $\mathbf{2 \times 2}$ matrix $\boldsymbol{Q}$ obtained by switching the rows in the identity matrix can be joined to the diagonal matrix with entries $(\mathbf{1}, -\mathbf{1})$ by the following continuous curve $\boldsymbol{P}(t)^{-1}\boldsymbol{Q}\boldsymbol{P}(t)$, where $\boldsymbol{P}(t)$ is the following orthogonal matrix:

$$\begin{pmatrix} \cos(\pi t/4) & -\sin(\pi t/4) \\ \sin(\pi t/4) & \cos(\pi t/4) \end{pmatrix}$$

This curve also lies in the group of orthogonal matrices.

We can use the second observation to say that our new product of elementary matrices lies in the same arc component as a product of matrices of the third type. This product will be a diagonal with $\pm\mathbf{1}$'s down the diagonal, and there must be an even number of them. We can now use the first observation to say that if the number of negative entries is positive, then this diagonal matrix lies in the same arc component as a diagonal matrix of the same type with two fewer positive entries; if repeat this process, we eventually conclude that all of these matrices must lie in the same arc component as the identity. Therefore we have shown that the original matrix lies in the same component as the identity, which was our goal. Note that we have can in fact construct a smooth curve joining the two matrices (verify this; the discussion in the next paragraph provides one means for doing so).

Proof of the Cerf – Palais Disk Theorem. The first step is to reduce everything to the special case where $f(\mathbf{0}) = g(\mathbf{0})$. We claim that if $\boldsymbol{p}$ and $\boldsymbol{q}$ are points of $\boldsymbol{M}$ then there is a smooth curve joining them. Define a binary relation on $\boldsymbol{M}$ such that two points are related if such a curve exists. This relation is clearly symmetric and reflexive, and we would like to show that it is also transitive so that we have an equivalence relation. Suppose we are given points such that a can be joined to $\boldsymbol{b}$ and $\boldsymbol{b}$ can be joined to $\boldsymbol{c}$. Then we have a **piecewise smooth** curve joining $\boldsymbol{a}$ to $\boldsymbol{c}$ with the only possible nonsmooth point at $\boldsymbol{b}$, and we would somehow like to smooth it out near $\boldsymbol{b}$ without changing it outside a small neighborhood of that point. This is really a local problem, and the details are explained in the following document:

http://math.ucr.edu/~res/math205A/nicecurves.pdf

The rest of the proof for the claim is fairly standard, for we have an equivalence relation which is locally constant on a connected space, so that the equivalence classes must be pairwise disjoint open sets, and by connectedness there can be only one of them.

The smooth curve joining a pair of points can be viewed as a smooth isotopy of embeddings for the one point space viewed as a $\mathbf{0}$ – dimensional manifold, and therefore the Isotopy Extension Theorem implies that this isotopy extends to an ambient isotopy H_t. Given an arbitrary pair of disk embeddings f and g, the ambient isotopy yields a new pair $H_1 f$ and g which send $\mathbf{0}$ to the same point in M, and it will suffice to prove the theorem for the embeddings $H_1 f$ and g. Thus we have the desired reduction of the proof to the case where $f(\mathbf{0}) = g(\mathbf{0})$.

By the uniqueness of closed tubular neighborhoods, we know that f is isotopic to $g\,Q$, where Q is some orthogonal matrix (the vector bundle isomorphism corresponds to an invertible linear transformation for vector bundles over a one point space). By the lemma, we know that Q can be joined to exactly one of the matrices I or S described in the statement of the theorem. Using a smooth curve $Q(t)$ joining Q to one of these matrices we obtain a smooth isotopy $g\,Q(t)$ from $g\,Q$ to either g or $g\,Q$.

Remark. There are examples of manifolds in which f and fS are ambient isotopic. The simplest one is the Möbius strip; if we take f so that its image is a disk on this surface for which the image of the x – axis is the curve through the middle of the strip, then if we transport this disk once around the middle curve it will be sent to itself by reflection about that axis. The following **YouTube** graphic illustrates this phenomenon; after the robot goes once around the center curve it is standing upside down with its right and left sides transposed.

http://www.youtube.com/watch?v=rYIyXPnWPXc